Algorithm Design and Analysis

CSE 565

LECTURE 25

Linear programming

- Definitions
- Duality

(Based on Erickson, Chapter 26)

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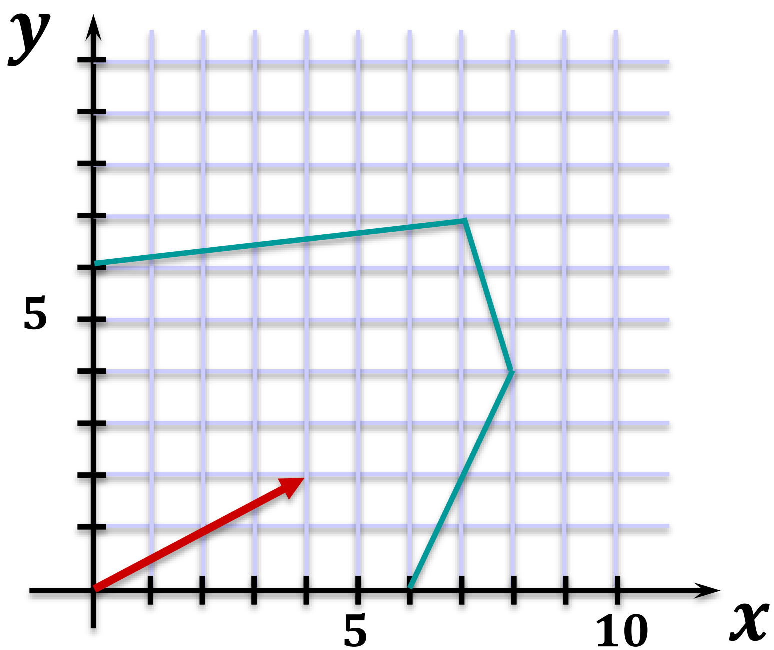
Review questions

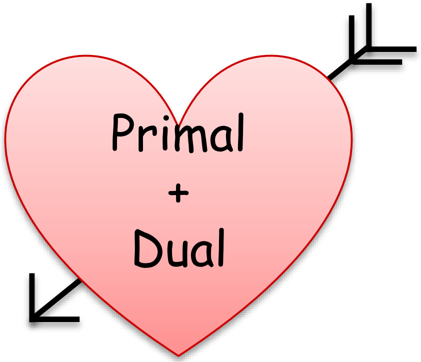
1. Maximaze $2x + y$ in the feasible region on the figure.
2. Is every optimal solution to a LP also a vertex of the feasible region?



3. What is the OPT of the following LP?
   Given a graph $G = (V, E)$ with edge lengths $\ell_{u,v}$
   LP: maximize $\sum_v d_v$
   s.t. $d_v - d_u \leq \ell_{u,v} \; \forall \, (u, v) \in E$
   $d_s = 0$

Duality of linear programs

- For every linear program Π, there is another linear program Π^* that is perfect for it
  - ➢ Π^* is "the" dual program of Π
    - The dual is not unique
  - ➢ The dual of Π^* is Π



- If we put Π in canonical form as a max. problem

$$\max \vec{c} \cdot \vec{x}$$
$$\text{s.t.} \quad Ax \leq \vec{b} \text{ and } x \geq 0$$

