CSE 565

LECTURE 12

Dynamic Programming

- Fibonacci Numbers
- Weighted Interval Scheduling
- Longest Common Subsequence

Sofya Raskhodnikova

Dynamic Programming

> “Those who cannot remember the past are doomed to repeat it."

George Santayana, *The Life of Reason, Book I: Introduction and Reason in Common Sense*

Design Techniques So Far

- Greedy. Build up a solution incrementally, myopically optimizing some local criterion.
- Recursion / divide &conquer. Break up a problem into subproblems, solve subproblems, and combine solutions.
- Dynamic programming. Break problem into *overlapping* subproblems, and build up solutions to larger and larger subproblems.

Dynamic Programming History

- Bellman. [1950s] Pioneered the systematic study of dynamic programming.

- Etymology.
  - Dynamic programming = planning over time.
  - Secretary of Defense was hostile to mathematical research.
  - Bellman sought an impressive name to avoid confrontation.

"it's impossible to use dynamic in a pejorative sense"
"something not even a Congressman could object to"

Reference: Bellman, R. E. *Eye of the Hurricane, An Autobiography.*

Dynamic Programming Applications

- Areas.
  - Bioinformatics.
  - Control theory.
  - Information theory.
  - Operations research.
  - Computer science: theory, graphics, AI, compilers, systems, ….
- Some famous dynamic programming algorithms.
  - Unix diff for comparing two files.
  - Viterbi for hidden Markov models / decoding convolutional codes
  - Smith-Waterman for genetic sequence alignment.
  - Bellman-Ford for shortest path routing in networks.
  - Cocke-Kasami-Younger for parsing context free grammars.

Fibonacci Sequence

Fibonacci Sequence

- Sequence defined by
  - $a_1 = 1$
  - $a_2 = 1$
  - $a_n = an_{-1} + an_{-2}$

1, 1, 3, 5, 8, 13, 21, 34, …

- How should you compute the Fibonacci sequence?
- Recursive algorithm:

```
Fib(n)
If n =1 or n=2, then
        return 1
Else
        a = Fib(n-1)
        b = Fib(n-2)
        return a+b
```

- Running Time?

Review Question

- Prove that the solution to the recurrence $T(n)=T(n-1)+T(n-2) + \Theta(1)$ is exponential in n.

Review Question

- Prove that the solution to the recurrence T(n)=T(n-1)+T(n-2) + Θ(1) is exponential in n.

Easy to show: $\Omega\left(\left(\sqrt{2}\right)^n\right)$ by inspecting the recursion tree

Later in the course (if time permits):

$\Theta(\phi^n)$ where $\phi \approx 1.618$ is the golden ratio

Computing Fibonacci Sequence Faster

- **Observation:** Lots of redundancy! The recursive algorithm only solves n-1 different subproblems
- "Memoization": Store the values returned by recursive calls in a sub-table
- Resulting Algorithm:

```
Fib(n)
If n =1 or n=2, then
        return 1
Else
        f[1]=1; f[2]=1
        For i=3 to n
            f[i] ← f[i-1]+f[i-2]
        return f[n]
```

- Running Time?

$O(n)$ if integer operations take constant time.

Computing Fibonacci Sequence Faster

- Observation: Fibonacci recurrence is linear

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} = \cdots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$$

- Can compute A^n using only O(log n) matrix multiplications; each one takes O(1) integer multiplications and additions.
- Total running time?

