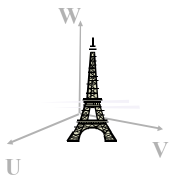
Lecture 13:
Camera Projection II

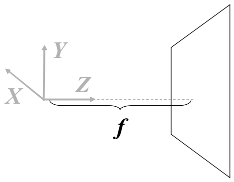
Reading: T&V Section 2.4

Recall: Imaging Geometry

Object of Interest in World Coordinate System (U,V,W)



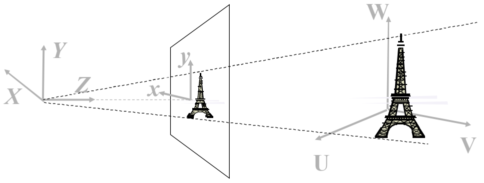
Imaging Geometry



Camera Coordinate System (X,Y,Z).

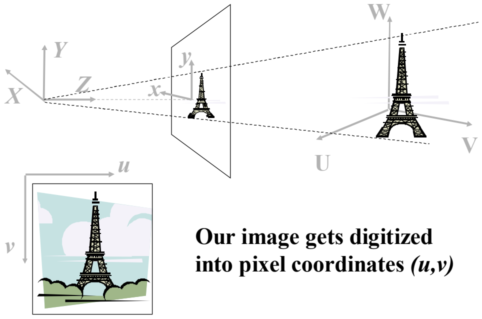
- Z is optic axis
- Image plane located f units out along optic axis
- f is called focal length

Imaging Geometry



Forward Projection onto image plane. 3D (X,Y,Z) projected to 2D *(x,y)*

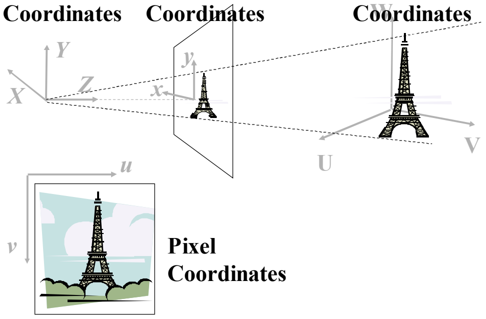
Imaging Geometry



Our image gets digitized into pixel coordinates *(u,v)*

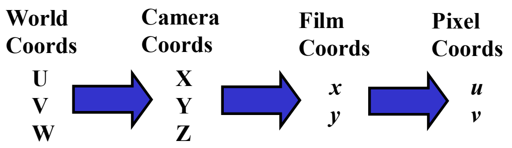
Imaging Geometry

Camera Coordinates **Image (film) Coordinates** **World Coordinates**



Pixel Coordinates

Forward Projection



World Coords
U V W

Camera Coords
X Y Z

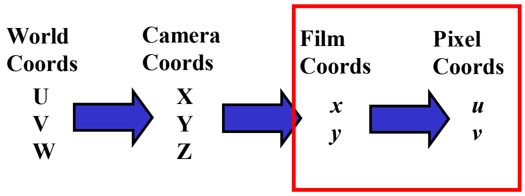
Film Coords
x y

Pixel Coords
u v

We want a mathematical model to describe how 3D World points get projected into 2D Pixel coordinates.

Our goal: describe this sequence of transformations by a big matrix equation!

Intrinsic Camera Parameters



World Coords
U V W

Camera Coords
X Y Z

Film Coords
x y

Pixel Coords
u v

Affine Transformation

Intrinsic parameters

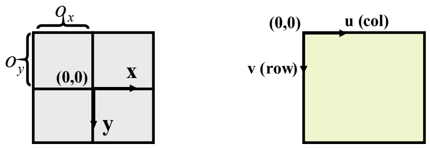
- Describes coordinate transformation between film coordinates (projected image) and pixel array
- Film cameras: scanning/digitization
- CCD cameras: grid of photosensors

still in T&V section 2.4

Intrinsic parameters (offsets)

film plane
(projected image)

pixel array



$$u = f\frac{X}{Z} + o_x \qquad v = f\frac{Y}{Z} + o_y$$

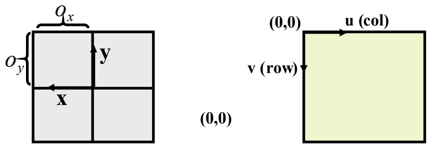
o_x and o_y called image center or principle point