n variables
m constraints

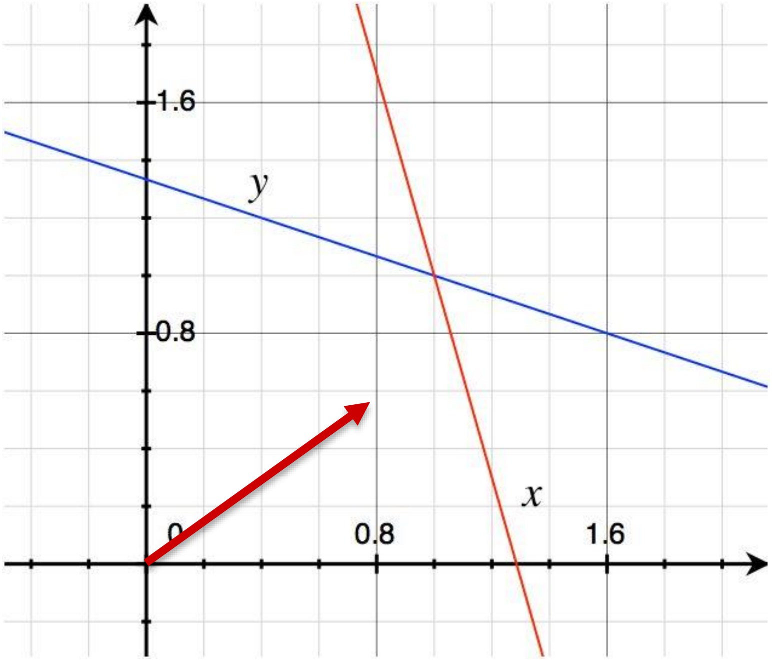
then we can write Π^* as a minimization problem

$$\min \vec{b} \cdot \vec{y}$$
$$\text{s.t.} \quad A^\top y \geq \vec{c} \text{ and } y \geq 0$$

m variables
n constraints

Examples: compute duals

1. $\max 4x + 3y$
   s.t. $\begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix}$
   and $x, y \geq 0$



2. Given $G = (V, E)$ with
   edge lengths $\ell_{u,v}$
   LP: maximize d_t
   s.t. $d_v - d_u \leq \ell_{u,v} \;\forall\, (u, v) \in E$
   $d_s = 0$

Examples: compute duals

1. $\max 4x + 3y$

   s.t. $\begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

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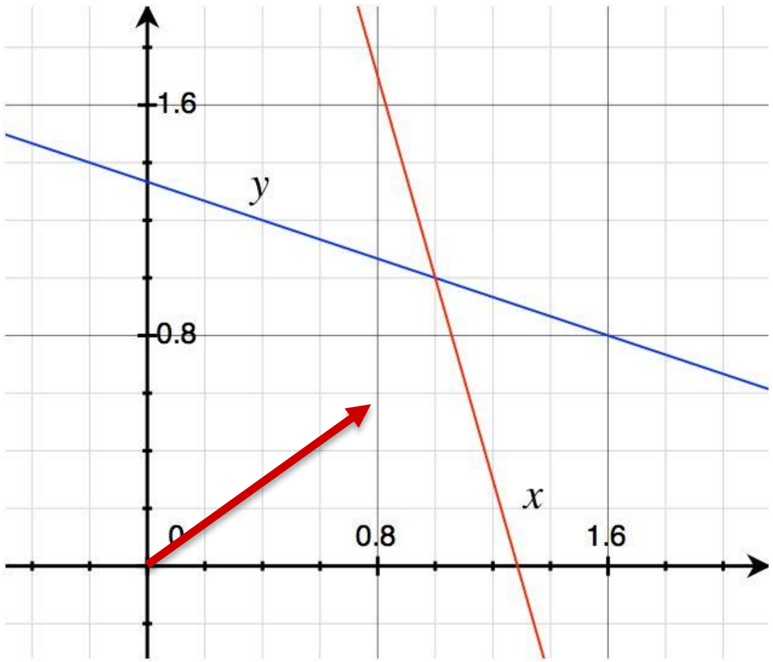


2. What is the OPT of the following LP?

   Given a graph $G = (V, E)$ with edge lengths $\ell_{u,v}$

   LP: maximize $\sum_v d_v$

   s.t. $d_v - d_u \leq \ell_{u,v} \; \forall \, (u, v) \in E$

   $d_s = 0$

Shortest path LP

There is another formulation of shortest paths as an LP minimization problem using an indicator variable $x_{u\to v}$ for each edge $u\to v$.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{u\to v} \ell_{u\to v} \cdot x_{u\to v} \\
\text{subject to} \quad & \sum_u x_{u\to s} - \sum_w x_{s\to w} = 1 \\
& \sum_u x_{u\to t} - \sum_w x_{t\to w} = -1 \\
& \sum_u x_{u\to v} - \sum_w x_{v\to w} = 0 \quad \text{for every vertex } v \neq s,t \\
& x_{u\to v} \geq 0 \quad \text{for every edge } u\to v
\end{aligned}
$$

