CSE 565

LECTURES 26

Linear programming

- Definitions
- Duality

(Based on Erickson, Chapter 26)

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Topics

- Definition of LP
  - Canonical form
  - Slack form
- Example
  - 2-dimensional
  - Shortest paths
- Exercise:
  - min-cost flow
  - Max-weight bipartite matching
- Dual LP
  - … in canonical form
  - … in dual form (homework?)
- Duality
  - Weak duality
  - Strong duality and Lagrange multipliers
- Simplex method: primal and dual versions
- Tying it all together: min-cost perfect matching
  - Primal-dual algorithm
  - Dual as prices on constraints

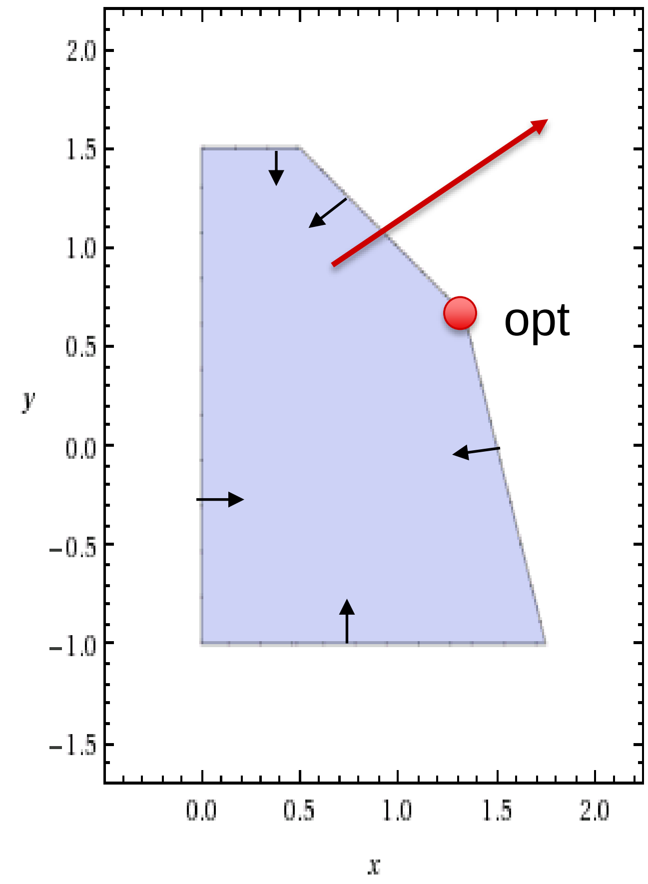
Linear Programming

- A linear program (LP) is described by
  - A set of n variables $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$
  - A set of m linear constraints of the form
    $\overrightarrow{a_j} \cdot \vec{x} \le b_j$ or
    $\overrightarrow{a_j} \cdot \vec{x} \ge b_j$ or
    $\overrightarrow{a_j} \cdot \vec{x} = b_j$
    where $\overrightarrow{a_j} \in \mathbb{R}^n$ and
    $b_j \in \mathbb{R}$ for $j = 1, \dots, m$.
  - Objective function: either $\max_{\vec{x}} \vec{c} \cdot \vec{x}$ or $\min_{\vec{x}} \vec{c} \cdot \vec{x}$

In matrix notation, we write
$A_1\vec{x} \le \overrightarrow{b_1}$ where $A_1 \in \mathbb{R}^{m_1 \times n}$ and $\overrightarrow{b_1} \in \mathbb{R}^{m_1}$
$A_2\vec{x} = \overrightarrow{b_2}$ where $A_2 \in \mathbb{R}^{m_2 \times n}$ and $\overrightarrow{b_2} \in \mathbb{R}^{m_2}$