Intrinsic parameters

sometimes one or more coordinate axes are flipped (e.g. T&V section 2.4)

film plane

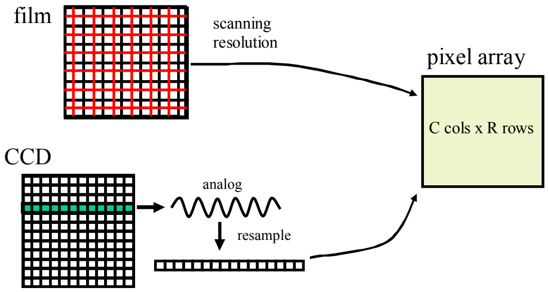
pixel array



$$u = -f\frac{X}{Z} + o_x \qquad v = -f\frac{Y}{Z} + o_y$$

Intrinsic parameters (scales)

sampling determines how many rows/cols in the image



Effective Scales: s_x and s_y

$$u = \frac{1}{s_x} f \frac{X}{Z} + o_x \qquad v = \frac{1}{s_y} f \frac{Y}{Z} + o_y$$

Note, since we have different scale factors in x and y, we don't necessarily have square pixels!

Aspect ratio is s_y / s_x

Perspective projection matrix

Adding the intrinsic parameters into the perspective projection matrix:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} f/s_x & 0 & o_x & 0 \\ 0 & f/s_y & o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

To verify:

$$u = \frac{x'}{z'} \qquad v = \frac{y'}{z'} \quad \Rightarrow \quad u = \frac{1}{s_x} f \frac{X}{Z} + o_x \qquad v = \frac{1}{s_y} f \frac{Y}{Z} + o_y$$

Note:

Sometimes, the image and the camera coordinate systems have opposite orientations: [the book does it this way]

$$f\frac{X}{Z} = \overset{\downarrow}{-}(u - o_x)s_x \qquad f\frac{Y}{Z} = \overset{\downarrow}{-}(v - o_y)s_y \quad \Rightarrow \quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -f/s_x & 0 & +o_x & 0 \\ 0 & -f/s_y & +o_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Note 2

In general, I like to think of the conversion as a separate 2D affine transformation from film coords (x,y) to pixel coordinates (u,v):

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}_{\mathbf{aff}}} \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{M}_{\mathbf{proj}}} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\mathbf{u} = \mathbf{M}_{\mathbf{int}}\, \mathbf{P}_{\mathbf{C}} = \mathbf{M}_{\mathbf{aff}}\, \mathbf{M}_{\mathbf{proj}}\, \mathbf{P}_{\mathbf{C}}$$

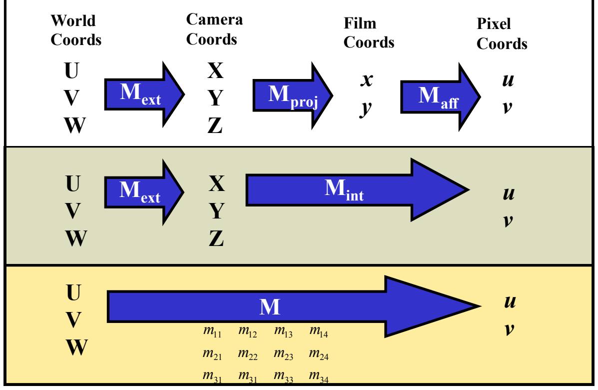
Huh?

Did he just say it was "a fine" transformation?

No, it was "affine" transformation, a type of 2D to 2D mapping defined by 6 parameters.

More on this in a moment...

