Sublinear Algorithms
Lecture 4

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Tentative Plan

Lecture 1. Background. Testing properties of images and lists.

Lecture 2. Testing properties of lists. Sublinear-time approximation for graph problems.

Lecture 3. Testing properties of functions. Linearity testing.

Lecture 4. Techniques for proving hardness. Other models for sublinear computation.

Query Complexity

- Query complexity of an algorithm is the maximum number of queries the algorithm makes.
  - Usually expressed as a function of input length (and other parameters)
  - Example: the test for sortedness (from Lecture 2) had query complexity O(log n) for constant ε.
  - running time $\geq$ query complexity
- Query complexity of a problem P, denoted $q(P)$, is the query complexity of the best algorithm for the problem.
  - What is $q(\text{testing sortedness})$? How do we know that there is no better algorithm?

Today: Two techniques for proving lower bounds on $q(P)$.

Yao's Principle

A Method for Proving Lower Bounds

A Lower Bound Game

Players: Evil algorithms designer Al and poor lower bound prover Lola.

Game1

Move 1. Al selects a **randomized** algorithm for the problem.

Move 2. Lola selects an input on which the algorithm is as slow as possible.

Game2

Move 1. Lola selects a distribution on inputs.

Move 2. Al selects a **deterministic** algorithm which works on Lola's distribution as fast as possible.

Yao's Minimax Principle (easy direction): Lola can perform in Game1 at least as well as she can perform in Game2.

A Lower Bound for Testing Sortedness

Input: a list of n numbers $x_1, x_2, \ldots, x_n$

Question: Is the list sorted or ε-far from sorted?

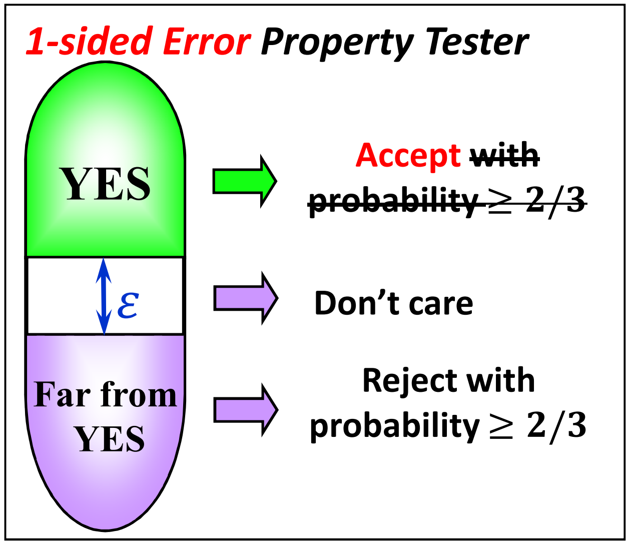
Already saw: two different $O((\log n)/\varepsilon)$ time testers.

Known [Ergün Kannan Kumar Rubinfeld Viswanathan 98, Fischer 01]:

$\Omega(\log n)$ queries are required for all constant $\varepsilon \leq 1/2$

Today: $\Omega(\log n)$ queries are required for all constant $\varepsilon \leq 1/2$
for every 1-sided error nonadaptive test.

- A test has 1-sided error if it always accepts all YES instances.
- A test is nonadaptive if its queries that do not depend on answers to previous queries.



1-Sided Error Tests Must Catch "Mistakes"

- A pair (x_i, x_j) is **violated** if $x_i < x_j$

Claim. A 1-sided error test can reject only if it finds a violated pair.

Proof: Every sorted partial list can be extended to a sorted list.