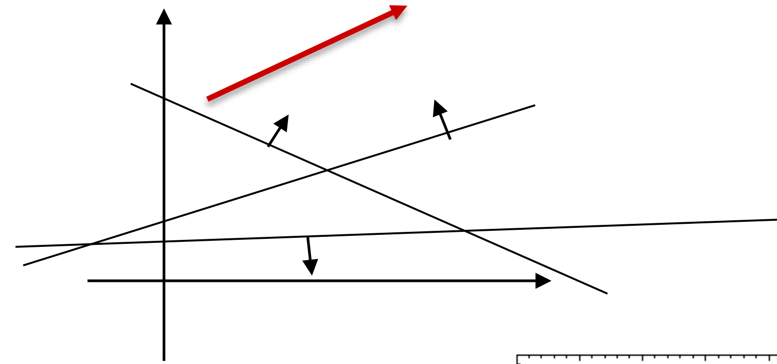
Example

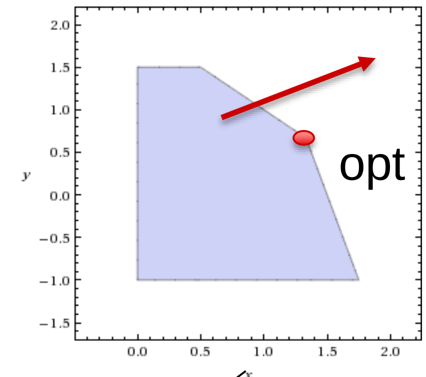
- Two variables: x, y
- Constraints
  - $-1 \leq y \leq 1.5$
  - $4x + 2y \leq 6$
  - $x + y \leq 2$
  - $x \geq 0$
- Maximize $2x + y$



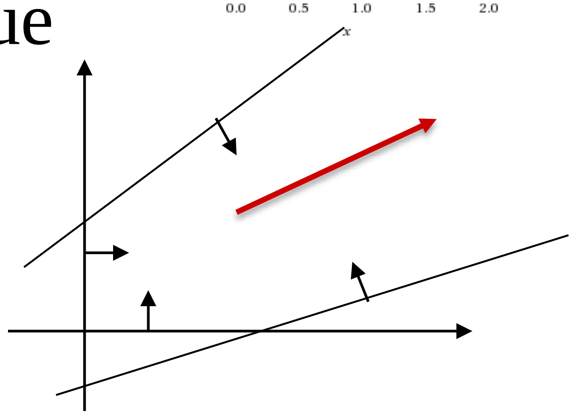
Possible behavior of maximization LP

- Infeasible
  - ➢ No point satisfies all constraints





- Feasible and bounded
  - ➢ Objective has a finite maximum
  - ➢ Maximum point may not be unique



- Feasible and unbounded
  - ➢ Objective can be arbitrarily large

Terminology

- Plane
- Halfspace
- Polytope = intersection of halfplanes
- Nonempty polytopes have
  - Vertices
  - Edges
  - Facets of various dimensions

Forms of LPs

- Many equivalent forms of "same" LP
  - Inequalities and variables may change
  - Informally: two LP's are called equivalent if there is an easy way to go from solutions of one to solutions of the other (and vice versa)
- Examples:
  - Canonical form
    - Maximization LP
    - Constraints: $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$
  - Slack form
    - Maximization LP
    - Constraints: $A'\vec{x} = \overrightarrow{b'}$ and $\vec{x} \geq 0$
  - Every LP can be formulated in canonical or slack form

Converting a LP into Canonical Form

- Getting rid of unconstraint variables

  Replace each variable x_j with two new variables x_j^+, x_j^-

  - add the inequalities $x_j^+ \geq 0$ and $x_j^- \geq 0$
  - substitute $x_j^+ - x_j^-$ instead of x_j

- Getting rid of equalities

  Replace every equality $\sum_j a_{ij} x_j \geq b_i$ with

  - $\sum_j a_{ij} x_j \geq b_i$ and $\sum_j a_{ij} x_j \leq b_i$

- Getting rid of $\geq$ signs

  Replace every lower bound $\sum_j a_{ij} x_j \geq b_i$ with

  - $\sum_j -a_{ij} x_j \leq -b_i$