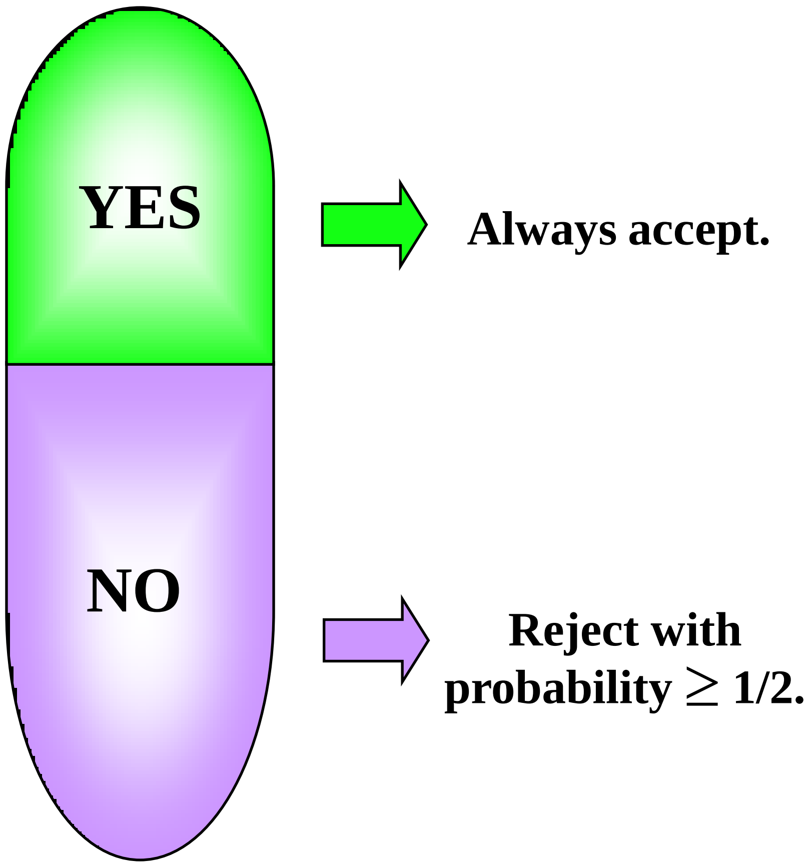
Monotonicity Testing

Yevgeniy Dodis, Oded Goldreich, Eric Lehman, Sofya Raskhodnikova, Dana Ron and Alex Samorodnitsky

Probabilistic Property Testing

Probabilistic Algorithm



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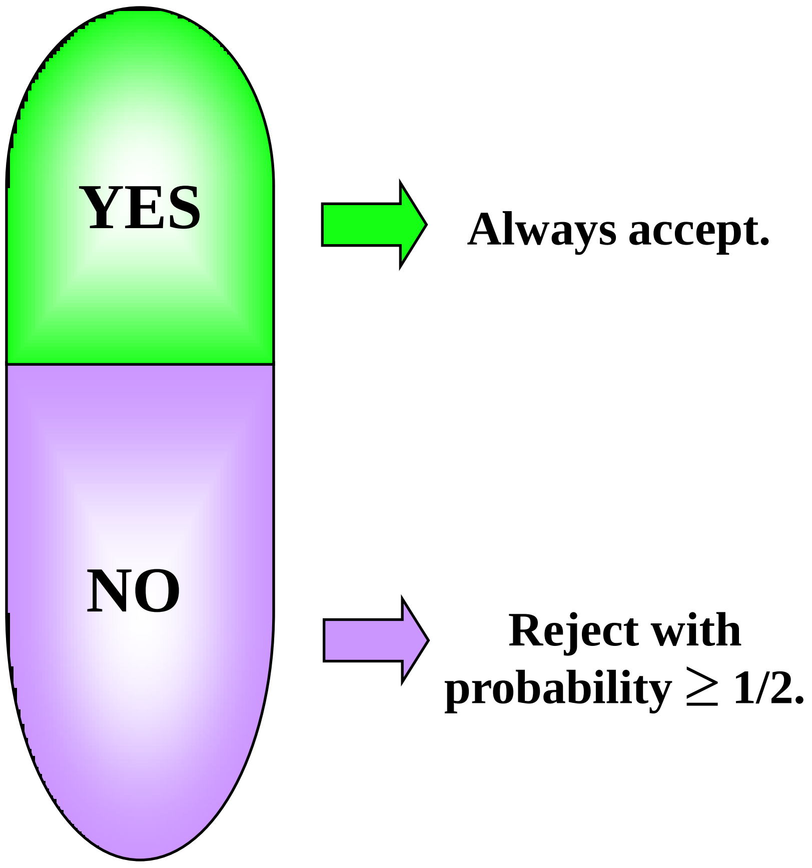
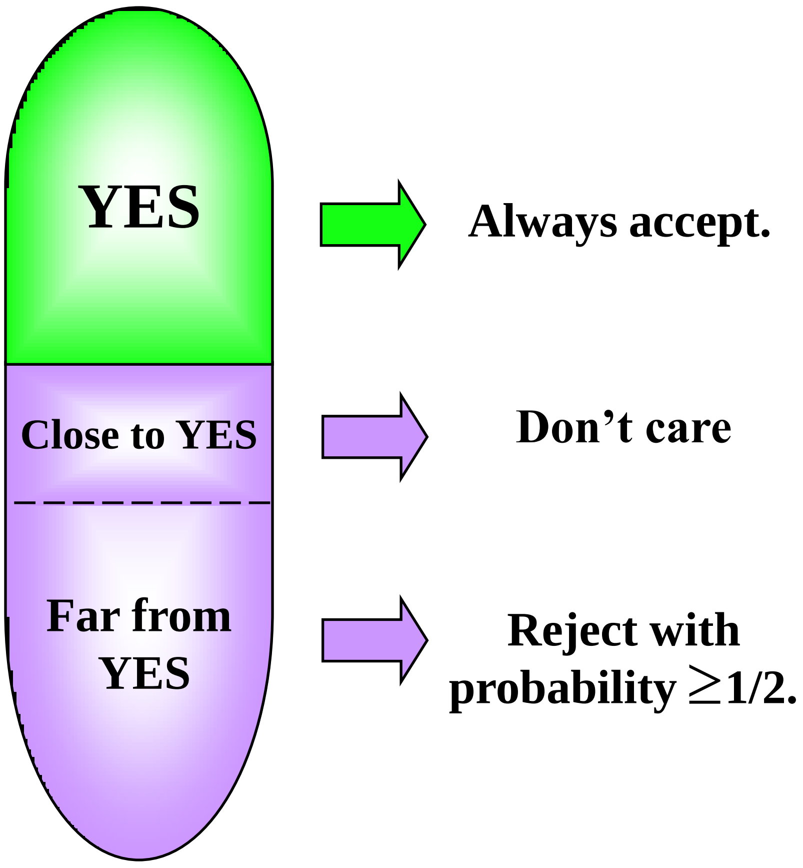