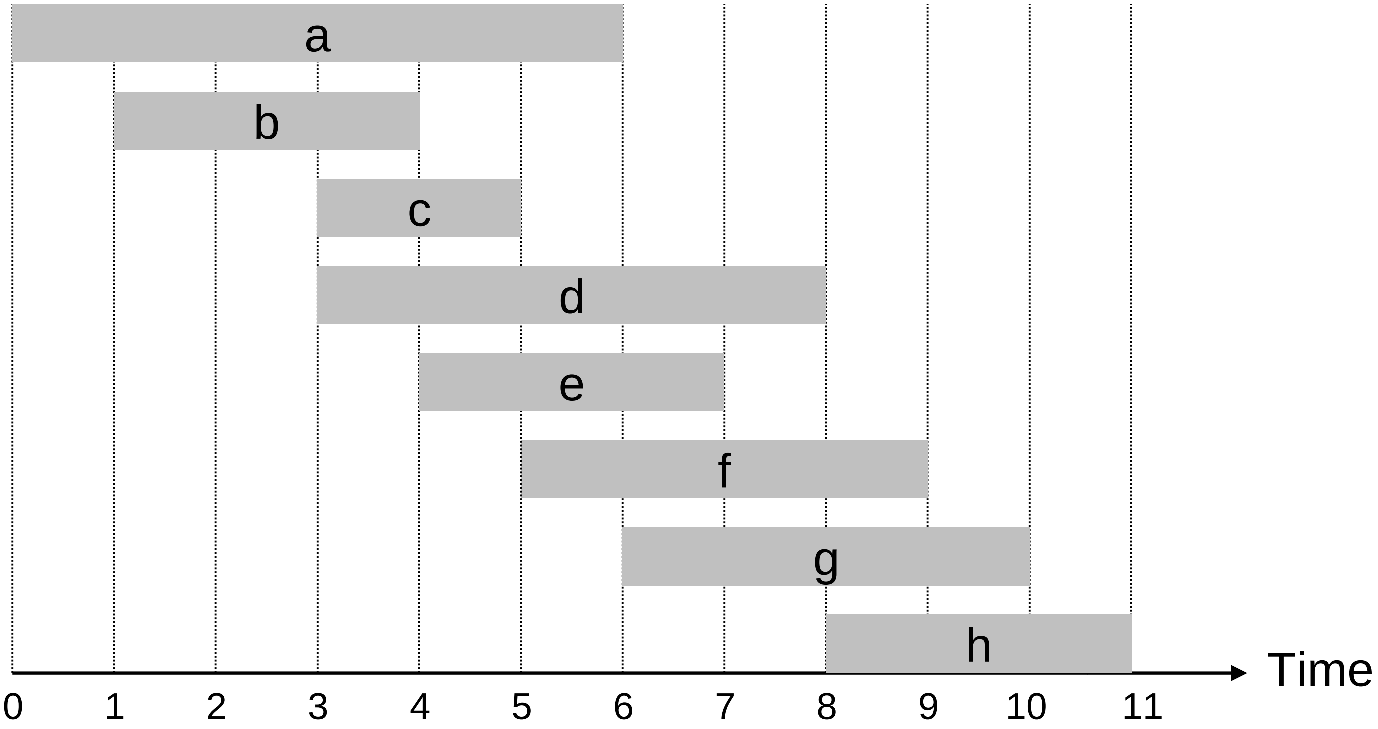
  O(log n) integer operations. Exponential improvement!

- Exercise: how big an improvement if we count bit operations?
  - Multiplying k-bit numbers takes O(k log k) time.
- How many bits needed to write down a_n?

Weighted Interval Scheduling

Weighted Interval Scheduling

- Weighted interval scheduling problem.
  - Job j starts at s_j, finishes at f_j, and has weight or value v_j .
  - Two jobs compatible if they don't overlap.
  - **Goal:** find maximum weight subset of mutually compatible jobs.

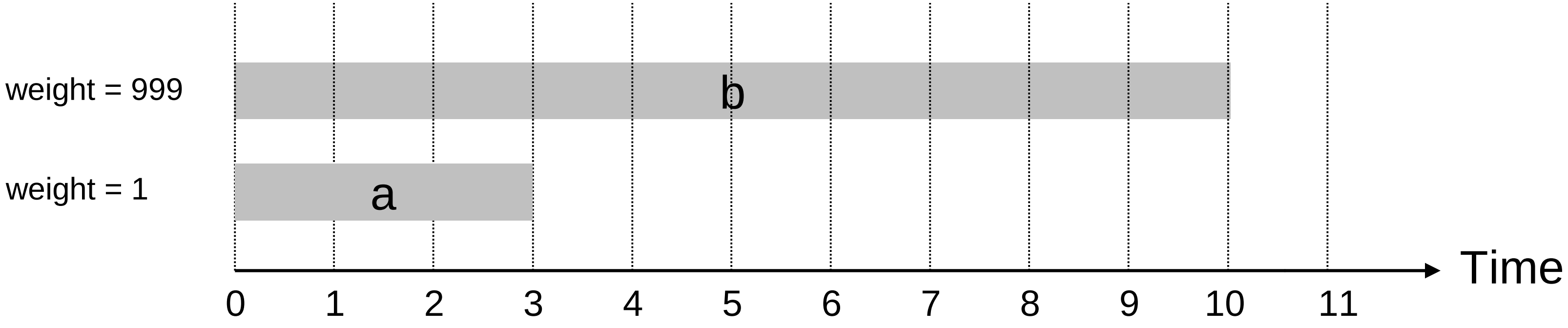


S. Raskhodnikova; based on slides by E. Demaine, C. Leiserson, A. Smith, K. Wayne

Unweighted Interval Scheduling Review

- **Recall.** Greedy algorithm works if all weights are 1.
  - Consider jobs in ascending order of finish time.
  - Add job to subset if it is compatible with previously chosen jobs.

