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Computational Symmetry
Today’s Theme:

Different types of symmetry and symmetry groups
Summary of last lecture

• Symmetry is a transformation
• Symmetry can only be defined with respect to a set S
• Mathematical definition of symmetry $g$ of set $S$ is: $g(S) = S$
• All the symmetries of $S$ form the symmetry group of $S$
• many different types of symmetries in Euclidean space?
  – Translation
  – Rotation
  – Reflection
  – Glide-reflection
  – ...?
Definition of Symmetry

If \( g \) is a distance preserving transformation in \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), and \( S \) is a subset of \( \mathbb{R}^n \), then \( g \) is a symmetry of \( S \) iff \( g(s) \in S \), where \( s \) is an element of \( S \).

Such that \( g(S) = \{ g(s) \mid s \in S \} = S \), i.e., \( S \) is setwise invariant under the automorphic transformation \( g \).

An example:

- Reflection axis
- 4-fold rotations

\[ g \]

a square plate in \( \mathbb{R}^2 \)
Symmetry Group

All symmetries of a subset S of Euclidean space $\mathbb{R}^n$ have a group structure G, and G is called the \textit{symmetry group} of S.
A **group** $G$ is a set of elements with a binary operation $*$ defined on the set that satisfy:

- $*$ is **associative** $a*(b*c) = (a*b)*c$
- there is an **identity** element $id$ in $G$, $id*a=a=a*id$
- any element $g$ in $G$ has an **inverse** $g^{-1}$ such that $g*g^{-1}=id$ and $g^{-1}*g=id$

An important “**closure**” property of a group implied by the binary operation $*$:

$G$ is **closed**, for any $a,b$ in $G$, $a*b$ also in $G$
Symmetry Group $G$ versus set $S$
An example:

$G = \text{symmetries of a square}$

* = transformation composition

$id = ?$

a square plate in $\mathbb{R}^2$
How to categorize, organize and compare different types of symmetry groups?
Euclidean Group

Euclidean group $E(n)$ is the symmetry group of $n$-dimensional Euclidean space. Its elements, the isometries associated with the Euclidean metric, are called Euclidean moves.
Euclidean Group of $\mathbb{R}^3$

All symmetries of $\mathbb{R}^3$ form the symmetry group of the 3D Euclidean space and is called the **Euclidean Group** $\text{E}(3)$

Examples of elements in the Euclidean Group are:
- Rotation
- Translation
- Reflection
- Glide-reflection
Proper Euclidean Group

All handedness-preserving isometries form the proper Euclidean group that excludes reflections
How to organize these infinitely many symmetry groups?
Subgroup

Definition:
Let \((G,\cdot)\) be a group and let \(H\) be a subset of \(G\).
Then \(H\) is a subgroup of \(G\) defined under the same operation if \(H\) is a group by itself (with respect to \(\cdot\)).

The subgroup is denoted likewise \((H, \cdot)\).
We denote \(H\) being a subgroup of \(G\) by writing \(H \subseteq G\).
An example:

\[ G = \text{symmetries of a square} \]

\[ * = \text{transformation composition} \]

All symmetries = (id, rot90, rot180, rot270, ref1, ref2, ref3, ref4)

A subgroup = (id, rot90, rot180, rot270)
Definitions of Different Types of Subgroups of the Euclidean Group
### Different subgroups of the proper Euclidean Group