Probabilistic Property Tester



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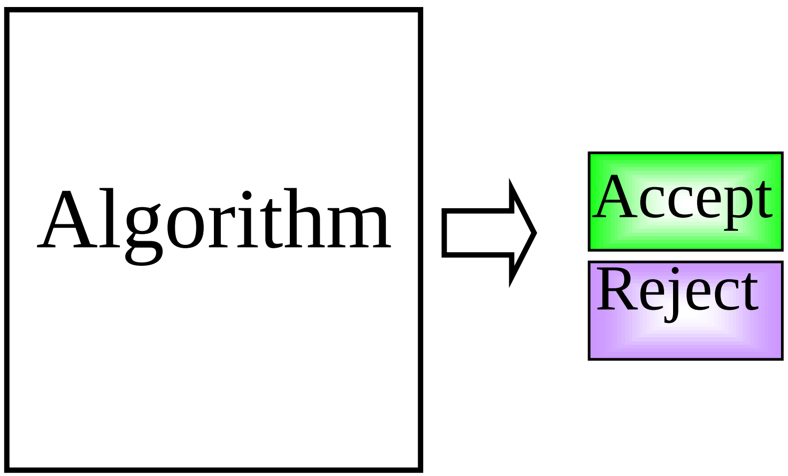
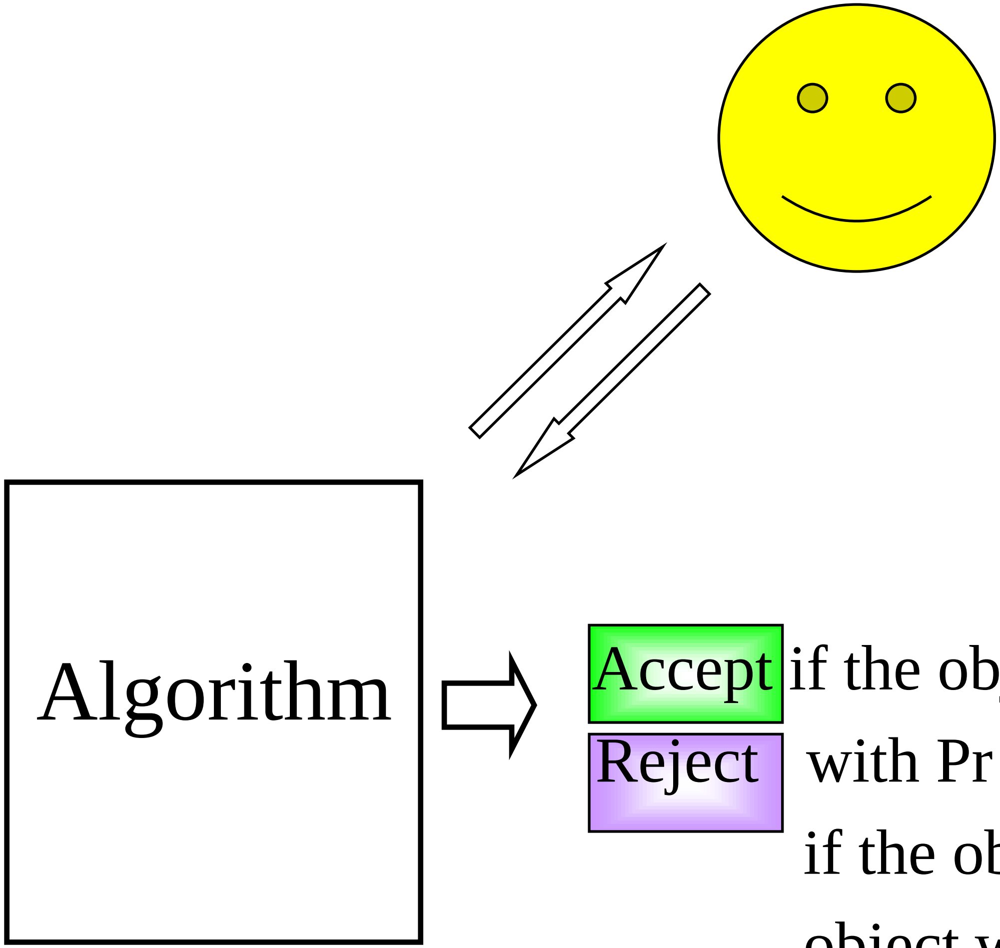


if the object has the property.

with Pr $\geq$ 1/2

if the object is far from any object with the property.

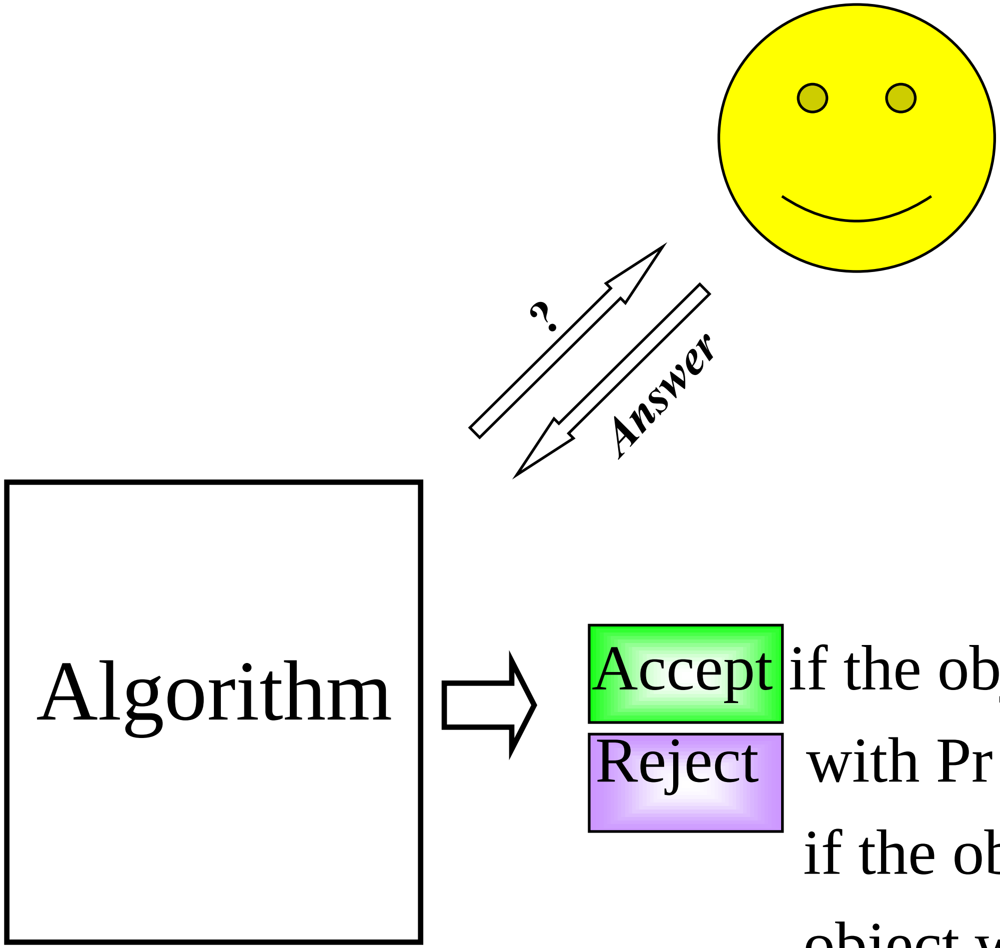
Probabilistic Property Tester



Accept if the object has the property.

Reject with Pr ≥ 1/2 if the object is far from any object with the property.