- Think of this as routing one unit of flow from s to t at minimum cost

Duality: Intuition

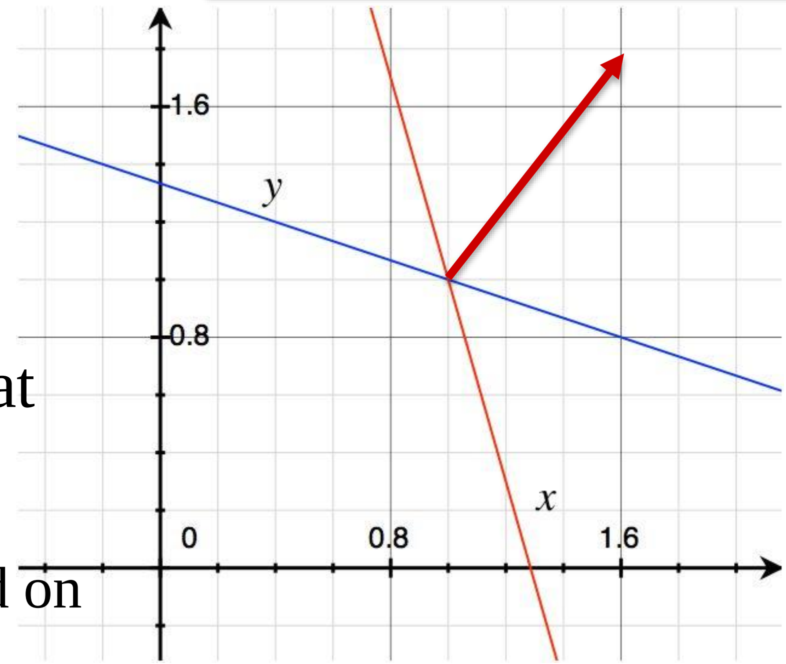
$\max 4x + 3y$

s.t. $\begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix}$

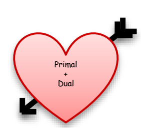
and $x, y \geq 0$

- We want an upper bound on OPT
  - ➢ Combine the inequalities we have?
  - ➢ For any feasible (x, y), and for all coefficients $w, z \geq 0$:

$$w(7x + 2y) + z(1x + 3y)$$
$$= (7w + z)x + (2w + 3z)y$$
$$\leq 9w + 4z$$



- Suppose we can find w, z such that $(7w + z) \geq 4$ and $(2w + 3z) \geq 3$
  - ➢ Then $9w + 4z$ is an upper bound on the value of the objective
  - ➢ How tight can we make this bound?
  - ➢ New LP! It's the dual!



$\min 9z + 4w$

s.t. $\begin{pmatrix} 7 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} \geq \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and

$z, w \geq 0$

Duality Theorems

n variables
$m + n$ constraints

$\max \vec{c} \cdot \vec{x}$
s.t. $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$

m variables
$n + m$ constraints

$\min \vec{y} \cdot \vec{b}$
s.t. $\vec{y}A \geq \vec{c}$ and $\vec{y} \geq 0$

- **Weak duality**: If $\vec{x}$ is feasible for Π and $\vec{y}$ is feasible for Π^* then $\vec{c} \cdot \vec{x} \leq \vec{y}A\vec{x} \leq \vec{y} \cdot \vec{b}$
  - Proof on board

- **Strong Duality**: A feasible point $\vec{x}$ for Π is optimal *if and only if* $\exists \vec{y}$ feasible for Π^* such that
$$\vec{c} \cdot \vec{x} = \vec{y}A\vec{x} = \vec{y} \cdot \vec{b}$$