- **Observation.** Greedy algorithm can fail spectacularly with weights.



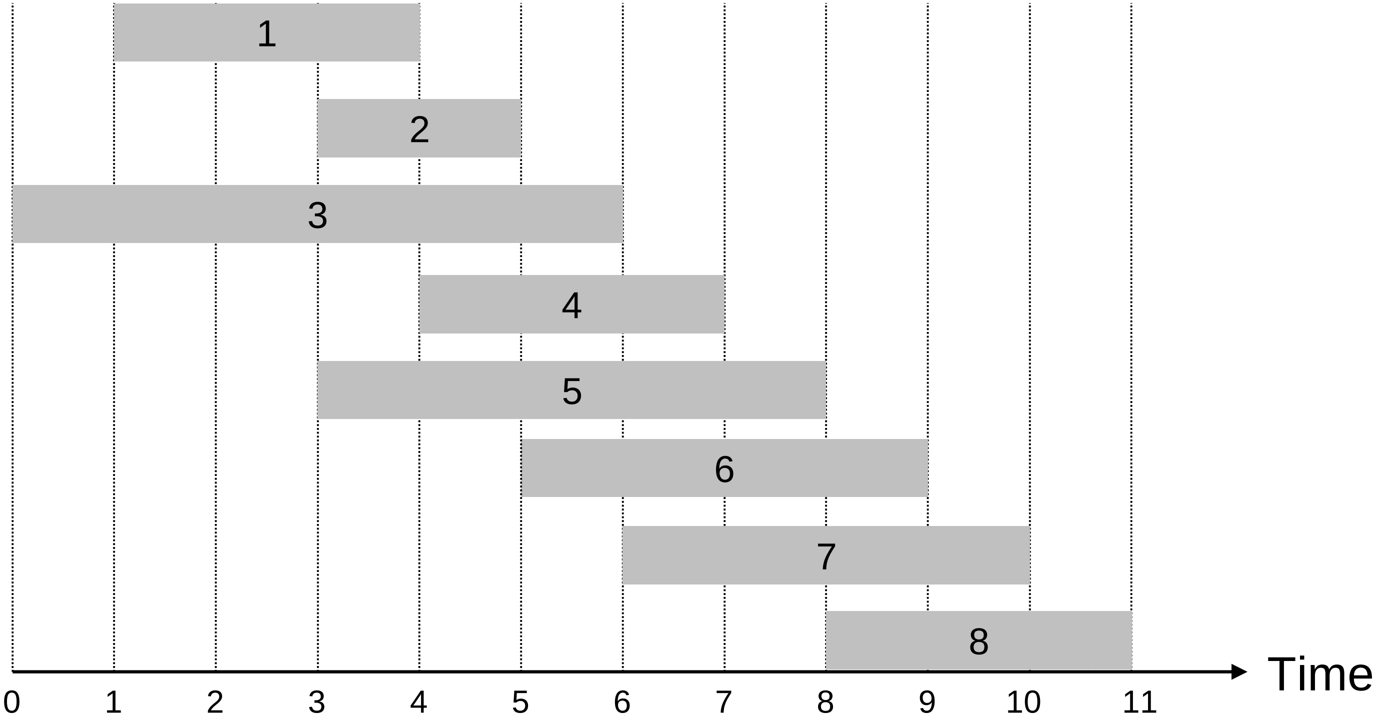
Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.

Define: p(j) = largest index i < j such that job i is compatible with j.

= largest i such that $f_i \leq s_j$

Example: p(8) = 5, p(7) = 3, p(2) = 0.