Probabilistic Property Tester



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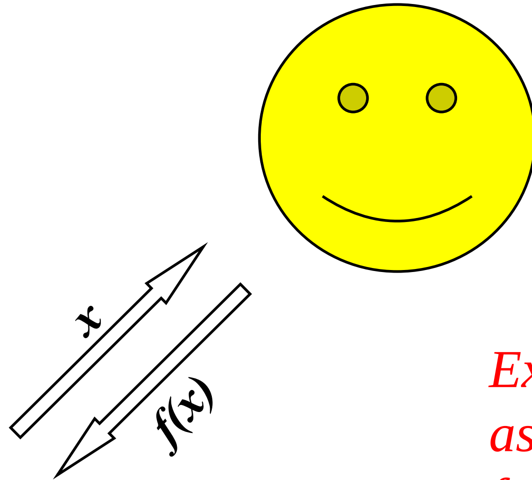


Express property testing as testing properties of functions.

Algorithm ⇒

Accept if the object has the property.

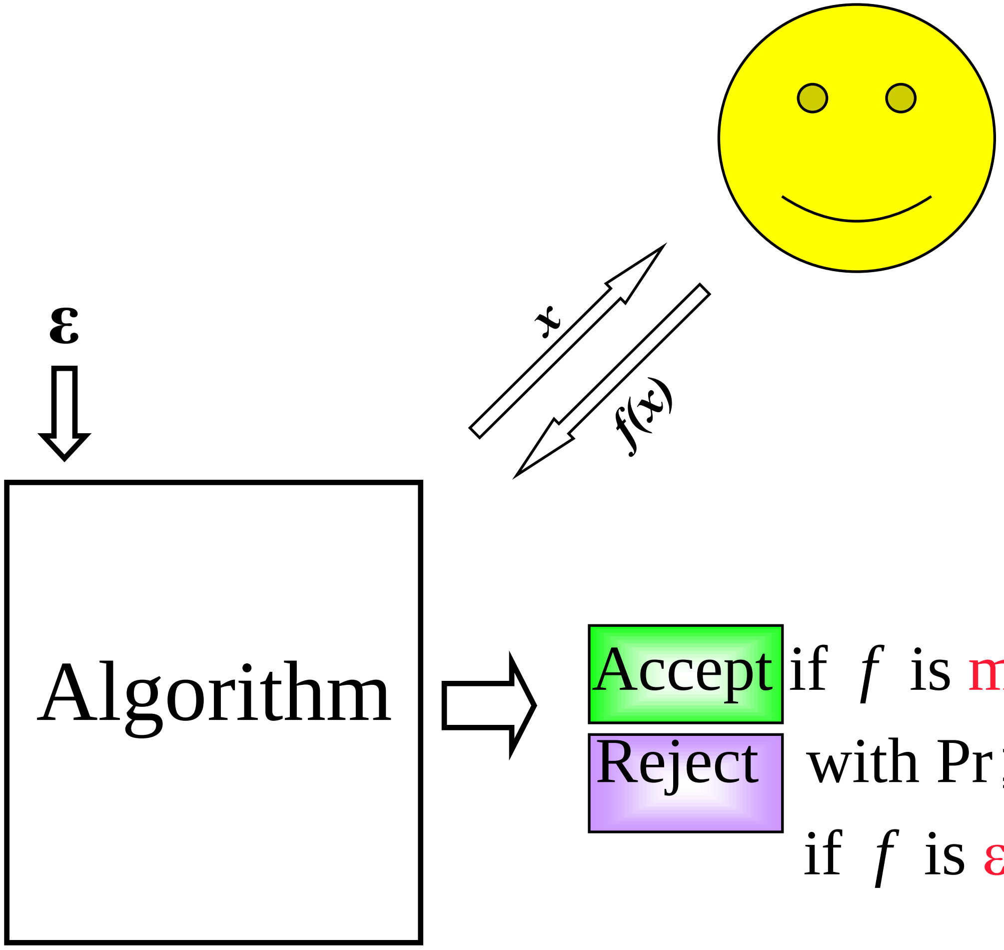
Reject with Pr $\geq$ 1/2 if the object is far from any object with the property.

Motivation

Probabilistic Property Tester **can be**

- much faster than an exact algorithm;
- the only option when the exact problem is not decidable;
- used for preprocessing;
- good enough in application where some errors are tolerable.

Problem Statement



Accept if f is monotone.

Reject with $\Pr \geq 1/2$ if f is ε-far from monotone.

Definitions for $f : \Sigma^n \mapsto R$

- For two n-symbol strings x and y we say $x \prec y$ if

  y is formed from x by increasing one or more symbols.

examples

001001
011001
011111

? 100000

Definitions for $f : \Sigma^n \mapsto R$

- For two n-symbol strings x and y we say $x \prec y$ if
  y is formed from x by increasing one or more symbols.

- f is monotone if
  $f(x) \le f(y)$ for all $x \prec y$.

examples

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x	0	1	2	3	4	5
$f(x)$	0	3	7	8	8	9

Definitions for $f : \Sigma^n \mapsto R$

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- For two n-symbol strings x and y we say $x \prec y$ if
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- f is monotone if
  $f(x) \le f(y)$ for all $x \prec y$.

x	0	1	2	3	4	5
$f(x)$	0	3	7	8	8	9

- f is ε-far from monotone if
  every monotone function disagrees with f on at least an ε-fraction of the domain.

x	0	1	2	3	4	5
$f(x)$	0	3	9	8	7	9

1/3-far from monotone

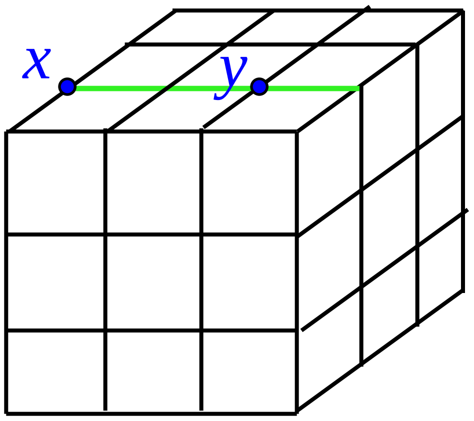
Results

Q = Query Complexity of Monotonicity Tests

[GGLR98] $$Q = O\left(\frac{n^2}{\varepsilon} \cdot |\Sigma|^2 \cdot |R|\right)$$

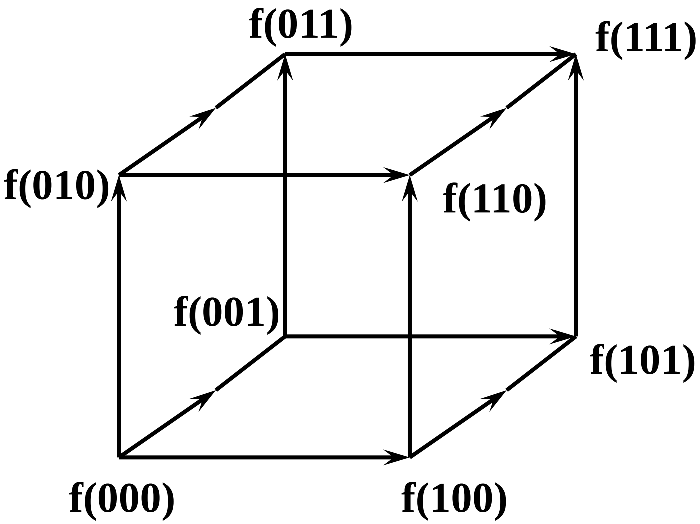
This work $$Q = O\left(\frac{n}{\varepsilon} \cdot \log|\Sigma| \cdot \log|R|\right)$$

Algorithm (Reduction to a simpler case)



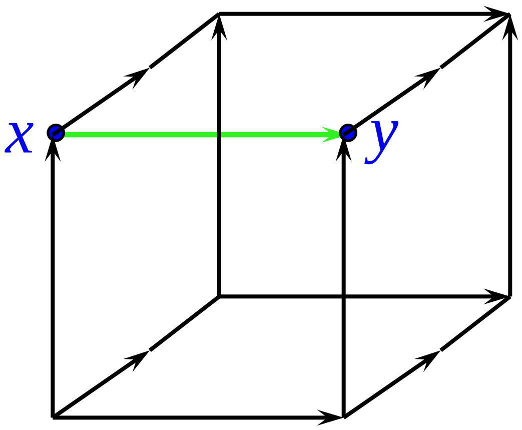
- INPUT:
  - ε and $f : \Sigma^n \mapsto R$
- Repeat several times:
  - Pick a **line** along the axes of the hyper-grid uniformly at random.
  - Use your favorite algorithm to test if the **line** is monotone [our paper, EKKRV98, Noga Alon] .
  - If a pair (x, y) of points on the **line** with $x \prec y$ and $f(x) > f(y)$ is found, then REJECT.
- Otherwise, ACCEPT.