Summary : Forward Projection



Summary: Projection Equation

$$\begin{bmatrix}u\\v\\1\end{bmatrix} \sim \underbrace{\underbrace{\underbrace{\overset{\text{Film plane to pixels}}{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\0&0&1\end{bmatrix}}}_{\mathbf{M}_{\text{aff}}}\underbrace{\overset{\text{Perspective projection}}{\begin{bmatrix}f&0&0&0\\0&f&0&0\\0&0&1&0\end{bmatrix}}}_{\mathbf{M}_{\text{proj}}}}_{\mathbf{M}_{\text{int}}}\underbrace{\overset{\text{World to camera}}{\begin{bmatrix}r_{11}&r_{12}&r_{13}&t_x\\r_{21}&r_{22}&r_{23}&t_y\\r_{31}&r_{32}&r_{33}&t_z\\0&0&0&1\end{bmatrix}}}_{\mathbf{M}_{\text{ext}}}}_{\mathbf{M}}\begin{bmatrix}U\\V\\W\\1\end{bmatrix}$$

Lecture 13/14:
Intro to Image Mappings

Image Mappings Overview

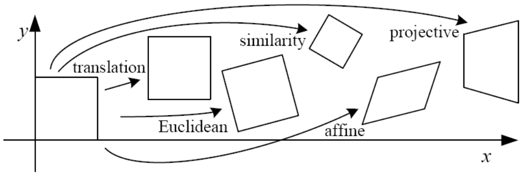
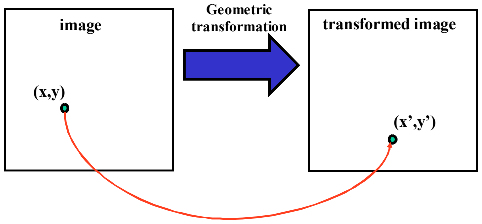


FIGURE 1. Basic set of 2D planar transformations

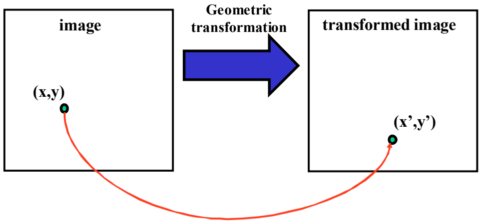
from R.Szeliski

Geometric Image Mappings



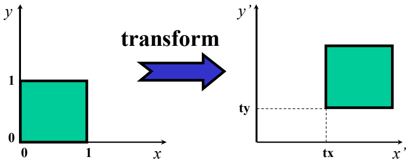
x' = f(x, y, {parameters})
y' = g(x, y, {parameters})

Linear Transformations
(Can be written as matrices)



$$\begin{bmatrix}x'\\y'\\1\end{bmatrix} = \left(M(\text{params})\right)\begin{bmatrix}x\\y\\1\end{bmatrix}$$

Translation



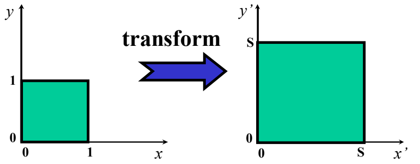
$$x' = x + t_x$$
$$y' = y + t_y$$

equations

$$\begin{bmatrix}x'\\y'\\1\end{bmatrix} = \begin{bmatrix}1&0&t_x\\0&1&t_y\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\1\end{bmatrix}$$

matrix form

Scale



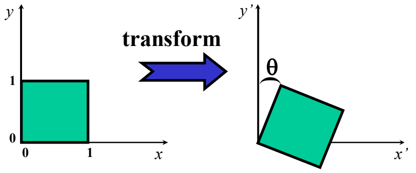
$$x' = s\,x_i \qquad y' = s\,y_i$$

equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Rotation



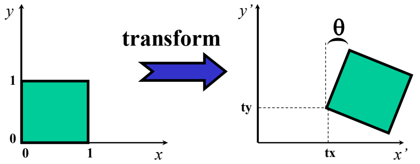
$$x' = x_i\cos\theta - y_i\sin\theta \qquad y' = x_i\sin\theta + y_i\cos\theta$$

equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Euclidean (Rigid)