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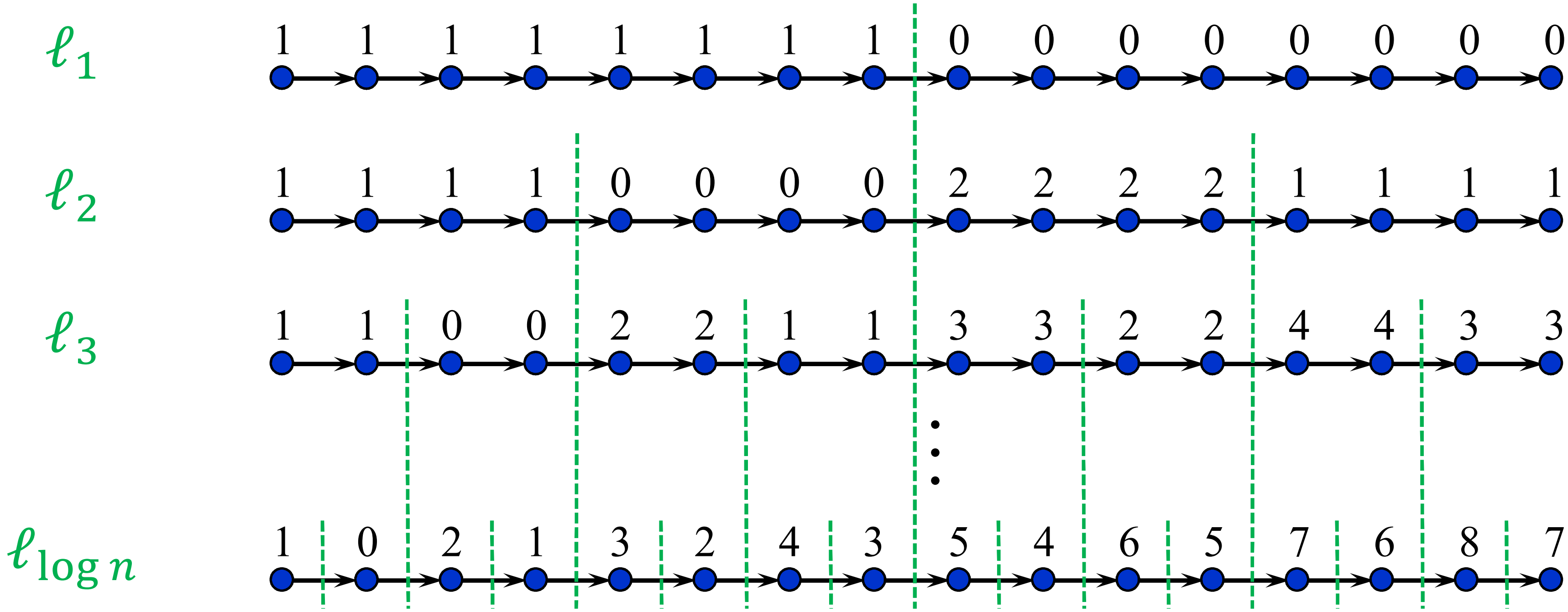
Yao's Principle Game [Jha]

Lola's distribution is uniform over the following $\log n$ lists:

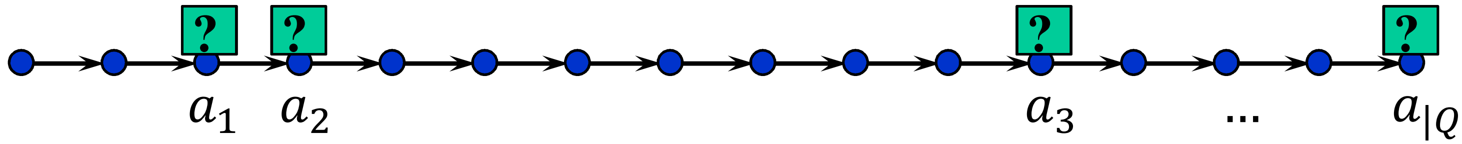


Claim 1. All lists above are 1/2-far from sorted.

Claim 2. Every pair (x_i, x_j) is violated in exactly one list above.

Yao's Principle Game: Al's Move

Al picks a set $Q = \{a_1, a_2, \ldots, a_{|Q|}\}$ of positions to query.