Special case: $f : \{0,1\}^n \mapsto R$



- Edge $x \rightarrow y$ iff $x \prec y$ and
  x and y differ in one coordinate
- Edge $x \rightarrow y$ is a violated edge of f
  if $f(x) > f(y)$.

Algorithm for $f : \{0,1\}^n \mapsto R$



- INPUT:
  - ε and $f : \{0,1\}^n \mapsto R$
- Repeat ***Q/2*** times:
  - Pick an **edge** $x \rightarrow y$ uniformly at random.
  - If $x \rightarrow y$ is violated (i.e. $f(x) > f(y)$), then REJECT.
- Otherwise, ACCEPT.

Intuition for Analysis

- If f is monotone, the algorithm always accepts.

YES

- If f is not monotone:
  - If f has few violated edges, we can make f monotone by changing its value at a few points.

Don't care

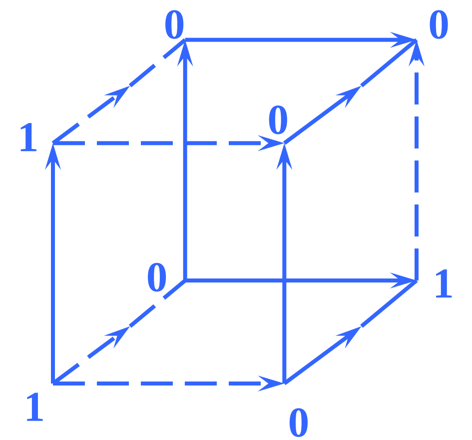
  - If f has many violated edges, the algorithm succeeds with high probability.

NO

Proof Plan

- BINARY RANGE ($f:\{0,1\}^n \mapsto \{0,1\}$)

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges}$$



THEOREM: If f is ε-far from monotone, then a random edge is violated with probability

$$\frac{\#\,\text{violated edges}}{n2^n} \ge \frac{\#\,\text{altered points}}{2 \cdot n2^n} \ge \frac{\varepsilon 2^n}{2 \cdot \text{n}2^n} \ge \frac{\varepsilon}{2\text{n}}.$$

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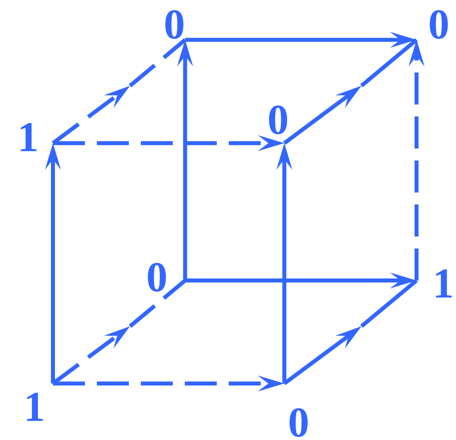

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- RANGE REDUCTION ($f:\{0,1\}^n \mapsto R$)

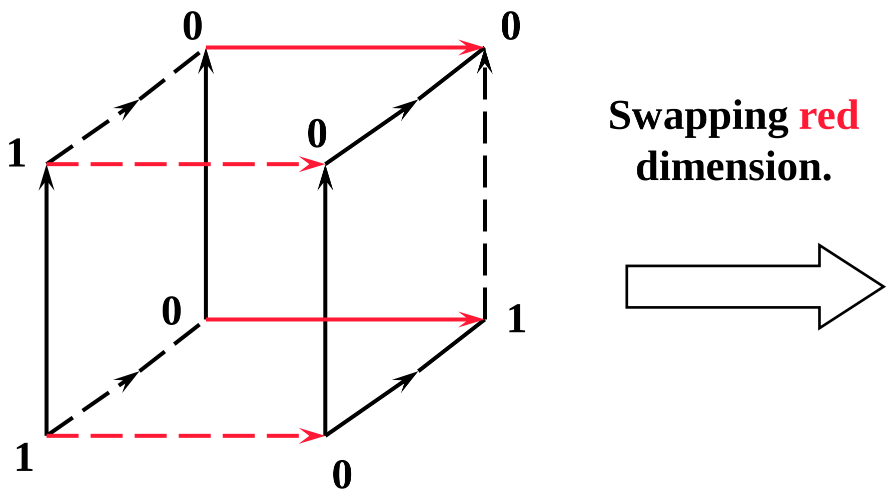
$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges} \cdot \log|R|$$

THEOREM: If f is ε-far from monotone, then a random edge is violated with probability

$$\frac{\#\,\text{violated edges}}{n2^n} \ge \frac{\#\,\text{altered points}}{n2^n \cdot 2\log|R|} \ge \frac{\varepsilon 2^n}{\mathrm{n}2^n \cdot 2\log|R|} \ge \frac{\varepsilon}{2\mathrm{n}\log|R|}.$$

Repairing Violated Edges in One Dimension

Swap violated edges $1 \rightarrow 0$ **in red dimension to** $0 \rightarrow 1$**.**