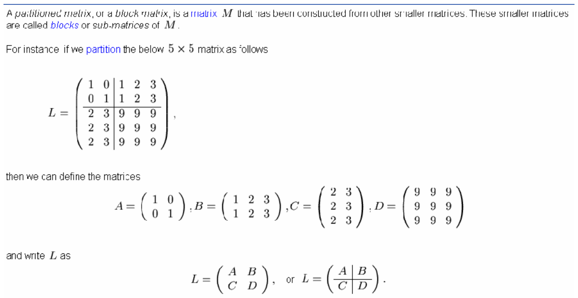
$$x' = x_i\cos\theta - y_i\sin\theta + t_x \qquad y' = x_i\sin\theta + y_i\cos\theta + t_y$$

equations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

matrix form

Partitioned Matrices



A *partitioned matrix*, or a *block matrix*, is a matrix M that has been constructed from other smaller matrices. These smaller matrices are called *blocks* or *sub-matrices* of M.

For instance if we partition the below 5×5 matrix as follows

$$L = \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 9 & 9 & 9 \\ 2 & 3 & 9 & 9 & 9 \\ 2 & 3 & 9 & 9 & 9 \end{array}\right),$$

then we can define the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}, D = \begin{pmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{pmatrix}$$

and write L as

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{or} \quad L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

http://planetmath.org/encyclopedia/PartitionedMatrix.html

Partitioned Matrices

$$\left[\begin{array}{c} x' \\ y' \\ \hline 1 \end{array}\right] = \left[\begin{array}{cc|c} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ \hline 1 \end{array}\right]$$

$$\begin{bmatrix} \overset{2x1}{p'} \\ \overset{1x1}{1} \end{bmatrix} = \begin{bmatrix} \overset{2x2}{R} & \overset{2x1}{t} \\ \overset{1x2}{0} & \overset{1x1}{1} \end{bmatrix} \begin{bmatrix} \overset{2x1}{p} \\ \overset{1x1}{1} \end{bmatrix}$$

matrix form

$$p' = Rp + t$$

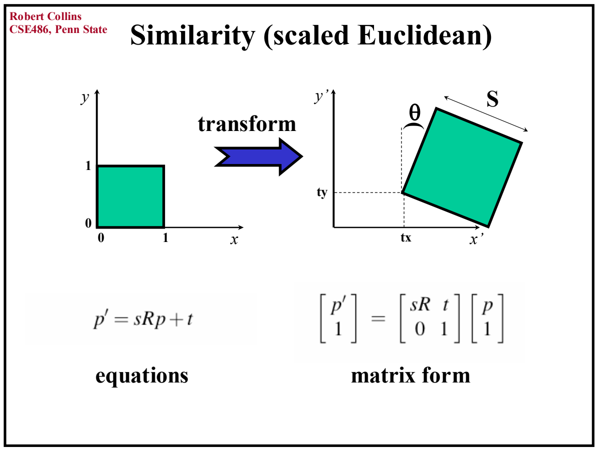
equation form

Another Example (from last time)

$$\left(\begin{array}{c} X \\ Y \\ Z \\ \hline 1 \end{array}\right) = \left[\begin{array}{ccc|c} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left(\begin{array}{c} U \\ V \\ W \\ \hline 1 \end{array}\right)$$

$$\begin{pmatrix} \overset{3x1}{P_C} \\ \overset{1x1}{1} \end{pmatrix} = \begin{pmatrix} \overset{3x3}{R} & \overset{3x1}{T} \\ \overset{1x3}{0} & \overset{1x1}{1} \end{pmatrix} \begin{pmatrix} \overset{3x1}{P_W} \\ \overset{1x1}{1} \end{pmatrix}$$

$$P_C = R\,P_W + T$$



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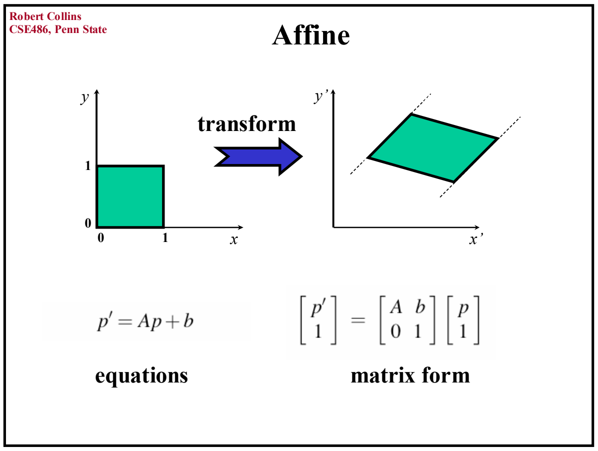
Similarity (scaled Euclidean)

