Algorithm Design and Analysis

CSE 565

LECTURES 18

Network Flow

- Algorithms:
  - Ford-Fulkerson
  - Capacity Scaling
- Applications

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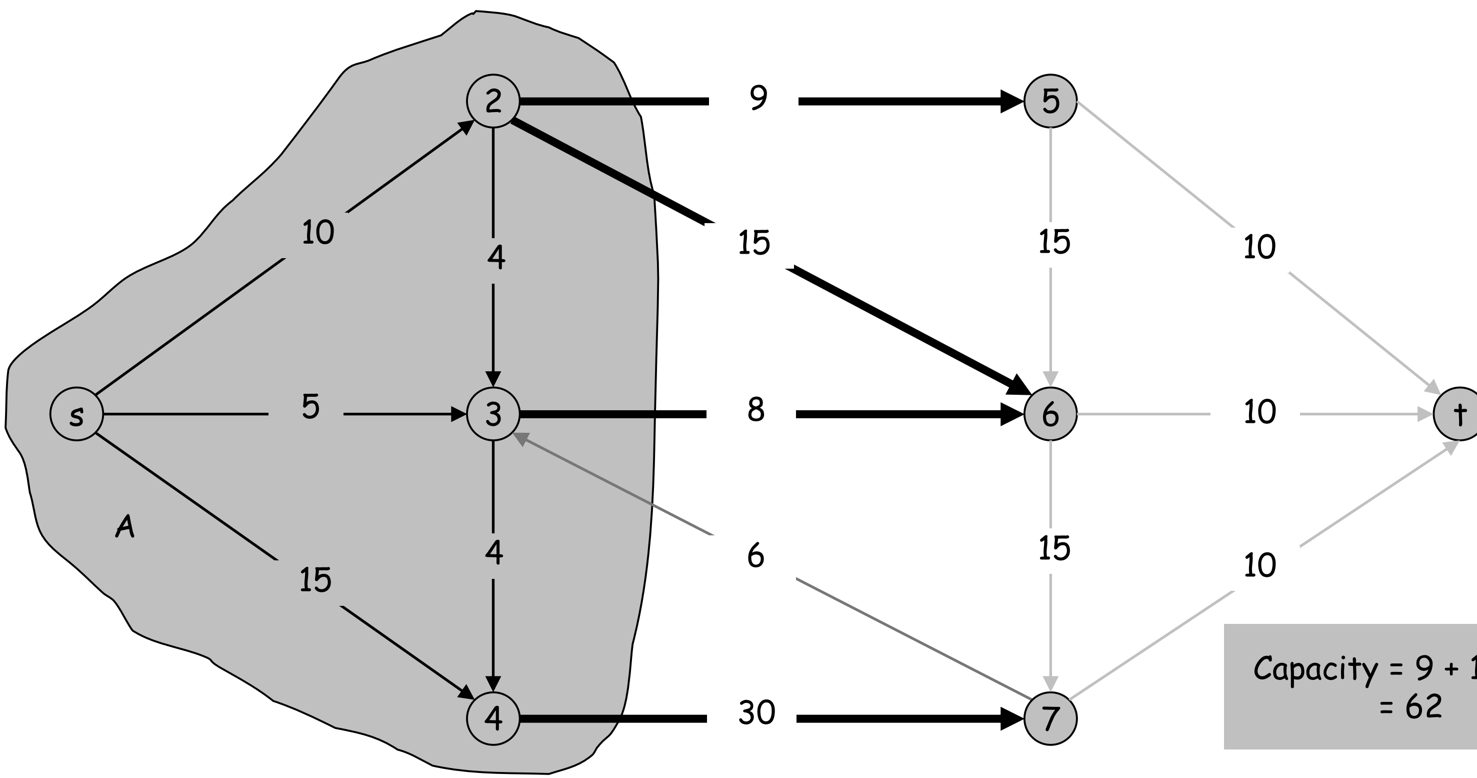
Network Flow

Minimum Cut Problem

Def. An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$.

Def. The **capacity** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$

Goal. Find an s-t cut of minimum capacity.