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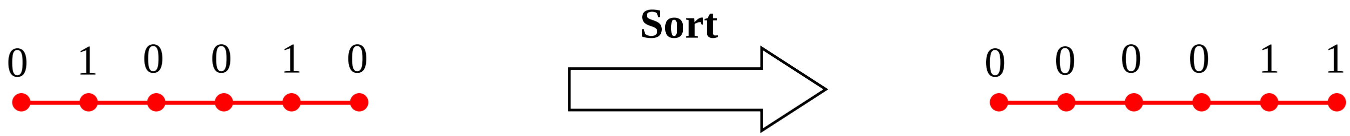
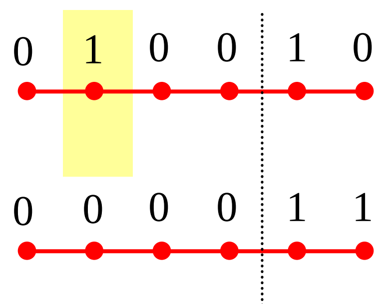
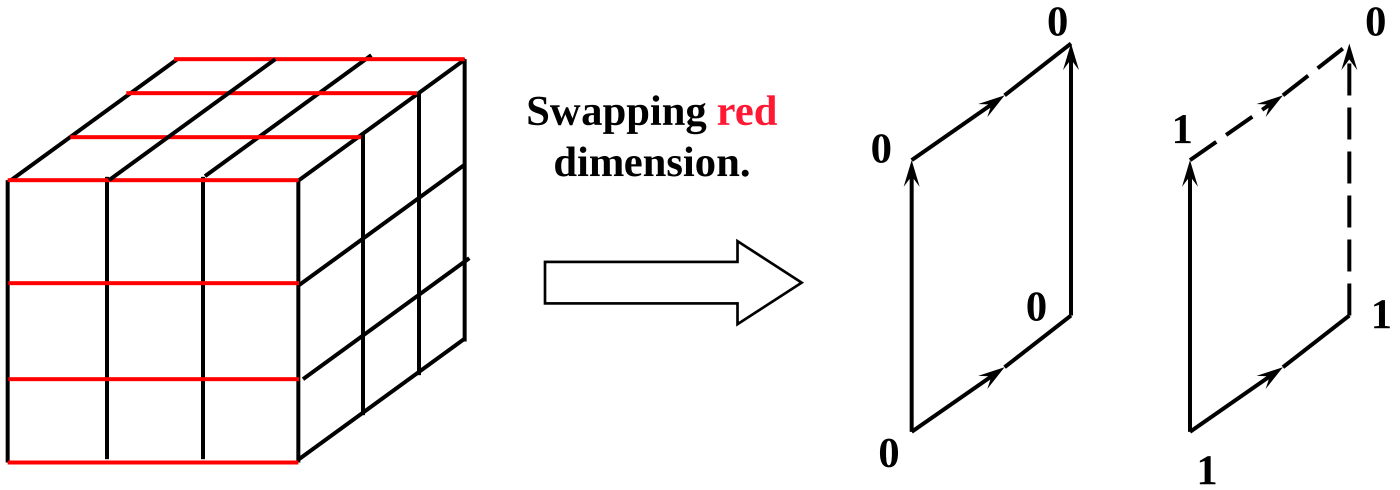




Repairing Violated Edges in One Dimension

Swap violated edges $1 \to 0$ **in red dimension to** $0 \to 1$.



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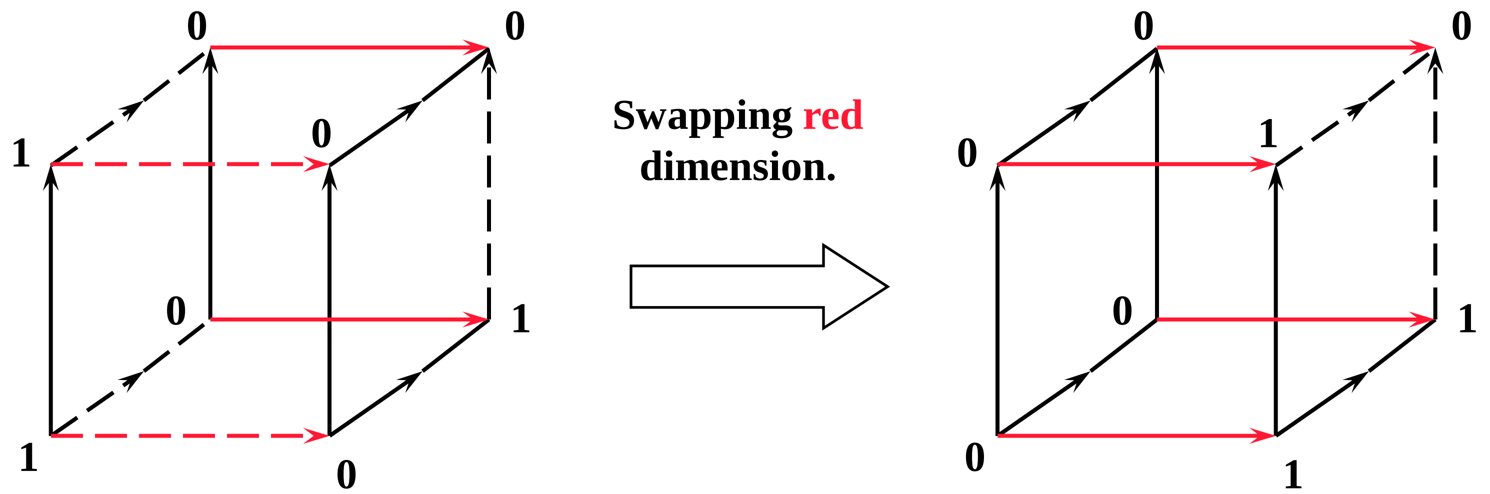
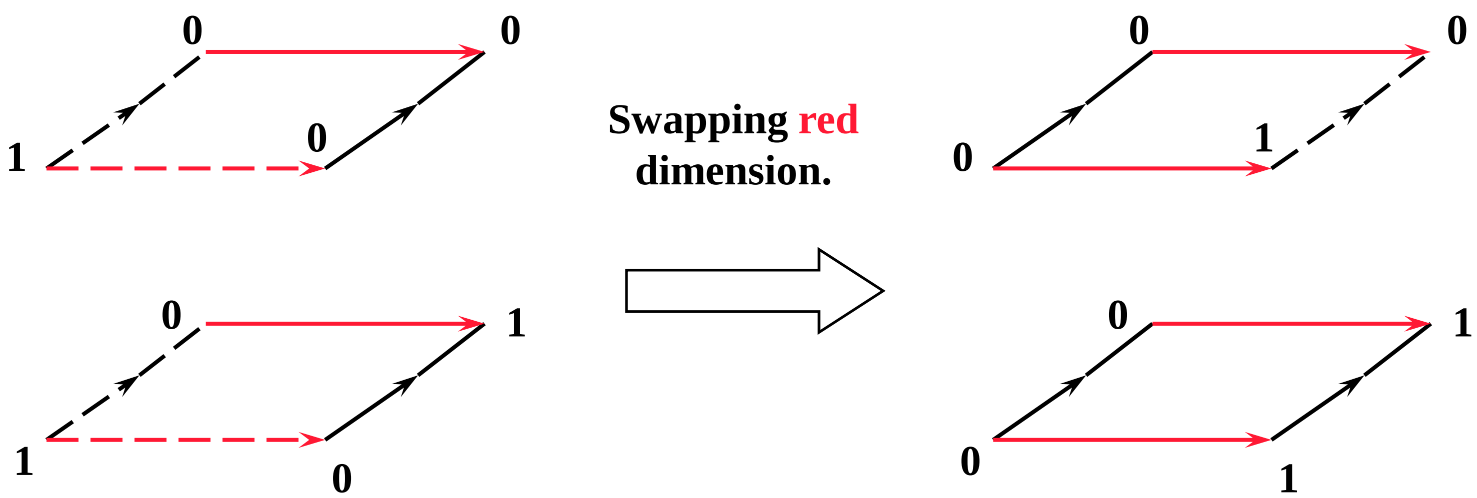


LEMMA. Swapping violated edges in dimension i

1. repairs all violated edges in dimension i;
2. does not increase the number of violated edges in dimension j, for all $j \neq i$.

Repairing Violated Edges in One Dimension

Swap violated edges *1→0* **in red dimension to** *0→1*.



LEMMA. Swapping violated edges in dimension *i*

1. repairs all violated edges in dimension *i*;
2. does not increase the number of violated edges in dimension *j*, for all $j \neq i$.

