Algorithm Design and Analysis

CSE 565

LECTURES 24

Linear programming

- Definitions
- Duality

(Based on Erickson, Chapter 26)

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Converting a LP into Canonical Form

- Replacing minimization with maximization

  Replace $\min_{\vec{x}} \vec{c} \cdot \vec{x}$ with $\max_{\vec{x}} -\vec{c} \cdot \vec{x}$

- Getting rid of unconstraint variables

  Replace each variable x_j with two new variables x_j^+, x_j^-

  - add the inequalities $x_j^+ \geq 0$ and $x_j^- \geq 0$
  - substitute $x_j^+ - x_j^-$ instead of x_j

- Getting rid of equalities

  Replace every equality $\sum_j a_{ij} x_j \geq b_i$ with

  - $\sum_j a_{ij} x_j \geq b_i$ and $\sum_j a_{ij} x_j \leq b_i$

- Getting rid of $\geq$ signs

  Replace every lower bound $\sum_j a_{ij} x_j \geq b_i$ with

  - $\sum_j -a_{ij} x_j \leq -b_i$

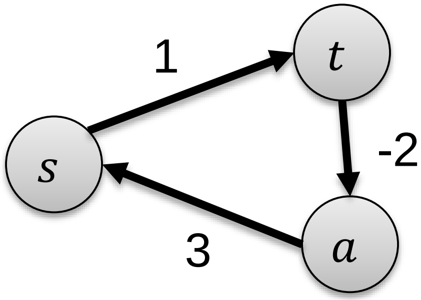
Example: Max flow as LP

- Given a graph, formulate an LP to determine the maximum *s*-*t* flow:
  - Variables: f_e for each edge $e \in E$
  - Constraints:
    - $f_e \geq 0$
    - $f_e \leq c_e$
    - $\sum_w f_{wv} - \sum_u f_{vu} = 0$ for all vertices $v \in V \backslash \{\mathrm{s}, \mathrm{t}\}$
  - Objective:
    - Maximize $\sum_u f_{su} - \sum_w f_{ws}$

Shortest Paths as an LP

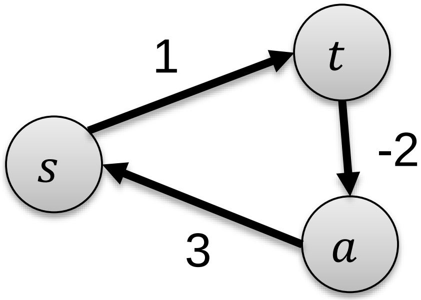
- Given: Graph $G = (V, E)$, lengths $\ell_{u,v}$, vertices $s, t \in V$
- Variables: d_v for each $v \in V$
- LP: maximize d_t
  s.t. $d_v - d_u \leq \ell_{u,v} \; \forall\, (u, v) \in E$
  $d_s = 0$
  - What is the value of this LP?
- **Claim:** LP value is the distance from s to t
  - Lower bound: If G has no negative cost cycles, then vector of shortest path distances from s is feasible.
  - Upper bound:
    Add the inequalities for edges along shortest s-t path:
    $d_t \leq$ (length of shortest s-t path)

Example: Shortest paths



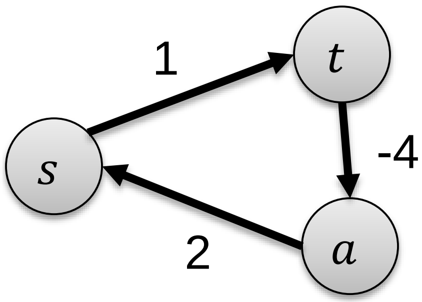
Shortest paths as an LP

Good case: shortest paths well-defined

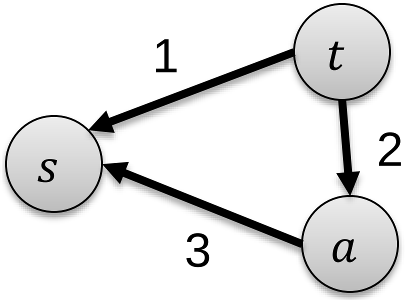


What happens to this LP if…

- G has a negative cost cycle?
  - LP is not feasible:
    Adding inequalities along cycle
    leads to contradiction



- G has no negative cost cycles, but t is not reachable from s in G?
  - LP is feasible but unbounded



Physical interpretation

- When G is undirected and edge lengths are positive, we can imagine
  - $|V|$ magnets on a whiteboard
  - For every edge $(u, v) \in E$, connect magnets u and v by a string of length $\ell_{u,v}$
  - Magnet s always at the bottom of the white board
- Interpreting the LP
  - d_v = height of magnet v on board (above s)
  - Feasible points: possible heights of magnets in some arrangement (with strings unbroken)
  - Value of LP: largest possible height for magnet t

