Sublinear Algorithms
Lecture 4

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Tentative Plan


Lecture 3. Testing properties of functions. Linearity testing.

Lecture 4. Techniques for proving hardness. Other models for sublinear computation.
Query Complexity

- Query complexity of an algorithm is the maximum number of queries the algorithm makes.
  - Usually expressed as a function of input length (and other parameters)
  - Example: the test for sortedness (from Lecture 2) had query complexity $O(\log n)$ for constant $\varepsilon$.
  - running time $\geq$ query complexity

- Query complexity of a problem $P$, denoted $q(P)$, is the query complexity of the best algorithm for the problem.
  - What is $q(\text{testing sortedness})$? How do we know that there is no better algorithm?

Today: Two techniques for proving lower bounds on $q(P)$. 
Yao’s Principle

A Method for Proving Lower Bounds
A Lower Bound Game

Players: Evil algorithms designer Al and poor lower bound prover Lola.

<table>
<thead>
<tr>
<th>Game1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Move 1. Al selects a <strong>randomized</strong> algorithm for the problem.</td>
</tr>
<tr>
<td>Move 2. Lola selects an input on which the algorithm is as slow as possible.</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Game2</th>
</tr>
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<tbody>
<tr>
<td>Move 1. Lola selects a distribution on inputs.</td>
</tr>
<tr>
<td>Move 2. Al selects a <strong>deterministic</strong> algorithm which works on Lola’s distribution as fast as possible.</td>
</tr>
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**Yao’s Minimax Principle (easy direction):** Lola can perform in Game1 at least as well as she can perform in Game2.
A Lower Bound for Testing Sortedness

Input: a list of $n$ numbers $x_1, x_2, \ldots, x_n$

Question: Is the list sorted or $\varepsilon$-far from sorted?

Already saw: two different $O((\log n)/\varepsilon)$ time testers.

Known [Ergün Kannan Kumar Rubinfeld Viswanathan 98, Fischer 01]:

$\Omega(\log n)$ queries are required for all constant $\varepsilon \leq 1/2$

Today: $\Omega(\log n)$ queries are required for all constant $\varepsilon \leq 1/2$

for every 1-sided error nonadaptive test.

- A test has 1-sided error if it always accepts all YES instances.
- A test is nonadaptive if its queries that do not depend on answers to previous queries.
**1-Sided Error Tests Must Catch “Mistakes”**

- A pair \((x_i, x_j)\) is **violated** if \(x_i < x_j\)

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**Claim.** A 1-sided error test can reject only if it finds a violated pair.

**Proof:** Every sorted partial list can be extended to a sorted list.

| 1 | ? | ? | 4 | ... | 7 | ? | ? | 9 |
Yao’s Principle Game [Jha]

Lola’s distribution is uniform over the following Θ log n lists:

\[ \ell_1, \ell_2, \ell_3, \ldots, \ell_{\log n} \]

Claim 1. All lists above are 1/2-far from sorted.

Claim 2. Every pair \((x_i, x_j)\) is violated in exactly one list above.
Yao’s Principle Game: Al’s Move

Al picks a set $Q = \{a_1, a_2, \ldots, a_{|Q|}\}$ of positions to query.

- His test must be correct, i.e., must find a violated pair with probability $\geq 2/3$ when input is picked according to Lola’s distribution.
- $Q$ contains a violated pair $\iff (a_i, a_{i+1})$ is violated for some $i$

$$\Pr_{\ell \leftarrow \text{Lola’s distribution}}[(a_i, a_{i+1}) \text{ for some } i \text{ is violated in list } \ell] \leq \frac{|Q| - 1}{\log n}$$

- If $|Q| \leq \frac{2}{3} \log n$ then this probability is $< \frac{2}{3}$
- So, $|Q| = \Omega(\log n)$
- By Yao’s Minimax Principle, every randomized 1-sided error nonadaptive test for sortedness must make $\Omega(\log n)$ queries.
Communication Complexity

A Method for Proving Lower Bounds [Blais
Brody Matulef 11]

Use known lower bounds for other models of computation

Partially based on slides by Eric Blais
(Randomized) Communication Complexity

**Input:** x

**Input:** y

**Goal:** minimize the number of bits exchanged.

- **Communication complexity of a protocol** is the maximum number of bits exchanged by the protocol.
- **Communication complexity of a function** $C$, denoted $R(C)$, is the communication complexity of the best protocol for computing $C$. 