$$p' = sRp + t$$

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

equations **matrix form**



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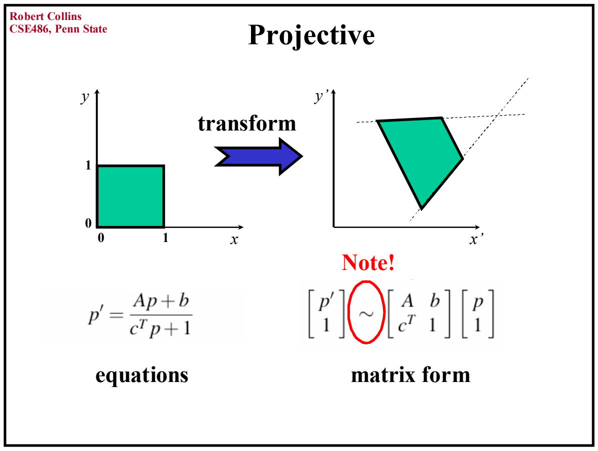
Affine

$$p' = Ap + b$$

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

equations **matrix form**



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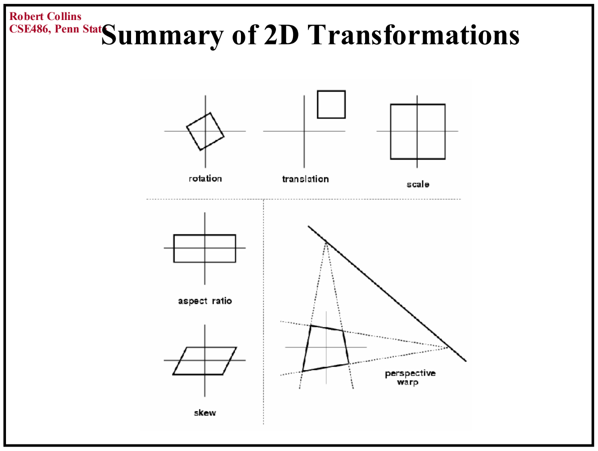
Projective

Note!

$$p' = \frac{Ap + b}{c^T p + 1}$$

$$\begin{bmatrix} p' \\ 1 \end{bmatrix} \sim \begin{bmatrix} A & b \\ c^T & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

equations **matrix form**



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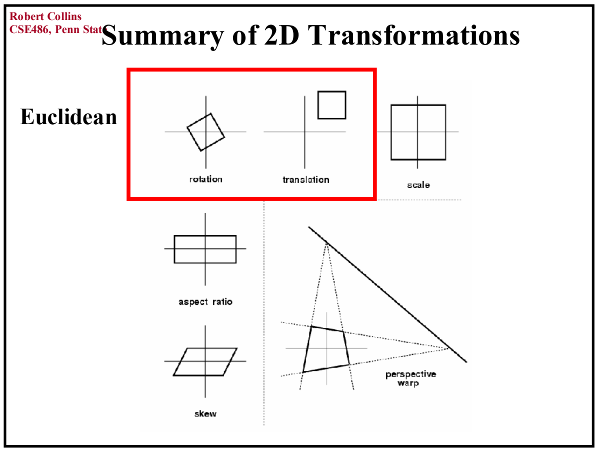
Summary of 2D Transformations



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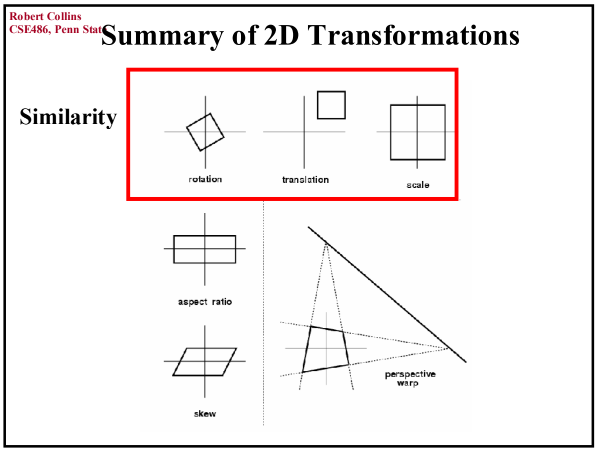
Summary of 2D Transformations