<table>
<thead>
<tr>
<th>Canonical Groups</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Identity Group</strong></td>
<td>$G_{id}$ {1}</td>
</tr>
<tr>
<td><strong>Rotation Subgroups</strong></td>
<td></td>
</tr>
<tr>
<td>$SO(3)$</td>
<td>$\text{gp}{\text{rot}(i, \theta)\text{rot}(j, \sigma)\text{rot}(k, \phi)</td>
</tr>
<tr>
<td>$O(2)$</td>
<td>$\text{gp}{\text{rot}(k, \theta)\text{rot}(l, n\pi)</td>
</tr>
<tr>
<td>$SO(2)$</td>
<td>$\text{gp}{\text{rot}(k, \theta)</td>
</tr>
<tr>
<td>$D_{2n}$</td>
<td>$\text{gp}{\text{rot}(k, 2\pi/n)\text{rot}(i, m\pi)</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$\text{gp}{\text{rot}(k, 2\pi/n)</td>
</tr>
<tr>
<td><strong>Translation Subgroups</strong></td>
<td></td>
</tr>
<tr>
<td>$\mathcal{T}^1$</td>
<td>$\text{gp}{\text{trans}(0, 0, z)</td>
</tr>
<tr>
<td>$\mathcal{T}_{\text{dis}}(t_0)$</td>
<td>$\text{gp}{\text{trans}(0, 0, t_0)</td>
</tr>
<tr>
<td>$\mathcal{T}^2$</td>
<td>$\text{gp}{\text{trans}(x, y, 0)</td>
</tr>
<tr>
<td>$\mathcal{T}^3$</td>
<td>$\text{gp}{\text{trans}(x, y, z)</td>
</tr>
<tr>
<td><strong>Mixed Subgroups</strong></td>
<td></td>
</tr>
<tr>
<td>$G_{cyl}$</td>
<td>$\text{gp}{\text{trans}(0, 0, z)\text{rot}(k, \theta)\text{rot}(i, n\pi)</td>
</tr>
<tr>
<td>$G_{\text{dir-cyl}}$</td>
<td>$\text{gp}{\text{trans}(0, 0, z)\text{rot}(k, \theta)</td>
</tr>
<tr>
<td>$G_{\text{plane}}$</td>
<td>$\text{gp}{\text{trans}(x, y, 0)\text{rot}(k, \theta)\text{rot}(i, n\pi)</td>
</tr>
<tr>
<td>$G_{\text{dir-plane}}$</td>
<td>$\text{gp}{\text{trans}(x, y, 0)\text{rot}(k, \theta)</td>
</tr>
<tr>
<td>$G_{\text{screw}}(p)$</td>
<td>$\text{gp}{\text{trans}(0, 0, z)\text{rot}(k, 2\pi/p)</td>
</tr>
<tr>
<td>$G_{T_1C_2}$</td>
<td>$\text{gp}{\text{trans}(0, 0, z)\text{rot}(i, n\pi)</td>
</tr>
<tr>
<td>$\mathcal{E}^+$</td>
<td>$\text{gp}{\text{trans}(x, y, z)\text{rot}(i, \theta)\text{rot}(j, \sigma)\text{rot}(k, \phi)</td>
</tr>
</tbody>
</table>
Symmetry or Near-Symmetry Patterns are ubiquitous

a square shape in $\mathbb{R}^2$
Symmetry or Near-Symmetry Patterns are ubiquitous
Symmetries in 3D

- Tetrahedron
- Cube (or Decahedron)
- Dodecahedron
- Octahedron
- Cylindrical
- Spherical
- Screw
- Revolute
- Planar
- Flat
Hierarchy of Symmetries
Q1. For each element in the set S, which one is kept **invariant**?
Types of Symmetry Groups G: for all $g$ in $G$, $g(S) = S$

- **Point group**: All symmetries $g$ in group $G$ leave at least one point in $\mathbb{R}^3$ invariant

- **Space group**: no point in $\mathbb{R}^3$ is left invariant by all the symmetries $g$ in $G$
Point groups

Platonic Solids
Discrete Point Groups as Subgroups of the Euclidean Group

| Group Name         | $|G|$ size |
|--------------------|----------|
| $G_{id}$           | 1        |
| $C_n = G_{cyclic}$ | $n$      |
| $G_{2m} = G_{dihedral}$ | $n = 2m$ |
| $G_{tetrahedral}$  | 12       |
| $G_{octahedral}$   | 24       |
| $G_{icosahedral}$  | 60       |
Q2. Given symmetry groups G1 and G2,

which one is more symmetrical?
Which one is more symmetrical?

or
Group Relations: Group Conjugate

G1, G2 are subgroups of G, we call $G1$ is a conjugate to $G2$ iff for some $g$ in $G$, such that $G1 = g * G2 * g^{-1}$
Definition: Orbits

An orbit of a point \( x \) in \( \mathbb{R}^3 \) under group \( G \) is \( G(x) = \{ g(x) \mid \text{all } g \text{ in } G \} \)
Types of Symmetry Groups $G$ of $S$

- $G$ is finite: $|G| = N$, $N$ is an integer
- $G$ is infinite (using orbit)
  - $G$ is discrete (Definition 1.1.10)
  - $G$ is continuous

**Definition 1.1.10** A discrete group $G$ is a subgroup of $\mathcal{E}$ such that for any $x \in \mathbb{R}^3$ and any sphere $B_r = \{y | y \in \mathbb{R}^3, \|y\| \leq r\}$ there are only a finite number of points in the $G$-orbit of $x$ that are contained in $B_r$. 
## Symmetry Group Categorization

<table>
<thead>
<tr>
<th>Type</th>
<th>Finite</th>
<th>Infinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discrete</td>
<td>$C_n$, $D_{2n}$, G of regular solids</td>
<td>Crystallographic groups</td>
</tr>
<tr>
<td>Non-discrete</td>
<td>???</td>
<td>$T_1C_2$ ...</td>
</tr>
<tr>
<td>Continuous</td>
<td>???</td>
<td>$S(2)$, $S(3)$, $SO(2)$, $SO(3)$, $T_{1-3}$</td>
</tr>
</tbody>
</table>
The hierarchy of the subgroups of The Euclidean Group

- O → orthogonal group
- SO → special orthogonal group
- T → translation group
- D → dihedral group
- C → cyclic group

Figure 1.3: Here the arrows $A \rightarrow B$ means $B$ is a subgroup of $A$. 
An Exploration of

Hierarchies of Symmetry Groups

- Surface Contact/Motion among solids (robotics assembly planning)
- Periodic Patterns (visual)
- Papercut Patterns (fold-then-cut)
Example I:
Symmetry in Contact Motions
Solids in Contact-Motion

Lower-pairs

- Revolute
- Planar
- Cylindrical
- Prismatic
- Spherical
- Screw
Insight:

The contacting **surface** pair from two different solids coincide, thus has the **same** symmetry group which determines their relative motions/locations.
Complete and unambiguous task specifications can be tedious for symmetrical objects.