- His test must be correct, i.e., must find a violated pair with probability $\geq 2/3$ when input is picked according to Lola's distribution.
- Q contains a violated pair $\Leftrightarrow$ (a_i, a_{i+1}) is violated for some i

$$\Pr_{\ell \leftarrow \text{Lola's distribution}}[(a_i, a_{i+1}) \text{ for some } i \text{ is vilolated in list } \ell] \leq \frac{|Q|-1}{\log n}$$

By the Union Bound

- If $|Q| \leq \frac{2}{3}\log n$ then this probability is $< \frac{2}{3}$
- So, $|Q| = \Omega(\log n)$
- By Yao's Minimax Principle, every randomized 1-sided error nonadaptive test for sortedness must make $\Omega(\log n)$ queries.

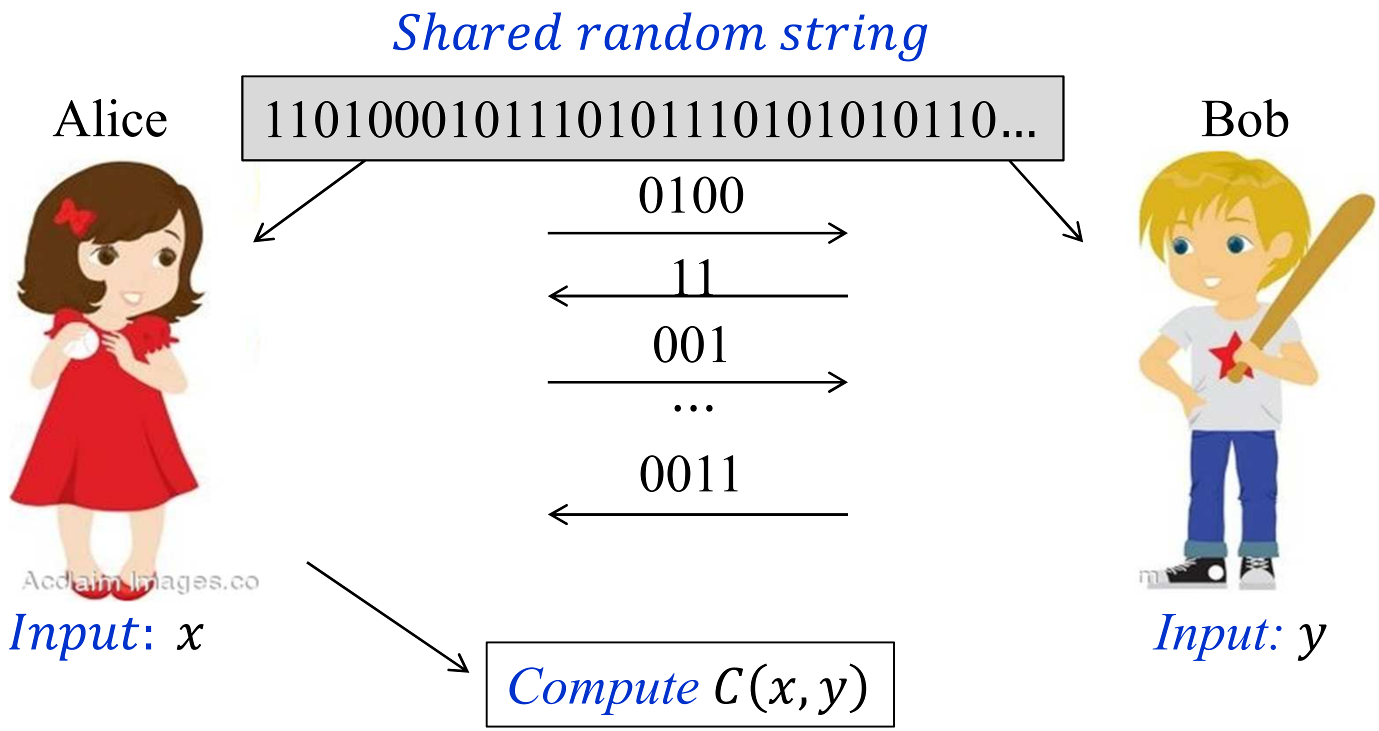
Communication Complexity

A Method for Proving Lower Bounds [Blais Brody Matulef 11]

Use known lower bounds for other models of computation

Partially based on slides by Eric Blais

(Randomized) Communication Complexity



Goal: minimize the number of bits exchanged.