Capacity = 9 + 15 + 8 + 30
= 62

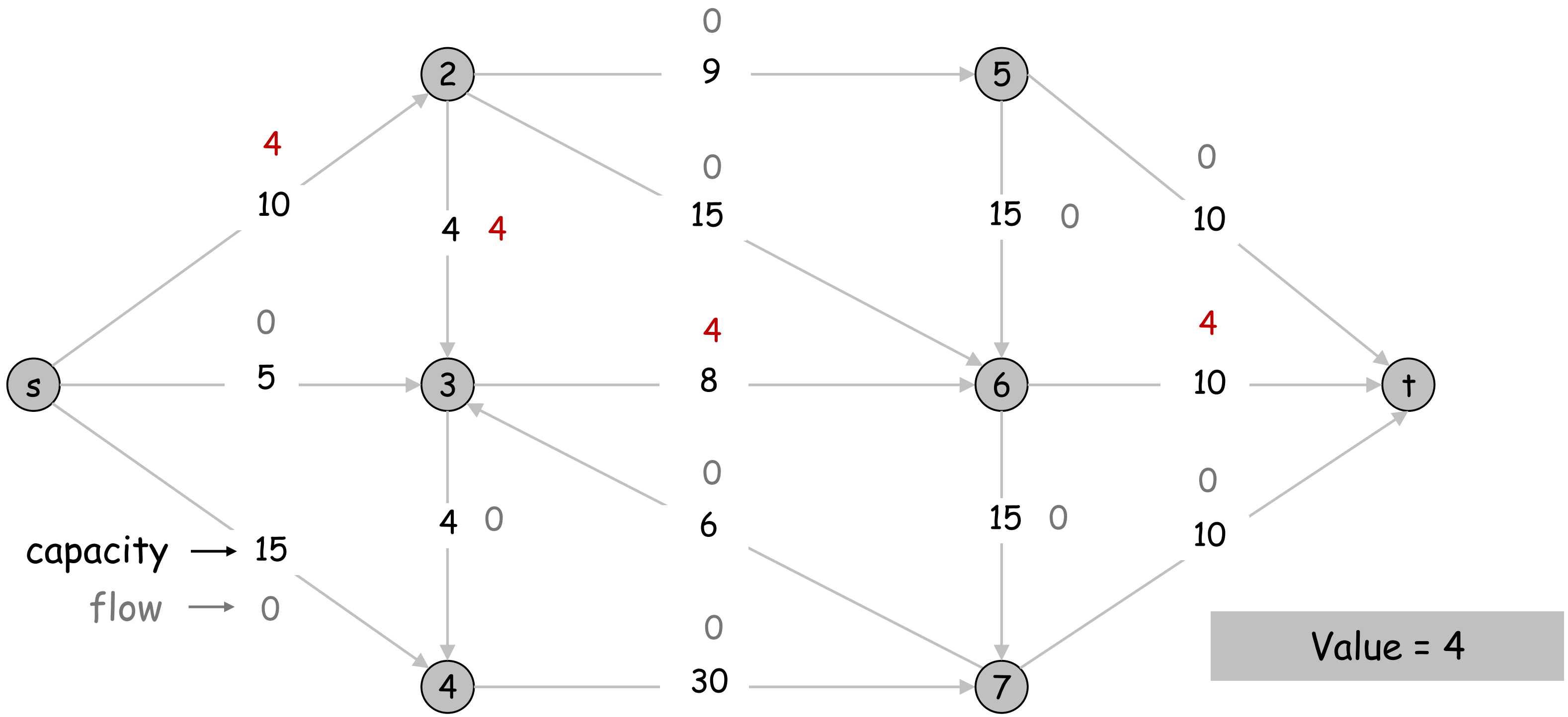
Maximum Flow Problem

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.

Goal. Find s-t flow of maximum value.



What we proved about flows and cuts

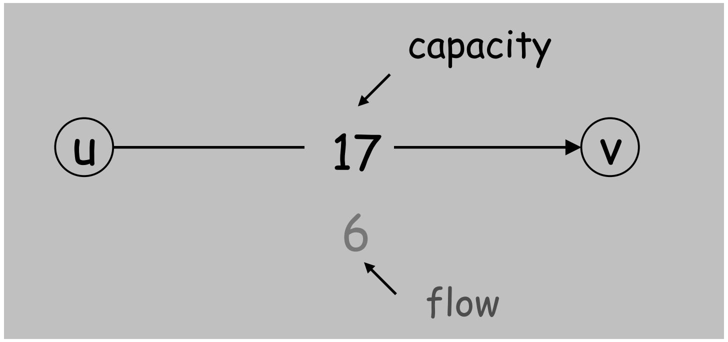
Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to A}} f(e) \; = \; v(f)$$

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

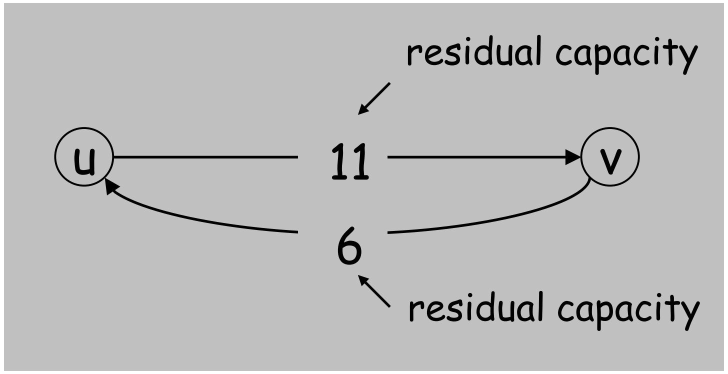
Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

Residual Graph



Original edge: $e = (u, v) \in E$.

- Flow f(e), capacity c(e).



Residual edge.

- "Undo" flow sent.
- $e = (u, v)$ and $e^R = (v, u)$.
- Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$

Residual graph: $G_f = (V, E_f)$.

- Residual edges with positive residual capacity.
- $E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

Ford-Fulkerson: Analysis

Ford-Fulkerson summary:

- **While** you can,
  - Greedily push flow
  - Update residual graph

Feasibility lemma: Ford-Fulkerson outputs a valid flow.

Optimality: If Ford-Fulkerson terminates then

- the output is a max flow;
- set of vertices reachable from s in residual graph forms a minimum cut.

Still to do:

- Running time (in particular, termination!)

Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \le nC$ iterations.
Proof. Each augmentation increases flow value by at least 1. ▪

Running time of Ford-Fulkerson on a graph with integer capacities?