Back to the Proof Plan

- BINARY RANGE ($f : \{0,1\}^n \mapsto \{0,1\}$)

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges}$$

- RANGE REDUCTION ($f : \{0,1\}^n \mapsto R$)

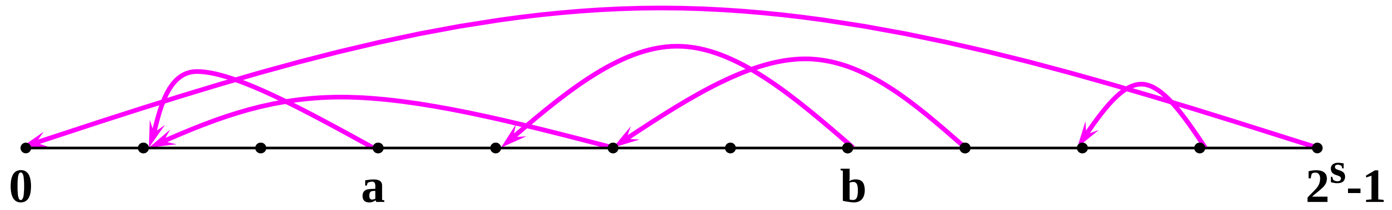
$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges} \cdot \log |R|$$

W.l.g. assume $R=\{0, 1, \ldots, 2^s-1\}$.

Prove $\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges} \cdot s$

by induction on s.

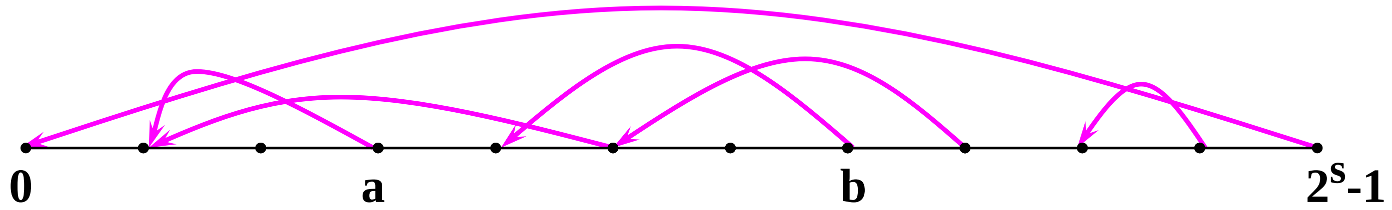
How can we make f monotone?



violated edges of *f*

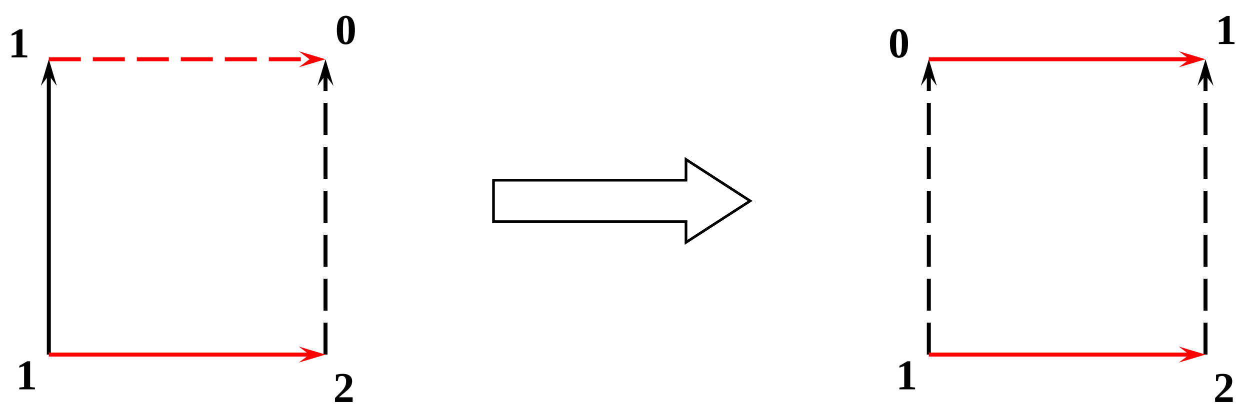
Swap violated edges in red dimension?

How can we make f monotone?

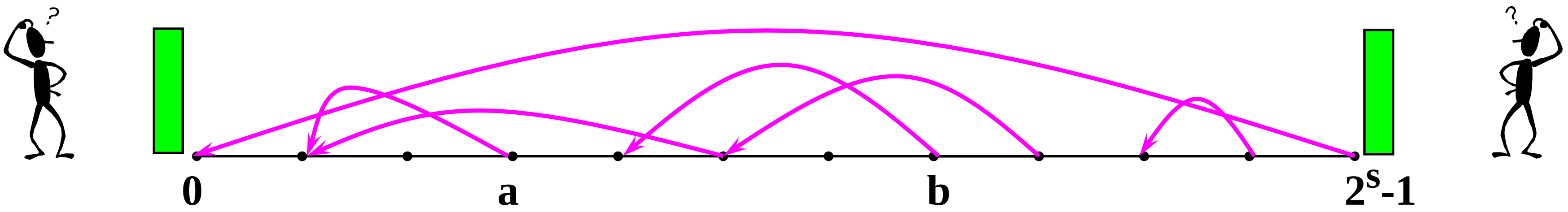


violated edges of *f*

Swap violated edges in red dimension?

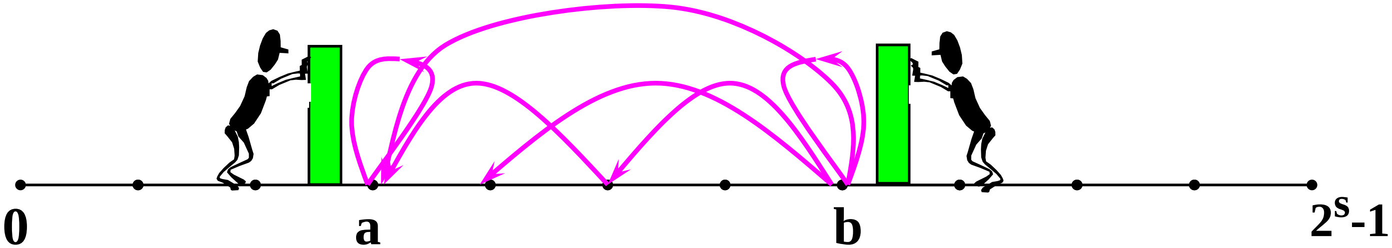


Operator SQUASH



violated edges of *f*

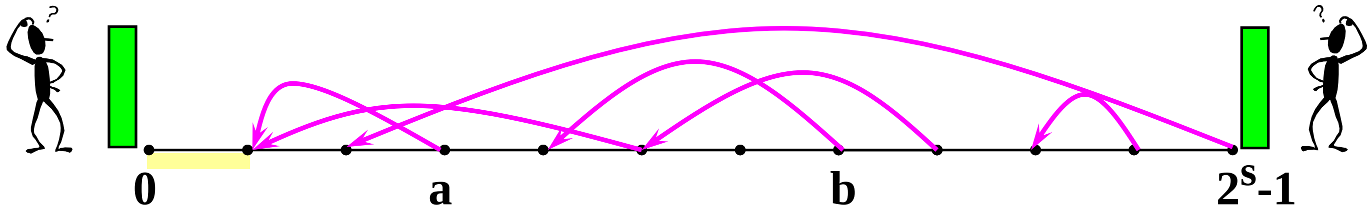
$$\text{SQUASH}[f,a,b](x)=\begin{cases} a & \text{if } f(x)\le a \\ b & \text{if } f(x)\ge b \\ f(x) & \text{otherwise} \end{cases}$$



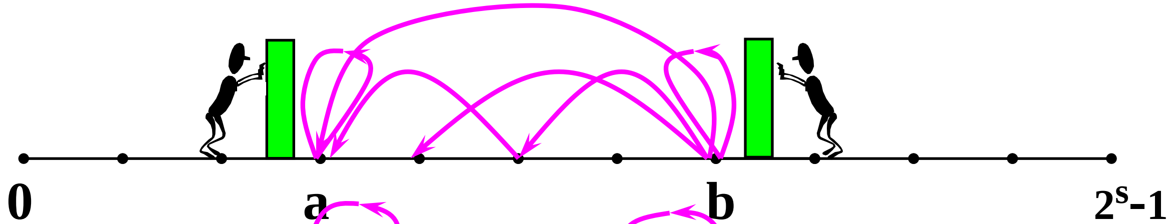
violated edges of
SQUASH[*f*,*a*,*b*]

PROPERTY: does not introduce new violated edges.