- Communication complexity of a protocol is the maximum number of bits exchanged by the protocol.
- Communication complexity of a function C, denoted $R(C)$, is the communication complexity of the best protocol for computing C.

Example: Set Disjointness $DISJ_{\boldsymbol{k}}$

Alice

11010001011101011101010110...

Bob

Input: $S \subseteq [n]$, $|S| = k$.

Input: $T \subseteq [n]$, $|T| = k$

Compute $DISJ_k(S, T)$
$$= \begin{cases} \boldsymbol{accept} & \text{if } S \cap T = \emptyset \\ \boldsymbol{reject} & \text{otherwise} \end{cases}$$

Theorem [Hastad Wigderson 07]

$$R(\mathrm{DISJ}_k) \geq \Omega(k) \text{ for all } k < \frac{n}{2}.$$

k-Parity Functions

Recall: $f : \{0,1\}^n \to \{0,1\}$ is *linear* if $f(x_1, \dots, x_n) = \sum_{i \in S} x_i$ for some $S \subseteq [n]$.

Last time: linearity is testable in $O(1/\varepsilon)$ time.

k-Parity Functions

A function $f : \{0,1\}^n \to \{0,1\}$ is a k-parity if

$$f(x) = \chi_S(x) = \sum_{i \in S} x_i$$

for some set $S \subseteq [n]$ of size $|S| = k$.

Testing if a Boolean Function is a k-Parity

Input: Boolean function $f: \{0,1\}^n \to \{0,1\}$ and an integer k

Question: Is the function a k-parity or ε-far from a k-parity ($\geq \varepsilon 2^n$ values need to be changed to make it a k-parity)?

Time:

$O(\min(k \log k, (n-k)\log(n-k), n))$ [Chakraborty Garcia−Soriano Matsliah]

$\Omega(\min(k, n-k))$ [Blais Brody Matulef 11]

- Today: $\Omega(k)$ for $k < n/2$

Today's bound implies $\Omega(\min(k, n-k))$

Reduction from $DISJ_{k/2}$ to Testing k-Parity

- Let T be the best tester for the k-parity property for $\varepsilon = 1/2$
  - query complexity of T is $q(\text{testing } k\text{−parity})$.
- We will construct a communication protocol for $DISJ_{k/2}$ that runs T and has communication complexity $2 \cdot q(\text{testing } k\text{−parity})$.