Augmenting Path Algorithm

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E,  f(e) ← 0
   G_f ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update G_f
   }
   return f
}
```

$O(m+n)$

$O(n)$

$O(n)$

```
Augment(f, c, P) {
   b ← bottleneck-capacity(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b
      else       f(e^R) ← f(e^R) - b
   }
   return f
}
```

$O(n)$

Min residual capacity of an edge in P

forward edge

reverse edge

Running Time

Assumption. All capacities are integers between 1 and C.

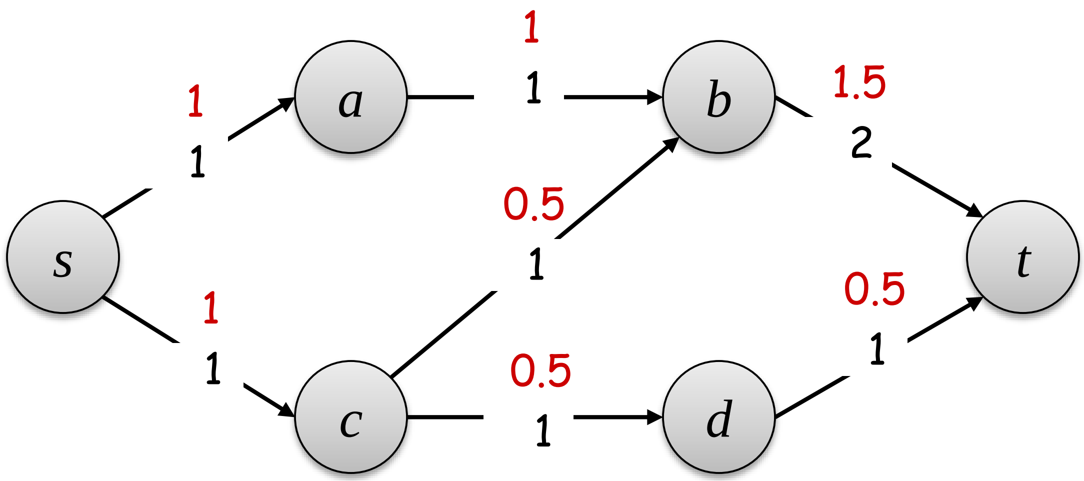
Invariant. Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations.
Proof. Each augmentation increases flow value by at least 1. ▪

Running time of Ford-Fulkerson on a graph with integer capacities: $O(mnC)$.
Space: $O(m+n)$.
Important special case. If $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Review Question

- Is this flow a maximum flow?



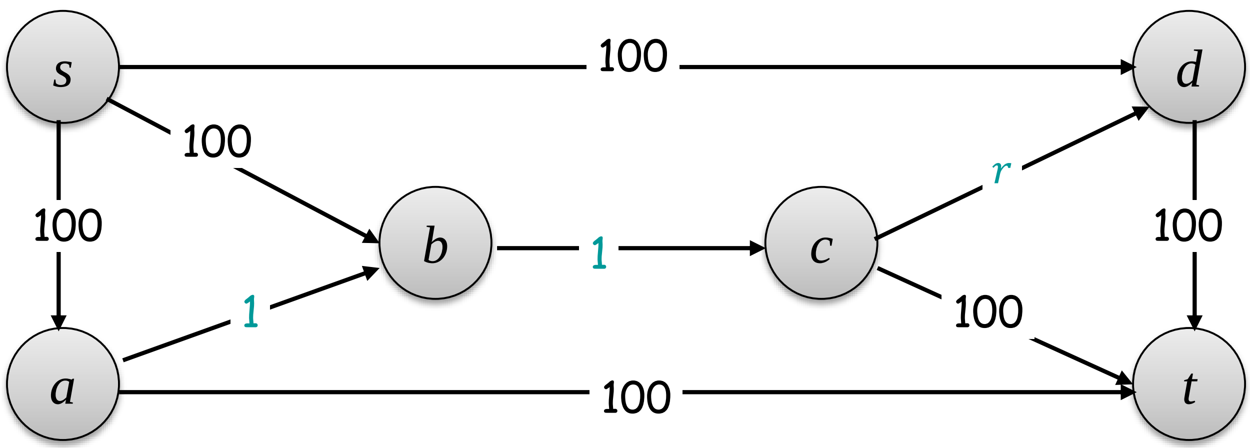
- **Def: Integral flow**: flows on all edges are integers
- Does this graph have an integral maximum flow?
- Does every graph with integer capacities have an integral maximum flow?

Ford-Fulkerson Summary

- **Assumption**: All capacities are integers between 1 and C.
- **Running time**: The FF algorithm terminates in at most $v(f^*) \leq nC$ iterations.
  Running time = $O(mnC)$. Space: $O(m+n)$.
- **Correctness**:
  - FF outputs a flow with maximum value
  - Set of vertices reachable from s in residual graph forms a minimum cut