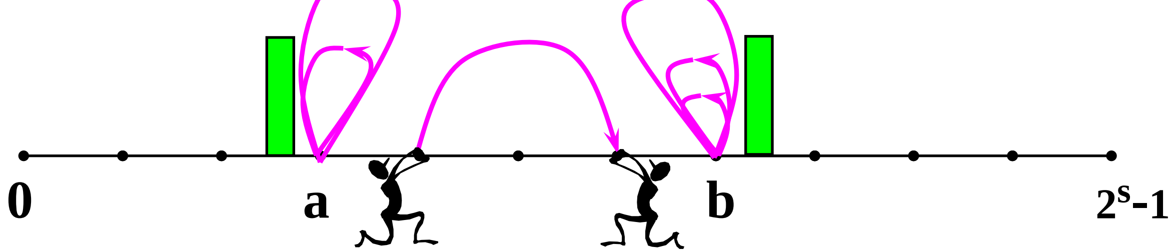
Operator CLEAR



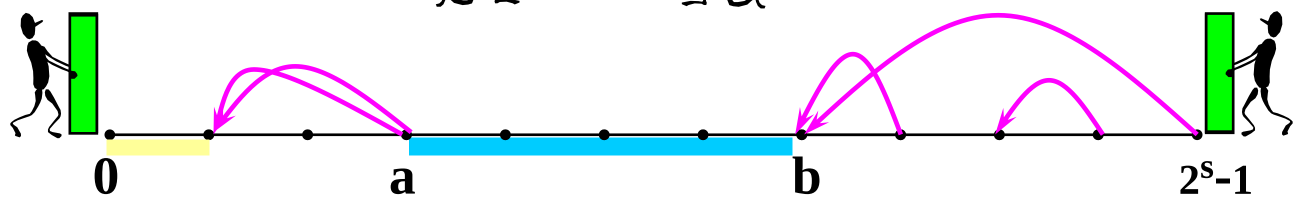
Squash:



Repair (**switch to closest monotone function with range {a,...,b}**):



Unsquash **values not altered by repair**:



PROPERTIES:

1. does not introduce new violated edges.
2. clears interval [a, b].
3. leaves clear intervals clear.