holds for CC of every protocol for $DISJ_k$

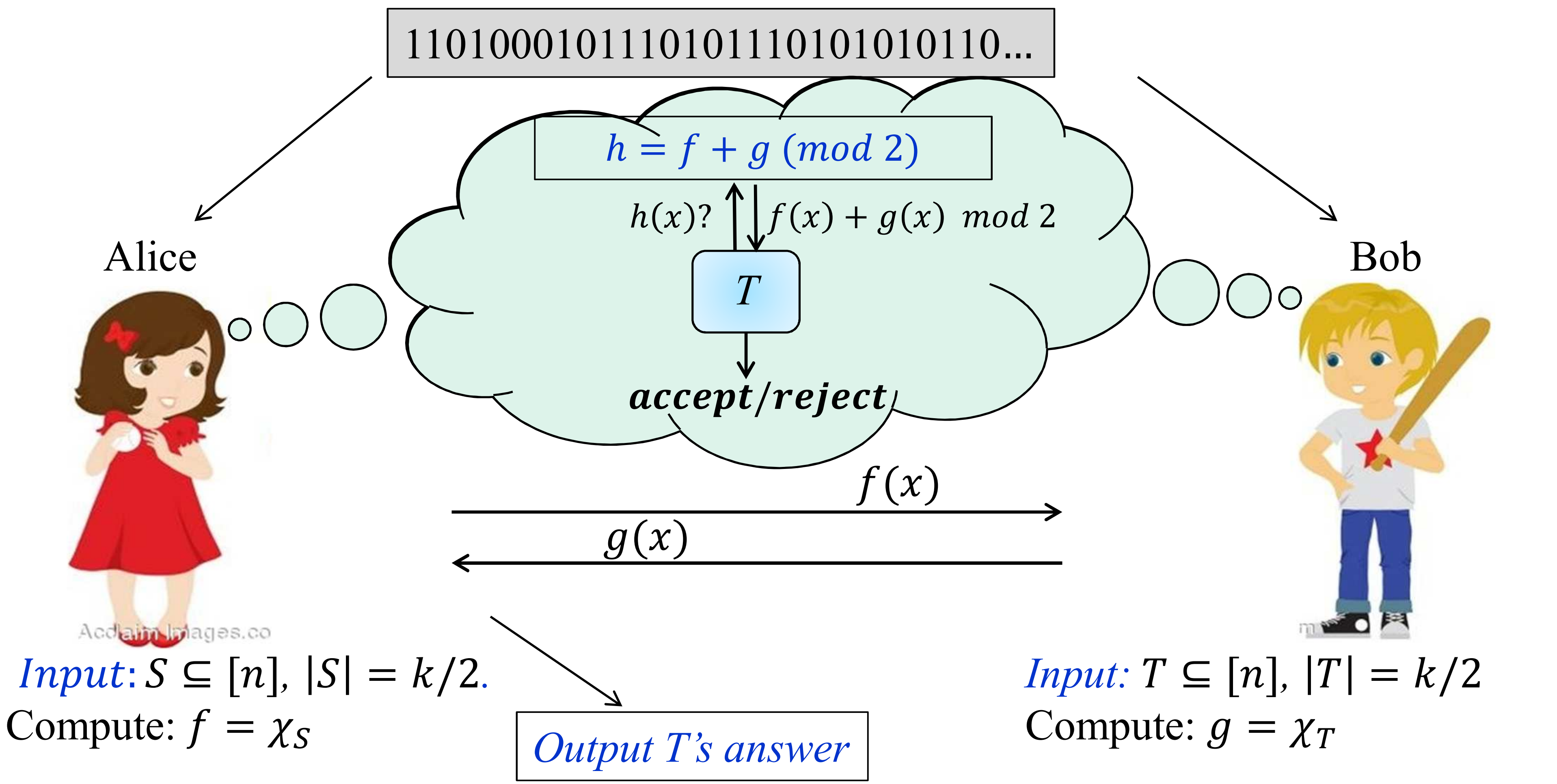
[Hastad Wigderson 07]

- Then $2 \cdot q(\text{testing } k\text{−parity}) \geq R\left(\mathrm{DISJ}_{k/2}\right) \geq \Omega(k/2)$ for $k \leq n/2$

$$\Downarrow$$

$$q(\text{testing } k\text{-parity}) \geq \Omega(k) \quad \text{for } k \leq n/2$$

Reduction from $DISJ_{k/2}$ to Testing k-Parity



- T receives its random bits from the shared random string.

Analysis of the Reduction

Queries: Alice and Bob exchange 2 bits for every bit queried by T

Correctness:

- $h = f + g \ (mod\ 2) = \chi_S + \chi_T \ (mod\ 2) = \chi_{S\Delta T}$
- $|S\Delta T| = |S| + |T| - 2|S \cap T|$
- $|S\Delta T| = \begin{cases} k & \text{if } S\cap T = \emptyset \\ \leq k-2 & \text{if } S\cap T \neq \emptyset \end{cases}$

$$h \text{ is} \begin{cases} k\text{−parity} & \text{if } S\cap T = \emptyset \\ k'\text{−parity where } k' \neq k & \text{if } S\cap T \neq \emptyset \end{cases}$$

1/2-far from every k-parity

- Recall that two different linear functions disagree on half of the values:

$\langle \chi_S, \chi_T \rangle = 1 - 2 \cdot$ (fraction of *disagreements* between χ_S and χ_T) $= 0$ for $S \neq T$