Euclidean



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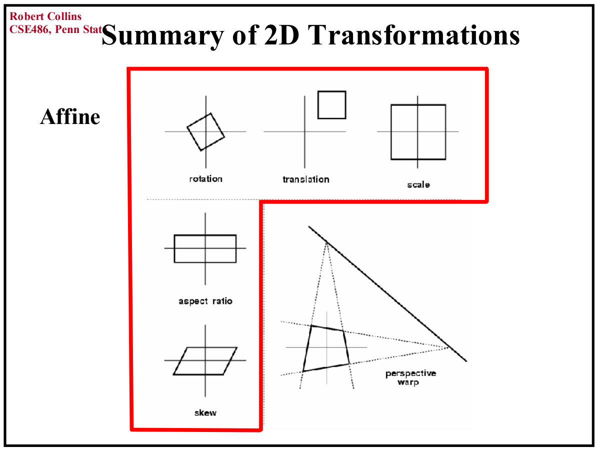
Summary of 2D Transformations

Similarity



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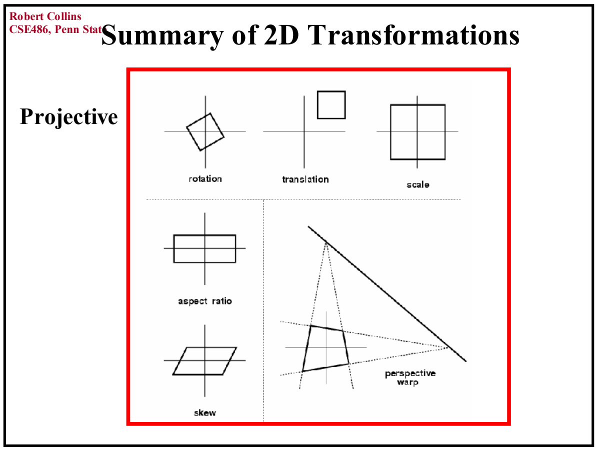
Summary of 2D Transformations

Affine



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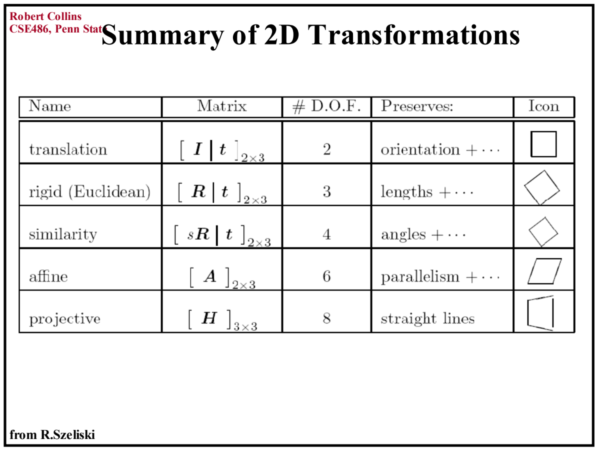
Summary of 2D Transformations

Projective



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Summary of 2D Transformations

Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\left[\begin{array}{c\|c} I & t \end{array}\right]_{2\times 3}$	2	orientation + $\cdots$	□
rigid (Euclidean)	$\left[\begin{array}{c\|c} R & t \end{array}\right]_{2\times 3}$	3	lengths + $\cdots$	◇
similarity	$\left[\begin{array}{c\|c} sR & t \end{array}\right]_{2\times 3}$	4	angles + $\cdots$	◇
affine	$\left[\begin{array}{c} A \end{array}\right]_{2\times 3}$	6	parallelism + $\cdots$	▱
projective	$\left[\begin{array}{c} H \end{array}\right]_{3\times 3}$	8	straight lines	⏢

from R.Szeliski