**Shared random string**

Alice

Bob

1101000101110101110101010110...

0100

11

001

...

0011

Compute $C(x, y)$
Example: Set Disjointness $\text{DISJ}_k$

**Input:** $S \subseteq [n], |S| = k$.

**Input:** $T \subseteq [n], |T| = k$

Compute $\text{DISJ}_k(S, T)$

$$= \begin{cases} \text{accept} & \text{if } S \cap T = \emptyset \\ \text{reject} & \text{otherwise} \end{cases}$$

**Theorem** [Hastad Wigderson 07]

$$R(\text{DISJ}_k) \geq \Omega(k) \text{ for all } k < \frac{n}{2}.$$
Recall: $f : \{0,1\}^n \to \{0,1\}$ is linear if $f(x_1, \ldots, x_n) = \sum_{i \in S} x_i$ for some $S \subseteq [n]$.

Last time: linearity is testable in $O(1/\varepsilon)$ time.

A function $f : \{0,1\}^n \to \{0,1\}$ is a $k$-parity if

$$f(x) = \chi_S(x) = \sum_{i \in S} x_i$$

for some set $S \subseteq [n]$ of size $|S| = k$. 

$k$-Parity Functions
Testing if a Boolean Function is a $k$-Parity

**Input:** Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ and an integer $k$

**Question:** Is the function a $k$-parity or $\epsilon$-far from a $k$-parity

($\geq \epsilon 2^n$ values need to be changed to make it a $k$-parity)?

**Time:**

$O(\min(k \log k, (n - k) \log(n - k), n))$ [Chakraborty Garcia–Soriano Matsliah]

$\Omega(\min(k, n - k))$ [Blais Brody Matulef 11]

- Today: $\Omega(k)$ for $k < n/2$

Today’s bound implies $\Omega(\min(k, n - k))$
Reduction from \(\text{DISJ}_{k/2}\) to Testing \(k\)-Parity

- Let \(T\) be the best tester for the \(k\)-parity property for \(\epsilon = 1/2\)
  - query complexity of \(T\) is \(q(\text{testing } k-\text{parity})\).
- We will construct a communication protocol for \(\text{DISJ}_{k/2}\) that runs \(T\) and has communication complexity \(2 \cdot q(\text{testing } k-\text{parity})\).

\[
2 \cdot q(\text{testing } k-\text{parity}) \geq R\left(\text{DISJ}_{k/2}\right) \geq \Omega\left(k/2\right) \text{ for } k \leq n/2
\]

\[
\Downarrow
\]

\[
q(\text{testing } k-\text{parity}) \geq \Omega(k) \text{ for } k \leq n/2
\]

[Hastad Wigderson 07]
Reduction from \( \text{DISJ}_{k/2} \) to Testing \( k \)-Parity

\[ h = f + g \pmod{2} \]

\( h(x) \)?

\( f(x) + g(x) \pmod{2} \)

Input: \( S \subseteq [n] \), \( |S| = k/2 \).
Compute: \( f = \chi_S \)

Output \( T \)'s answer

- \( T \) receives its random bits from the shared random string.