‘Put that cube in the corner with face 1 on top!’ (4 different ways)

‘Put that cube in the corner!’ (how many different ways?)
Relative Motion and Contacting Surface Symmetry
Constructive Solid Geometry (CSG) Representation

Different types of surfaces associated with different types of groups

The concept of “conjugated symmetry groups” are used here
Algebraic Surface feature and its coordinates

World coordinates

Solid coordinates

Surface coordinates

Contacting surfaces
Example II:
Symmetry in Periodic Patterns
Hilbert’s 18th Problem

Question:

In n-dimensional euclidean space is there … only a finite number of essentially different kinds of (symmetry) groups of motions with a fundamental region?

Answer:

Yes! (Bieberbach and Frobenius, published 1910-1912)
Examples of Seven Frieze Patterns and their symmetry groups
Table 1. Symmetries of frieze pattern tiles (N is number of pixels in one tile)

<table>
<thead>
<tr>
<th>Symmetry Group</th>
<th>translation</th>
<th>2-fold rotation</th>
<th>Horizontal reflection</th>
<th>Vertical reflection</th>
<th>Glide reflection</th>
<th>Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>N</td>
</tr>
<tr>
<td>F2</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>N/2</td>
</tr>
<tr>
<td>F3</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>N/2</td>
</tr>
<tr>
<td>F4</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>N/2</td>
</tr>
<tr>
<td>F5</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>N/4</td>
</tr>
<tr>
<td>F6</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>N/2</td>
</tr>
<tr>
<td>F7</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>N/4</td>
</tr>
</tbody>
</table>
Hierarchy of Frieze Groups
Examples of 17 Wallpaper Patterns and Their Symmetry Groups

From a web page by:
David Joyce, Clark Univ.
lattice units of the 17 wallpaper groups
Subgroup Relationship Among the 17 Wallpaper Groups (Coxeter)
Where does the symmetry group of an affinely deformed pattern migrate to?
Examples of Symmetry Group Migration

- cm
- cm
- p1
- p3m1
Potential Symmetry

= largest Euclidean subgroup in AGA-1

An inherent, affine-invariant, non-face-value property of a pattern
Example III: Reflection Symmetries in Papercut Patterns
SIGGRPHA 2005 Technical Sketch Slides
Symmetry Groups are not just decorative but (computationally and physically) functional

reflection symmetry $\rightarrow$ folding line
Global versus Local Symmetries
Fold and Cut Example
Formal treatment of papercutting

\[ P = \bigcup S_i \]
\[ S_i \cap S_j = \emptyset, i \neq j \]
Characterization by Symmetry Groups

Symmetrical

Frieze Group

Dihedral Group

Asymmetrical
Dihedral group examples

$n = 3$

\[
\text{\begin{tikzpicture}
\draw (0,0) -- (1,0) -- (0.5,0.866) -- cycle;
\end{tikzpicture}}
\]
Dihedral group examples

$n = 3$

[Diagram of a triangle with lines indicating symmetries]
Dihedral group examples

\[ n = 3 \]

\[ \text{Dihedral group example} \]
Dihedral group examples
Dihedral group examples

$n = 3$ $n = 4$ $n = 5$ $n = 6$ $n = 7$ $n = 8$
Dihedral group examples

Fundamental Regions
Dihedral group examples

3 folds 3 folds 4 folds 4 folds 4 folds 4 folds

$n = 3$ $n = 4$ $n = 5$ $n = 6$ $n = 7$ $n = 8$

Fundamental Regions

3 folds 3 folds 4 folds 4 folds 4 folds 4 folds
Dihedral group examples

\[
\begin{align*}
\text{Fundamental Regions} & \\
3 \text{ folds} & & 3 \text{ folds} & & 4 \text{ folds} & & 4 \text{ folds} & & 4 \text{ folds} & & 4 \text{ folds} & & 4 \text{ folds}
\end{align*}
\]

Work required = \( \frac{1}{2^f} \)
Summary

- Mathematical definition of symmetry group $G$ of set $S$
- A group has three properties: associative, identity, inverse
- Group closure property is very important
- Symmetry group $G$ of $S$ and $S$ co-exist
- Symmetry group $G$ has an inner structure: its subgroups
  - Subgroups of Euclidean group
  - Point vs. space groups
  - Finite vs. infinite
  - Discrete vs. continuous
- Different types of symmetry groups and their hierarchy
  - Euclidean group hierarchy
  - Point groups: cyclic, dihedral, regular solids
  - Space groups
    - Frieze group
    - Wallpaper group