Min-Cost Flow Flow

- Input: G, s, t, c, cost function \$, and demand value d
- Output: flow f of value d that minimizes $\sum_{e \in E} \$(e) \cdot f(e)$
- Give an LP for this problem
  - Variables: f_e for each edge $e \in E$
  - Constraints:
    - $f_e \geq 0$ for all edges $e \in E$
    - $f_e \leq c_e$ for all edges $e \in E$
    - $\sum_w f_{wv} - \sum_u f_{vu} = 0$ for all vertices $v \in V \setminus \{s, t\}$
    - $\sum_u f_{su} - \sum_w f_{ws} = d$
  - Objective:
    - Maximize $\sum_{e \in E} \$(e) \cdot f_e$

Multicommodity Flow

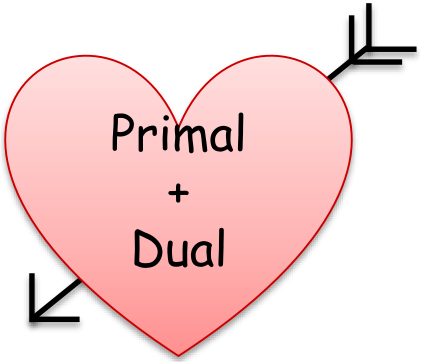
- Input: G, c, for each of k commodities: (s_i, t_i, d_i), where s_i is the source, t_i is the sink, d_i is the demand for commodity i
- Output: flow f_i for each commodity i
- Give an LP for this problem
  - Variables: f_e^i for each edge $e \in E$ and commodity i
  - Constraints:
    - $f_e^i \geq 0$ for each i, e
    - $\sum_i f_e^i \leq c_e$ for each edge e
    - $\sum_w f_{wv}^i - \sum_u f_{vu}^i = 0$ for all vertices $v \in V \backslash \{\mathrm{s}, \mathrm{t}\}$ and commodities i
    - $\sum_u f_{su}^i - \sum_w f_{ws}^i = d_i$ for all commodities i
  - Objective:
    - Maximize 0

Min cut as LP

- Given a graph, formulate an LP to determine the minimum s-t cut:
  - Variables:
    - s_v for each vertex $v \in V$
      - Intuition: s_v=1 if $v \in S$ and 0 if $v \in T$
    - x_{uv} for each edge (u, v)
      - Intuition: x_{uv}=1 if $u \in S$ and $v \in T$
  - Constraints:
    - $x_{uv} + s_v - s_u \geq 0$ for all edges $(u, v) \in E$
    - $x_{uv} \geq 0$ for all edges $(u, v) \in E$
    - $s_s = 1$ and $s_t = 0$
  - Objective:
    - Minimize $\sum_{uv} c_{uv} \cdot x_{uv}$
- Requires some work to prove correctness

Duality of linear programs

- For every linear program Π, there is another linear program Π^* that is perfect for it
  - ➢ Π^* is "the" dual program of Π
    - The dual is not unique
  - ➢ The dual of Π^* is Π



- If we put Π in canonical form as a max. problem

$$\max \vec{c} \cdot \vec{x}$$
$$\text{s.t.} \quad Ax \leq \vec{b} \text{ and } x \geq 0$$

n variables
m constraints

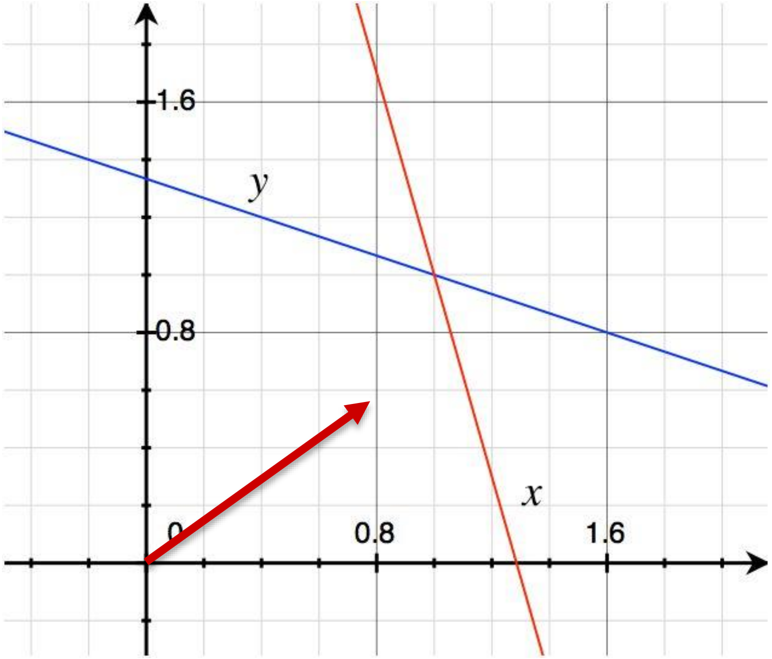
then we can write Π^* as a minimization problem

$$\min \vec{b} \cdot \vec{y}$$
$$\text{s.t.} \quad A^\top y \geq \vec{c} \text{ and } y \geq 0$$

m variables
n constraints

Examples

1. $\max 4x + 3y$
   s.t. $\begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 9 \\ 4 \end{pmatrix}$
   and $x, y \geq 0$



2. (Shortest paths variant)
   LP: maximize $\sum_v d_v$
   s.t. $d_v - d_u \leq \ell_{u,v} \; \forall \, (u,v) \in E$
   $d_s = 0$