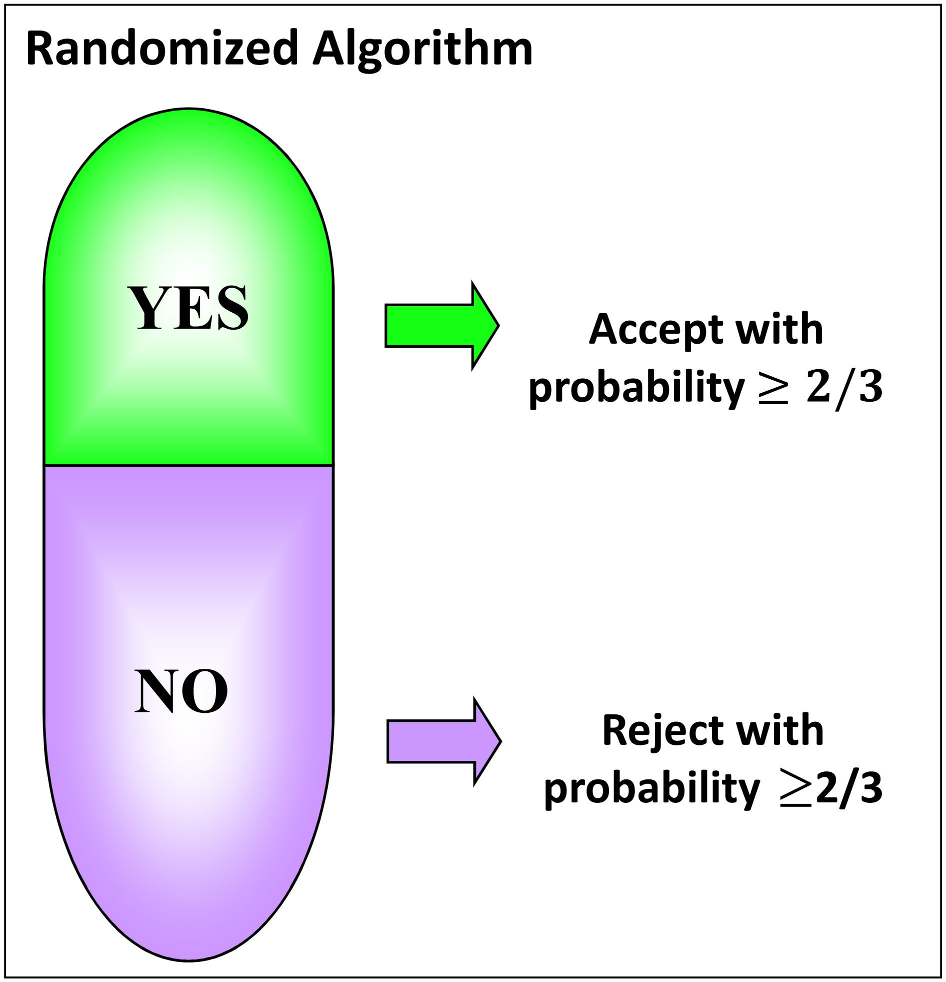
Summary: q(testing k-parity) $\geq \Omega(k)$ for $k \leq n/2$