Input: \( T \subseteq [n] \), \( |T| = k/2 \).
Compute: \( g = \chi_T \)
Analysis of the Reduction

Queries: Alice and Bob exchange 2 bits for every bit queried by $T$

Correctness:

- $h = f + g \pmod{2} = \chi_S + \chi_T \pmod{2} = \chi_{S \Delta T}$
- $|S \Delta T| = |S| + |T| - 2|S \cap T|$

- $|S \Delta T| = \begin{cases} k & \text{if } S \cap T = \emptyset \\ \leq k - 2 & \text{if } S \cap T \neq \emptyset \end{cases}$

- $h$ is\begin{cases} k\text{-parity} & \text{if } S \cap T = \emptyset \\ k'\text{-parity where } k' \neq k & \text{if } S \cap T \neq \emptyset \end{cases}$

- Recall that two different linear functions disagree on half of the values: $\langle \chi_S, \chi_T \rangle = 1 - 2 \cdot \text{(fraction of disagreements between } \chi_S \text{ and } \chi_T ) = 0 \text{ for } S \neq T$

Summary: $q(\text{testing } k\text{-parity}) \geq \Omega(k)$ for $k \leq n/2$
Summary of Lower Bounds

- **Yao’s Principle**
  - testing sortedness

- **Reductions from communication complexity problems**
  - testing if a function is a $k$-parity
Other Models of Sublinear Computation
Tolerant Property Tester [Rubinfeld Parnas Ron]

Randomized Algorithm

YES

Accept with probability $\geq \frac{2}{3}$

NO

Reject with probability $\geq \frac{2}{3}$

Tolerant Property Tester

YES

Accept with probability $\geq \frac{2}{3}$

$\delta$-close to YES

Don’t care

$\varepsilon$-far from YES

Reject with probability $\geq \frac{2}{3}$
**Sublinear-Time “Restoration” Models**

**Local Decoding**
*Input:* a slightly corrupted codeword  
*Requirement:* recover a given bit of the closest codeword with a constant number of queries.

**Program Checking**
*Input:* a program $P$ computing $f$ with a small error probability.  
*Requirement:* self-correct program $P$ – for a given argument $x$, compute $f(x)$ by making a few calls to $P$.

**Local Reconstruction**
*Input:* Function $f$ nearly satisfying some property $P$  
*Requirement:* Reconstruct function $f$ to ensure that the reconstructed function $g$ satisfies $P$, changing $f$ only when necessary. For a given argument $x$, compute $g(x)$ with a few queries to $f$. 
Sublinear-Space Algorithms

What if we cannot get a sublinear-time algorithm?
Can we at least get sublinear space?

Note: sublinear space is broader (for any algorithm, space complexity \( \leq \) time complexity)
**Data Stream Model**

- **Motivation:** network traffic, database transactions, sensor networks, satellite data feed

Model the stream as $m$ elements from $[n]$, e.g.,

$$\langle x_1, x_2, \ldots, x_m \rangle = 3, 5, 3, 7, 5, 4, \ldots$$

**Goal:** Compute a function of the stream, e.g., median, number of distinct elements, longest increasing sequence.

Streaming Puzzle

A stream contains $n - 1$ distinct elements from $[n]$ in arbitrary order.

Problem: Find the missing element, using $O(\log n)$ space.
Sampling from a Stream of Unknown Length

**Problem:** Find a uniform sample \( s \) from a stream \( \langle x_1, x_2, \ldots, x_m \rangle \) of unknown length \( m \)

**Algorithm**

1. Initially, \( s \leftarrow x_1 \)
2. On seeing the \( t^{\text{th}} \) element, \( s \leftarrow x_t \) with probability \( 1/t \)

**Analysis:**

What is the probability that \( s = x_i \) at some time \( t \geq i \)?

\[
\Pr[s = x_i] = \frac{1}{i} \cdot \left(1 - \frac{1}{i+1}\right) \cdot \ldots \cdot \left(1 - \frac{1}{t}\right)
\]

\[
= \frac{1}{i} \cdot \frac{i}{i+1} \cdot \ldots \cdot \frac{t-1}{t} = \frac{1}{t}
\]

**Space:** \( O(k \log n) \) bits to get \( k \) samples.
Conclusion

Sublinear algorithms are possible in many settings

- simple algorithms, more involved analysis
- nice combinatorial problems
- unexpected connections to other areas
- many open questions