  - **Integrality theorem**: FF outputs an integral flow, so every graph with integer capacities has an integral maximum flow.
- **Important special case**: if $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Review Question

- Does Ford-Fulkerson always terminate if capacities are rational?
- Does Ford-Fulkerson always terminate if capacities are irrational?



$$r = \frac{\sqrt{5}-1}{2} \implies r^2 = 1 - r$$

- Exercise: Find a sequence of augmenting paths so that FF does not terminate and does not converge to max flow.

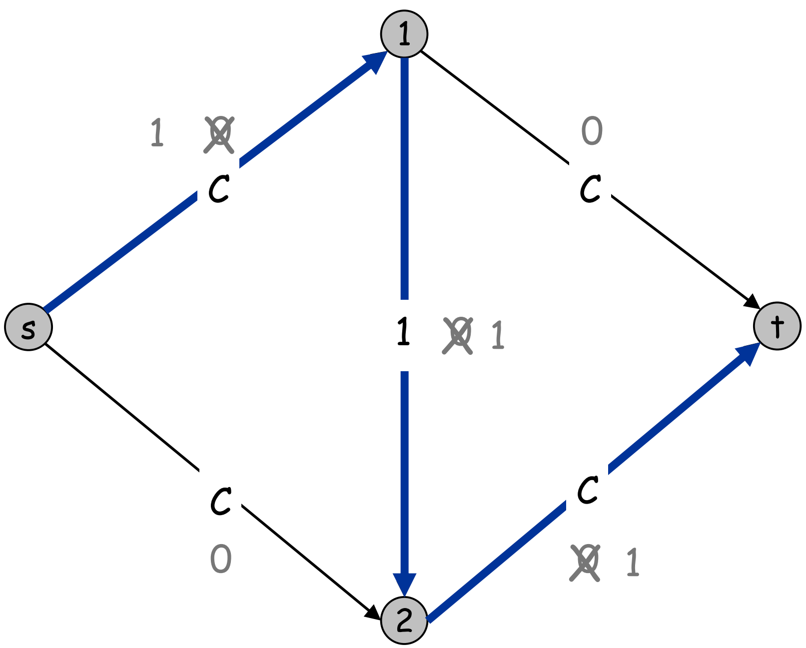
Faster algorithms when capacities are large

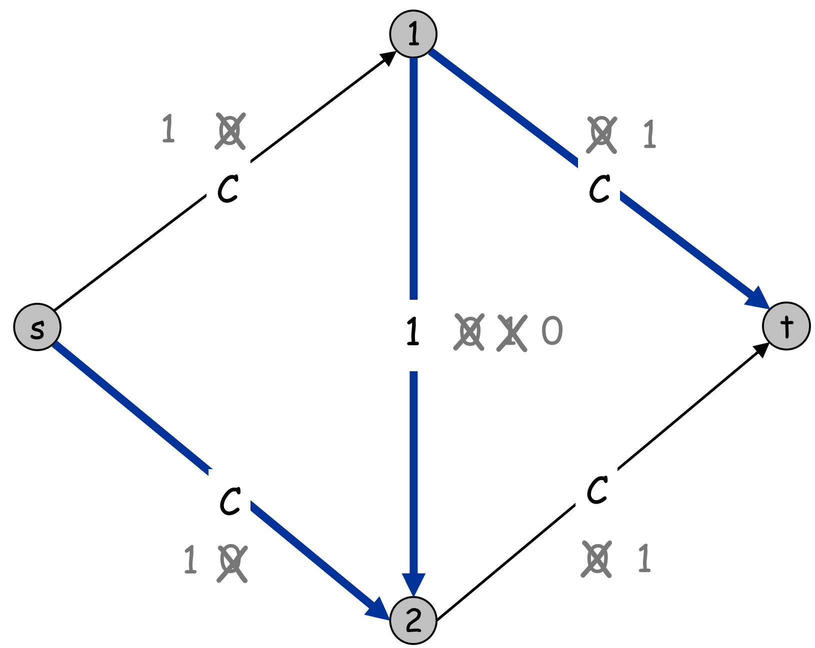
Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C

A. No. If max capacity is C, then algorithm can take C iterations.





Intuition: We're choosing the wrong paths!

Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

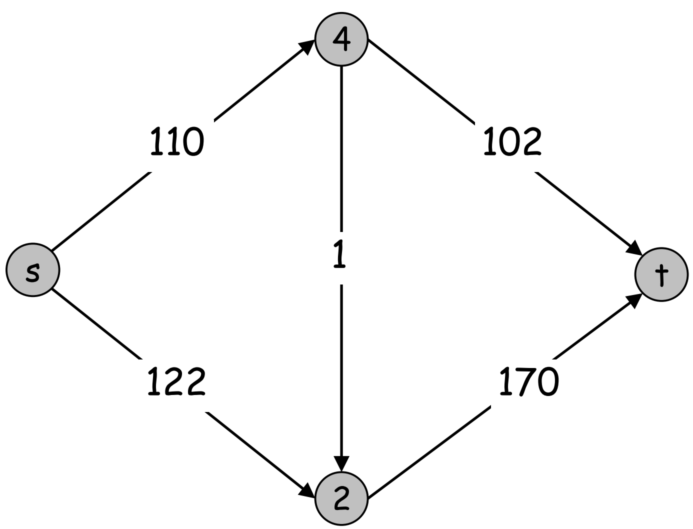
Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

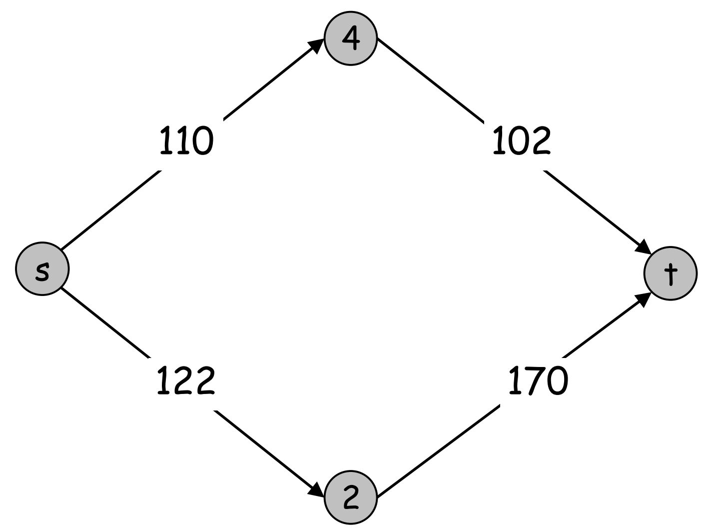
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of **only** arcs with capacity at least Δ.



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   Δ ← smallest power of 2 greater than or equal to C
   G_f ← residual graph

   while (Δ ≥ 1) {
      G_f(Δ) ← Δ-residual graph
      while (there exists augmenting path P in G_f(Δ)) {
         f ← augment(f, c, P) // augment flow by ≥ Δ
         update G_f(Δ)
      }
      Δ ← Δ / 2
   }
   return f
}
```

Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C.

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Proof.

- By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.
Proof. Initially $C \leq \Delta < 2C$; Δ decreases by a factor of 2 each iteration. ▪

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow is at most $v(f) + m\,\Delta$. ← proof on next slide

Lemma 3. There are at most 2m augmentations per scaling phase.
- Let f be the flow at the end of the previous scaling phase.
- Lemma 2 $\Rightarrow$ $v(f^*) \leq v(f) + m\,(2\Delta)$.
- Each augmentation in a Δ-phase increases $v(f)$ by at least Δ. ▪

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ▪

Capacity Scaling: Running Time

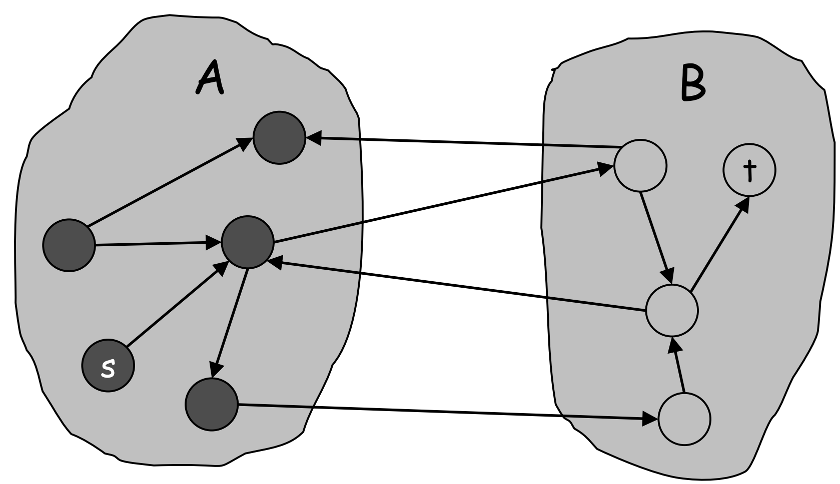
Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then value of the maximum flow is at most $v(f) + m\,\Delta$.

Proof. (almost identical to proof of max-flow min-cut theorem)

- We show that at the end of a Δ-phase, there exists a cut (A, B) such that $cap(A, B) \le v(f) + m\,\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of A, source $s \in A$.
- By definition of f, sink $t \notin A$.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\ &\ge \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\ &= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\ &\ge cap(A, B) - m\Delta \end{aligned}$$

So, $v(f^*) \le cap(A,B) \le v(f) + m\Delta$. ▪



original network

General Principle

- Let
  - $G = (V, E)$ be a directed graph with capacities $\{c_e\}_{e \in E}$
  - f be any valid flow in G
  - G_f be the residual graph for f in G
  - f^* be any maximum flow in G
- Then we have
  $v(f^*) = v(f) +$ (value of max s-t flow in G_f)

- In particular, for any cut A, B:
  $v(f^*) \leq v(f) +$ (capacity of A, B in G_f)
- Applications:
  - Correctness of Ford-Fulkerson
  - Running time analysis for capacity scaling

Best Known Algorithms For Max Flow

- Reminder: The scaling max-flow algorithm runs in $O(m^2 \log C)$ time.
- There are algorithms that run in time
  - $O(mn)$ (Orlin, 2013)
  - $O(m^{\frac{10}{7}} \log^a m)$ for constant a and $C = 1$ (Madry, 2013)
  - $O\left(\min\left(n^{\frac{2}{3}}, m^{\frac{1}{2}}\right) \cdot m \cdot \log n \cdot \log C\right)$
- Active topic of research:
  - Flow algorithms for specific types of graphs
  - Special cases (bipartite matching, etc)
  - Multi-commodity flow
  - …