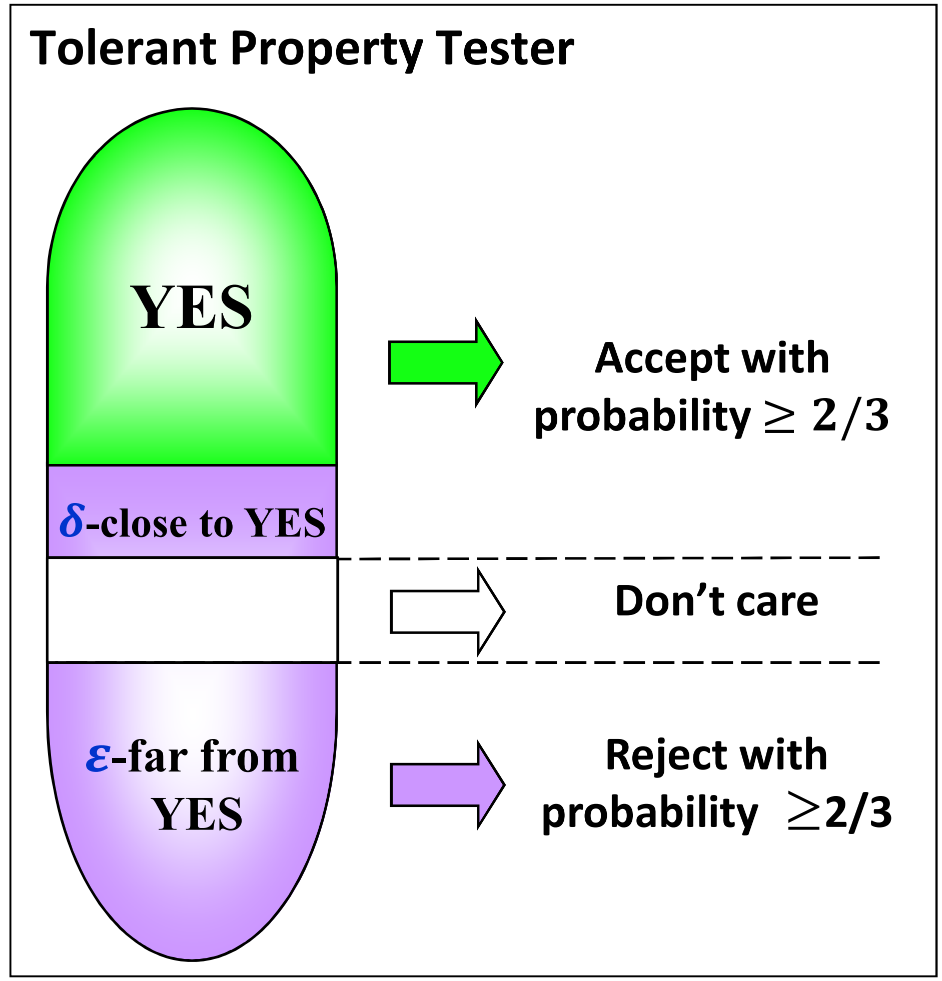
Summary of Lower Bounds

- Yao's Principle
  - testing sortedness
- Reductions from communication complexity problems
  - testing if a function is a k-parity

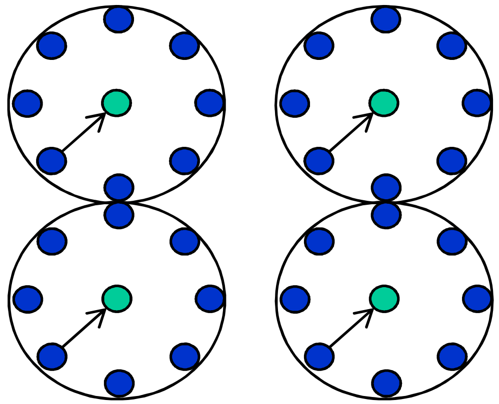
Other Models of Sublinear Computation

Tolerant Property Tester [Rubinfeld Parnas Ron]





Sublinear-Time "Restoration" Models

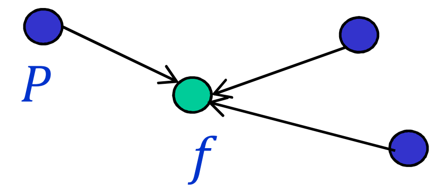


Local Decoding

Input: a slightly corrupted codeword

Requirement: recover a given bit of the closest codeword with a constant number of queries.

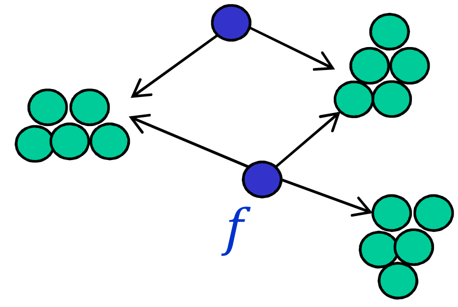
Program Checking



Input: a program P computing f with a small error probability.

Requirement: self-correct program P – for a given argument x, compute $f(x)$ by making a few calls to *P*.

Local Reconstruction



Input: Function f nearly satisfying some property P

Requirement: Reconstruct function f to ensure that the reconstructed function g satisfies P, changing f only when necessary. For a given argument x, compute $g(x)$ with a few queries to f.