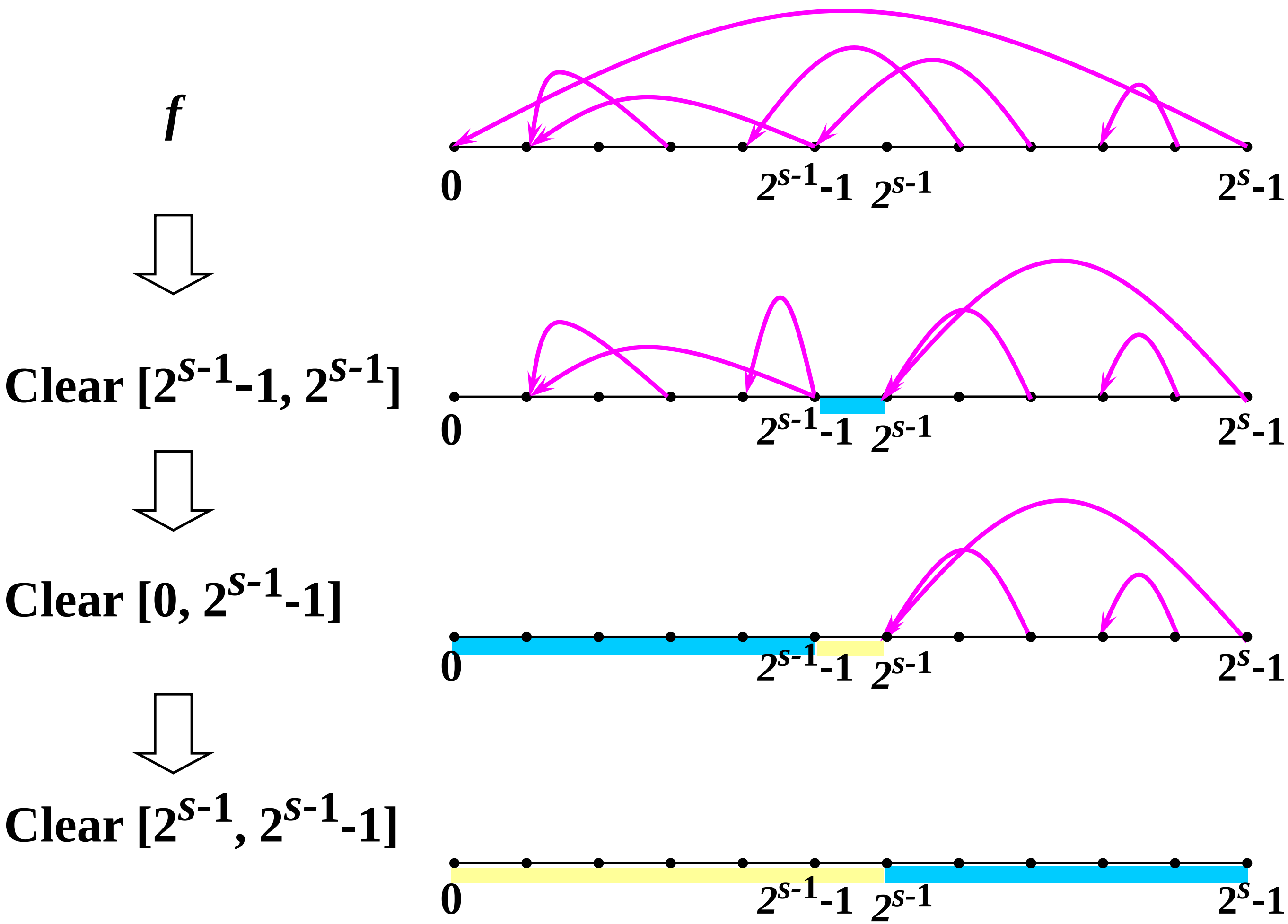
Making f monotone

f

Clear $[2^{s-1}-1, 2^{s-1}]$

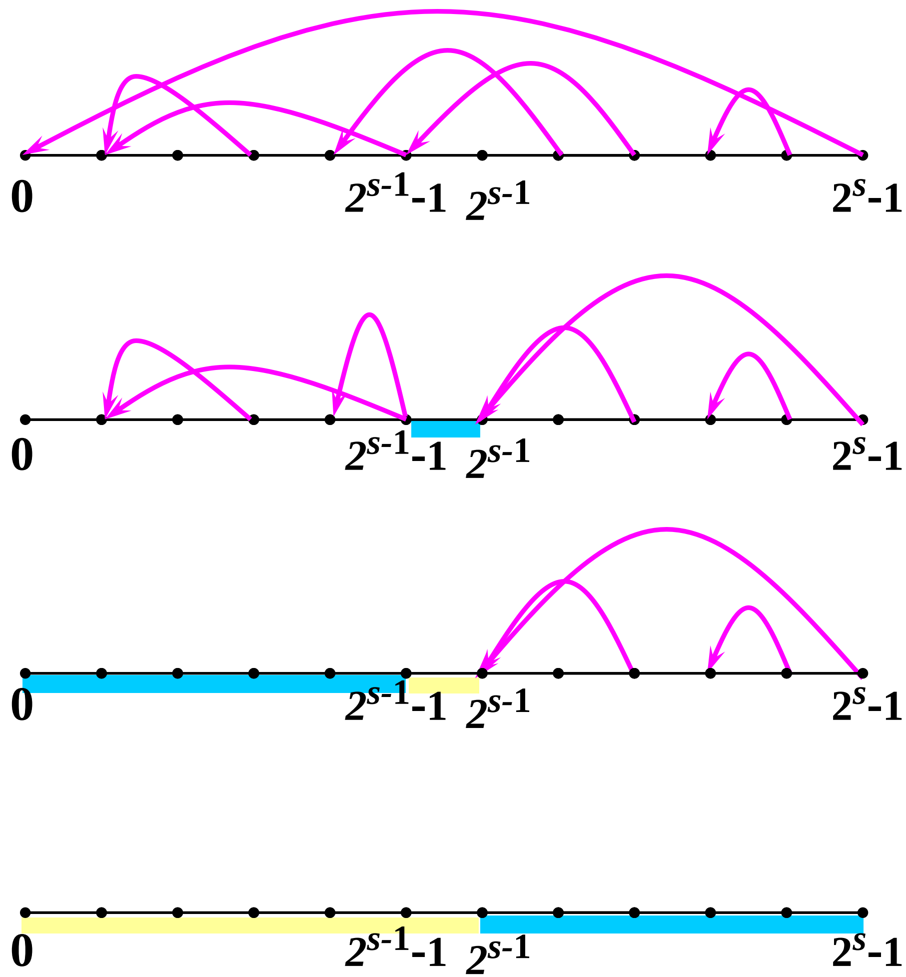
Clear $[0, 2^{s-1}-1]$

Clear $[2^{s-1}, 2^{s-1}-1]$



Making f monotone

		#violated edges
f		$2v$
Clear $[2^{s-1}-1, 2^{s-1}]$		$2(l+r) \le 2v$
Clear $[0, 2^{s-1}-1]$		$2r$
Clear $[2^{s-1}, 2^{s-1}-1]$		0



Reminder

We are proving (by induction on S)

that for functions with a range of size 2^S,

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges} \cdot S.$$

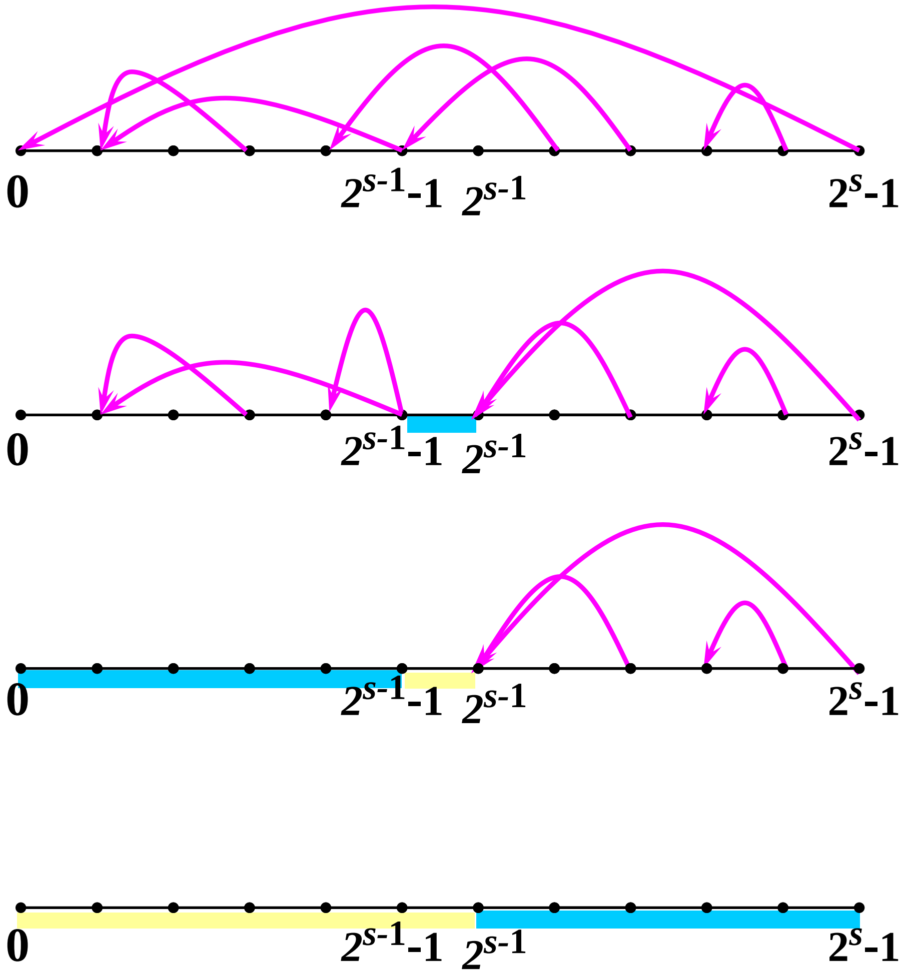
Base case [functions with a range of size 2]:

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges}.$$

Induction hypothesis [functions with a range of size 2^{S-1}]:

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges} \cdot (S-1).$$

Making f monotone



	#violated edges	#points changed
f	$2v$	**0**
Clear [2^{s-1}-1, 2^{s-1}]	$2(l+r) \le 2v$	$\le 2v$
Clear [0, 2^{s-1}-1]	$2r$	$\le 2l(s-1)$
Clear [2^{s-1}, 2^{s-1}-1]	0	$\le 2r(s-1)$
	total	$\le 2v \cdot s$

We are done

- BINARY RANGE ($f : \{0,1\}^n \mapsto \{0,1\}$)

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges}$$

- RANGE REDUCTION ($f : \{0,1\}^n \mapsto R$)

$$\#\,\text{altered points} \le 2 \cdot \#\,\text{violated edges} \cdot \log |R|$$

W.l.g. assume $R=\{0, 1,\ldots, 2^s\text{-}1\}$.

Proof by induction on s.

Conclusions

- SUMMARY OF RESULTS IN THIS TALK
  - Monotonicity test for $f:\{0,1\}^n \mapsto \{0,1\}$ [switching argument].
  - Monotonicity test for $f:\{0,1\}^n \mapsto R$ [SQUASH and CLEAR argument].
- SUMMARY OF RESULTS IN THE PAPER
  - Designed good monotonicity tests for $f:\Sigma \mapsto \{0,1\}$.
  - Reduced testing monotonicity of $f:\Sigma^n \mapsto \{0,1\}$ to the case $n=1$ [sorting argument (a generalization of the switching argument)].
  - Reduced testing monotonicity of $f:\Sigma^n \mapsto R$ to the case $f:\Sigma^n \mapsto \{0,1\}$ [SQUASH and CLEAR argument].
- OPEN PROBLEMS
  - Query complexity independent of the size of the range?
  - $f:D \mapsto R$, where D is any partially ordered set.
  - Tests for other properties.