Applications when C=1

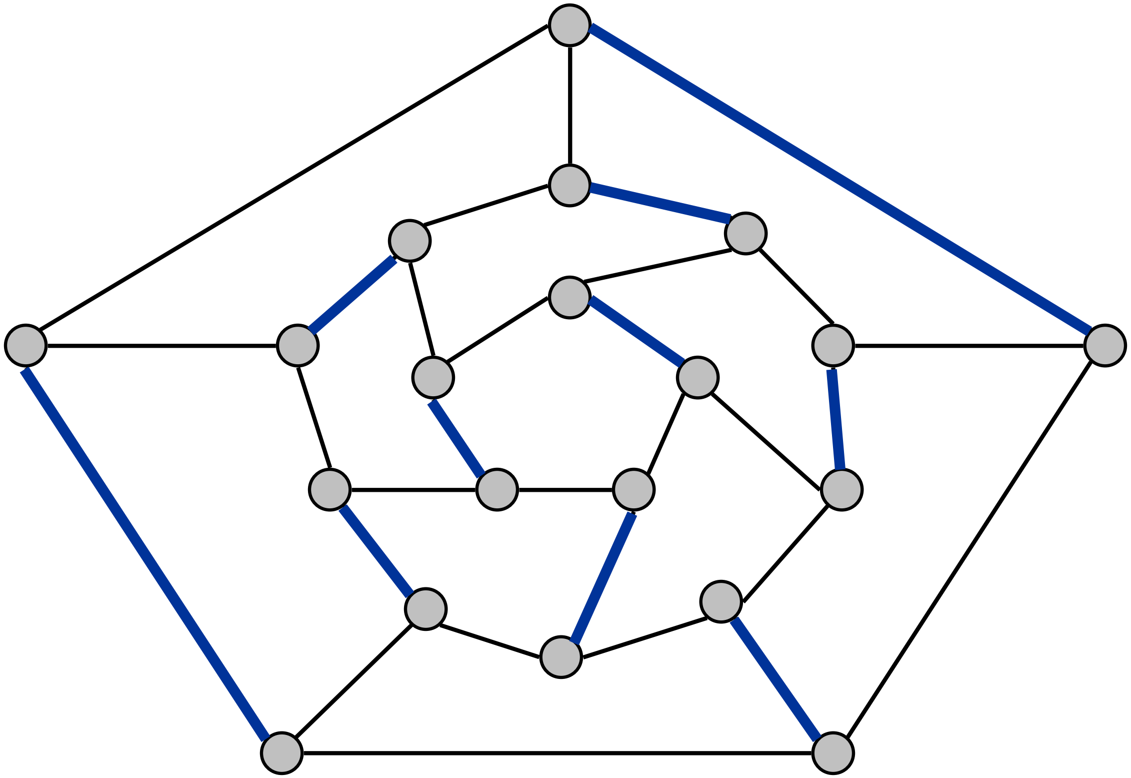
- Maximum bipartite matching
  – Reducing MBM to max-flow
  – Hall's theorem

- Edge-disjoint paths
  – another reduction

Matching

Matching.

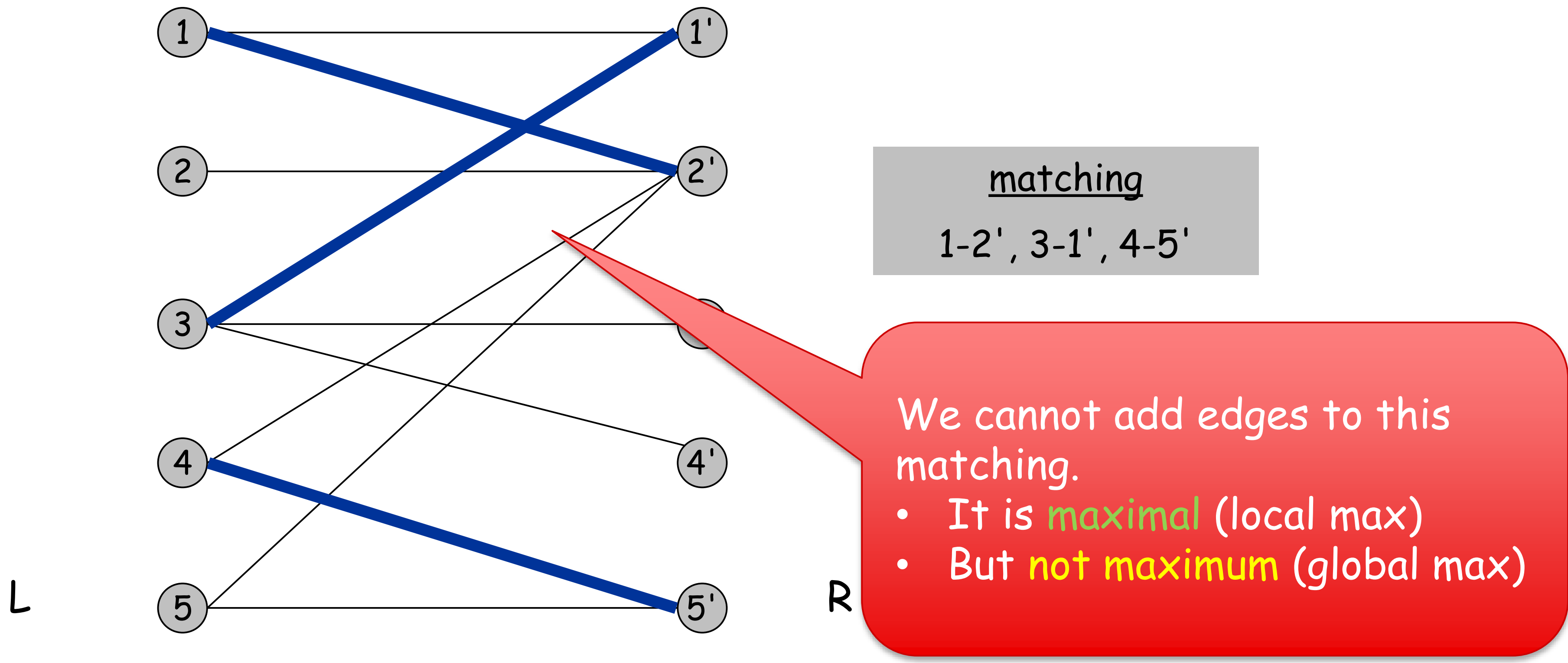
- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most 1 edge in M.
- Maximum matching: find a matching with as many edges as possible.



Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Maximum matching: find a matching with as many edges as possible.



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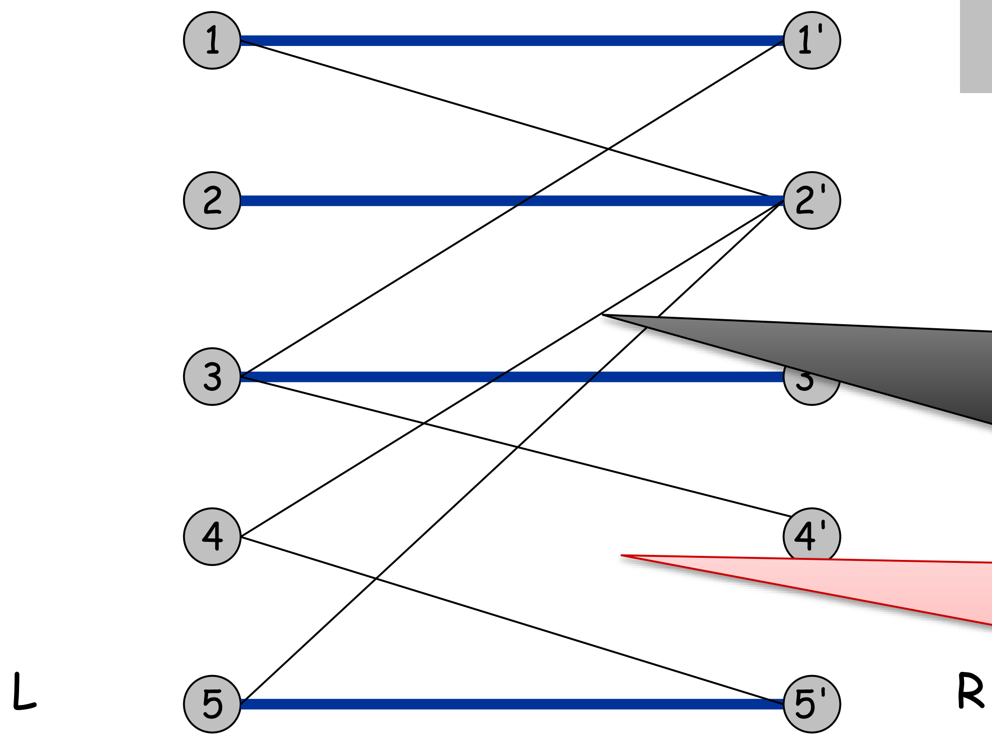


max matching

1-1', 2-2', 3-3' 4-4'

There is no matching in this graph with more than 4 edges

- This matching is both maximal (local max) and maximum (global max)

Do not confuse with stable matching (different inputs and goals)