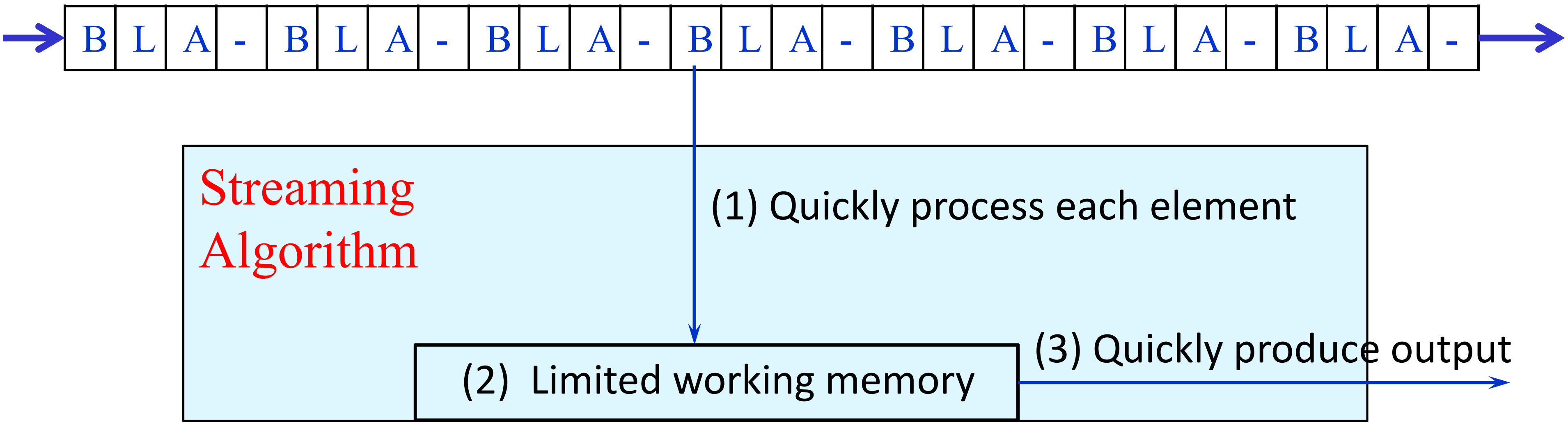
Sublinear-Space Algorithms

What if we cannot get a sublinear-time algorithm?

Can we at least get sublinear space?

Note: sublinear space is broader (for any algorithm, space complexity $\leq$ time complexity)

Data Stream Model



- Motivation: network traffic, database transactions, sensor networks, satellite data feed

Model the stream as m elements from $[n]$, e.g.,

$$\langle x_1, x_2, \dots, x_m \rangle = 3, 5, 3, 7, 5, 4, \dots$$

Goal: Compute a function of the stream, e.g., median, number of distinct elements, longest increasing sequence.

Based on Andrew McGregor's slides: http://www.cs.umass.edu/~mcgregor/slides/10-jhu1.pdf

Streaming Puzzle

A stream contains $n - 1$ **distinct** elements from $[n]$ in arbitrary order.

Problem: Find the missing element, using $O(\log n)$ space.

Sampling from a Stream of Unknown Length

Problem: Find a uniform sample s from a stream $\langle x_1, x_2, \dots, x_m \rangle$ of unknown length m

Algorithm

1. Initially, $s \leftarrow x_1$
2. On seeing the t^{th} element, $s \leftarrow x_t$ with probability $1/t$

Analysis:

What is the probability that $s = x_i$ at some time $t \geq i$?

$$\Pr[s = x_i] = \frac{1}{i} \cdot \left(1 - \frac{1}{i+1}\right) \cdot \dots \cdot \left(1 - \frac{1}{t}\right)$$

$$= \frac{1}{i} \cdot \frac{i}{i+1} \cdot \dots \cdot \frac{t-1}{t} = \frac{1}{t}$$

Space: $O(k \log n)$ bits to get k samples.

Conclusion

Sublinear algorithms are possible in many settings

- simple algorithms, more involved analysis
- nice combinatorial problems
- unexpected connections to other areas
- many open questions