The Rules of Single-Player Craps

- First roll a pair of (assumed fair) six-sided dice.
  - If the outcome (dice sum) of the first roll X is 7 or 11 then the player wins,
  - else if the $X \in \{2, 3, 12\}$ then the player loses ("craps out"),
  - else define the "point" $X \in \{4, 5, 6, 8, 9, 10\}$ and the player continues to roll the dice.
- For each subsequent (independent) roll of the dice:
  - If the player rolls the point X then they win,
  - else if the player rolls 7 then they lose,
  - else they roll again.

Single-Player Craps - Preliminaries

- Define T as the number of rolls in the craps game.
- Note that $\{T > 1\} = \{X \in A\}$, where $A = \{4, 5, 6, 8, 9, 10\}$.
- Let Y be the outcome of a dice toss independent of X.
- The outcome of (last) toss $T > 1$ is $\sim (Y | Y \in \{X, 7\})$, *i.e.*, is distributed as Y conditioned on $Y \in \{X, Y\}$.
- Let W be the winning event.
- Generally, all games of chance in casinos have the property that the probability of winning is less than half, $\mathrm{P}(W) < 0.5$, *i.e.*, one is more likely to lose.

The probability of winning craps

$$
\begin{aligned}
\mathrm{P}(W) &= P(W, T=1) + \mathrm{P}(W, T>1) \\
&= \mathrm{P}(X \in \{7,11\}) + \mathrm{P}(Y = X | Y \in \{X, 7\}) \\
&= \mathrm{P}(X \in \{7,11\}) + \sum_{i \in A} \mathrm{P}(X = i, Y = i | Y \in \{i, 7\}) \\
&= \mathrm{P}(X \in \{7,11\}) + \sum_{i \in A} \mathrm{P}(X = i)\mathrm{P}(Y = i | Y \in \{i, 7\}) \\
&= \mathrm{P}(X \in \{7,11\}) + \sum_{i \in A} \mathrm{P}(X = i)\mathrm{P}(Y = i)/\mathrm{P}(Y \in \{i, 7\}) \\
&= \mathrm{P}(X \in \{7,11\}) + \sum_{i \in A} \mathrm{P}(X = i)^2/\mathrm{P}(Y \in \{i, 7\}) \\
&= \frac{6+2}{36} + \frac{(\frac{3}{36})^2}{\frac{3+6}{36}} + \frac{(\frac{4}{36})^2}{\frac{4+6}{36}} + \frac{(\frac{5}{36})^2}{\frac{5+6}{36}} + \frac{(\frac{5}{36})^2}{\frac{5+6}{36}} + \frac{(\frac{4}{36})^2}{\frac{4+6}{36}} + \frac{(\frac{3}{36})^2}{\frac{3+6}{36}} \\
&= \frac{244}{495} = 0.493
\end{aligned}
$$

Exercises

- Find the distribution of T. Hint: It is geometric.
- Argue that W and T are conditionally independent given $\{T > 1\}$.
- Simulate 1000 independent, complete craps games and plot the running sample mean and sample standard deviation of the empirical probability of winning.