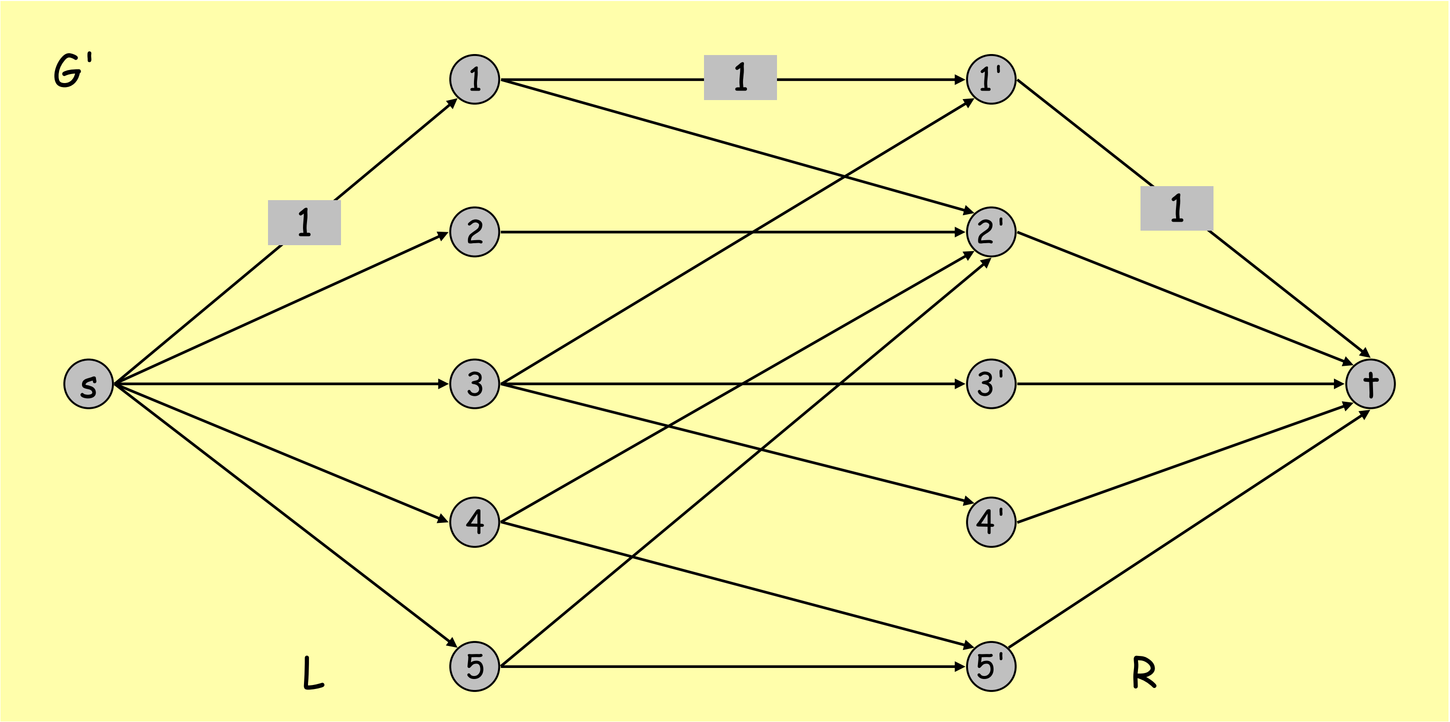
Reductions

- "Problem A reduces to problem B"
  - Rough meaning: there is a simple algorithm for A that uses an algorithm for B as a subroutine.
  - Denote $A \leq B$
- Usually:
  - Given instance x of problem A
    we find a instance x' of problem B
  - Solve x'
  - Use the solution to build a solution to x
- Useful skill: quickly identify problems where existing solutions may be applied.
  - Good programmers do this all the time

Reducing Bipartite Matching to Maximum Flow

Reduction to Max flow.

- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from L to R, and assign capacity 1.
- Add source s, and capacity 1 edges from s to each node in L.
- Add sink t, and capacity 1 edges from each node in R to t.



Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Proof: We need two statements

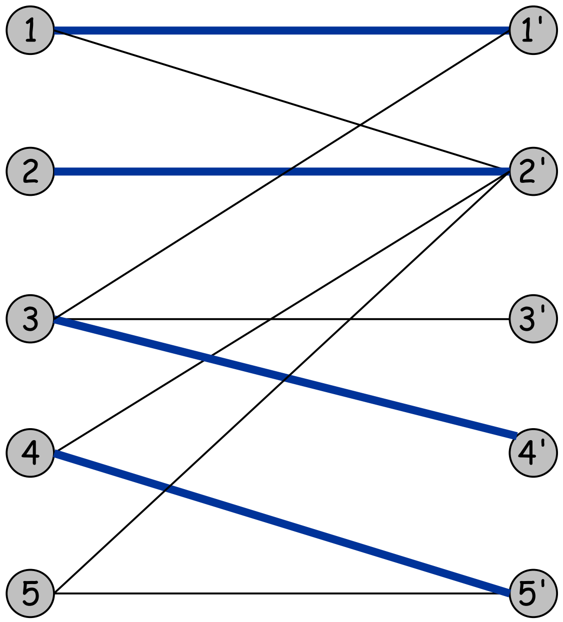
- max. matching in $G \leq$ max flow in G'
- max. matching in $G \geq$ max flow in G'

Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\leq$

- Given max matching M of cardinality k.
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has value k. ▪



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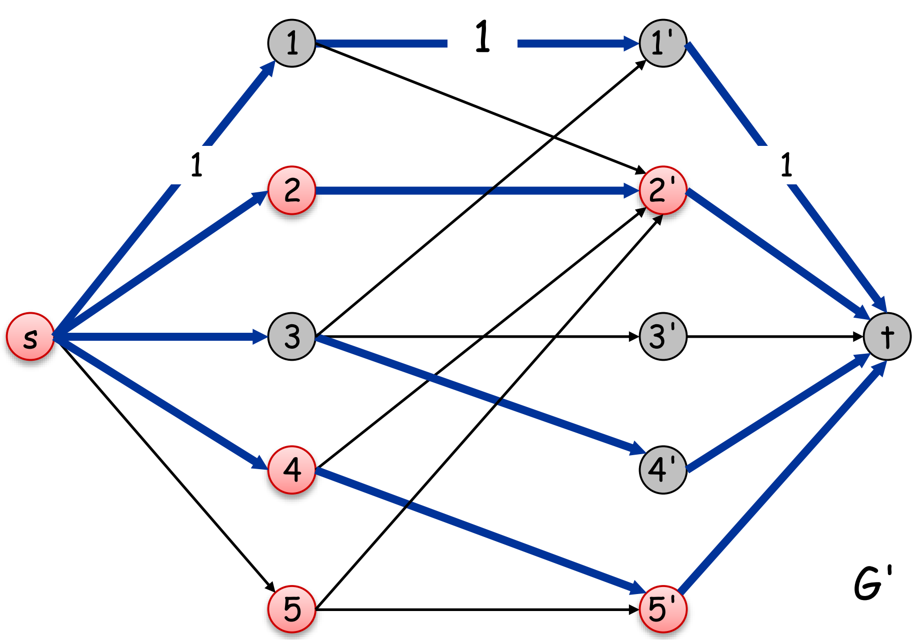

G'

Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\geq$

- Let f be a max flow in G' of value k.
- Integrality theorem $\Rightarrow$ we can find a max flow f that is integral;
  - all capacities are 1 $\Rightarrow$ f takes values only in {0,1}
- Consider M = set of edges from L to R with f(e) = 1.
  - Each node in L and R participates in at most one edge in M
    - ✎ Because all capacities are 1 and flow must be conserved
  - |M| = k: consider cut ($\{s\}$, $S \cup R \cup t$)