S. Raskhodnikova; based on slides by E. Demaine, C. Leiserson, A. Smith, K. Wayne

Dynamic Programming: Binary Choice

- **Notation.** OPT(j) = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.
  - **Case 1:** OPT selects job j.
    - collect profit v_j
    - can't use incompatible jobs { p(j) + 1, p(j) + 2, ..., j - 1 }
    - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., p(j)
  - **Case 2:** OPT does not select job j.
    - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

optimal substructure

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)),\ OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

- Brute force algorithm.

```
Input: n, $s_1,…,s_n$ , $f_1,…,f_n$ , $v_1,…,v_n$

Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

Compute p(1), p(2), …, p(n)

Compute-Opt(n)

Compute-Opt(j) {
    if (j = 0)
        return 0
    else
        return max($v_j$ + Compute-Opt(p(j)), Compute-Opt(j-1))
}
```

Weighted Interval Scheduling: Memoization

- **Memoization**. Store results of each sub-problem in a table; lookup as needed.

```
Input: n, $s_1,…,s_n$ , $f_1,…,f_n$ , $v_1,…,v_n$

Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty
M[0] = 0

Compute-Opt(n)

M-Compute-Opt(j) {
   if (M[j] is empty)
      M[j] = max($w_j$ + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
}
```

global array

Weighted Interval Scheduling: Run Time

- **Claim.** Memoized version of algorithm takes O(n log n) time.
  - Sort by finish time: O(n log n).
  - Computing p(·): O(n log n) via repeated binary search.
  - `M-Compute-Opt(j)`: each invocation takes O(1) time and either
    - (i) returns an existing value `M[j]`
    - (ii) fills in one new entry `M[j]` and makes two recursive calls
  - Case (ii) occurs at most n times ⇒ at most 2n recursive calls overall
  - Overall running time of `M-Compute-Opt(n)` is O(n). ▪
- **Remark.** O(n) if jobs are pre-sorted by start and finish times.

S. Raskhodnikova; based on slides by E. Demaine, C. Leiserson, A. Smith, K. Wayne

Equivalent algorithm: Bottom-Up

- Bottom-up dynamic programming. Unwind recursion.

```
Input: n, $s_1,…,s_n$ , $f_1,…,f_n$ , $v_1,…,v_n$

Sort jobs by finish times so that $f_1 \le f_2 \le ... \le f_n$.

Compute p(1), p(2), …, p(n)

Iterative-Compute-Opt {
   M[0] = 0
   for j = 1 to n
      M[j] = max($v_j$ + M[p(j)], M[j-1])
}

return M[n]
```

Total time = (sorting +computing p(j)) + $O(n) = O(n \log(n))$

Weighted Interval Scheduling: Finding a Solution

- Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?
- A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
   if (j = 0)
      output nothing
   else if ($v_j$ + M[p(j)] > M[j-1])
      print j
      Find-Solution(p(j))
   else
      Find-Solution(j-1)
}
```

  - # of recursive calls $\leq n \Rightarrow O(n)$.

Longest Common Subsequence

A.k.a. “sequence alignment”

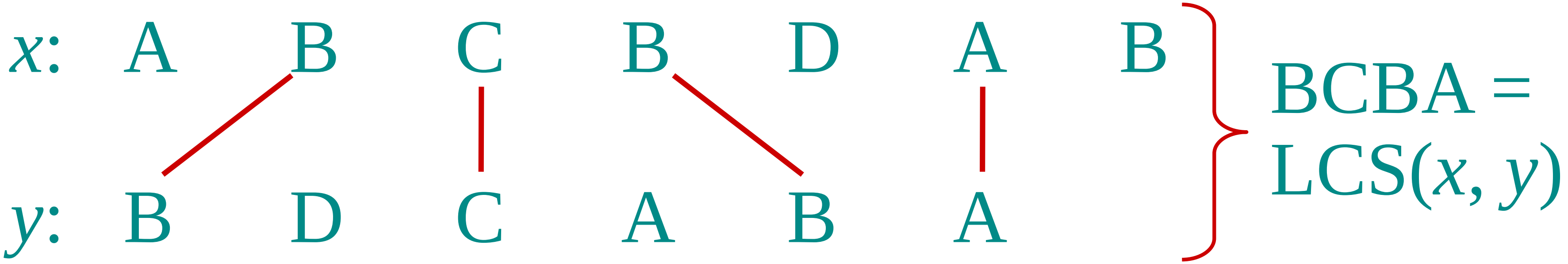
“edit distance”

…

Longest Common Subsequence (LCS)

- Given two sequences $x[1\,..\,m]$ and $y[1\,..\,n]$, find a longest subsequence common to them both.

“a” *not* “the”



x: A B C B D A B

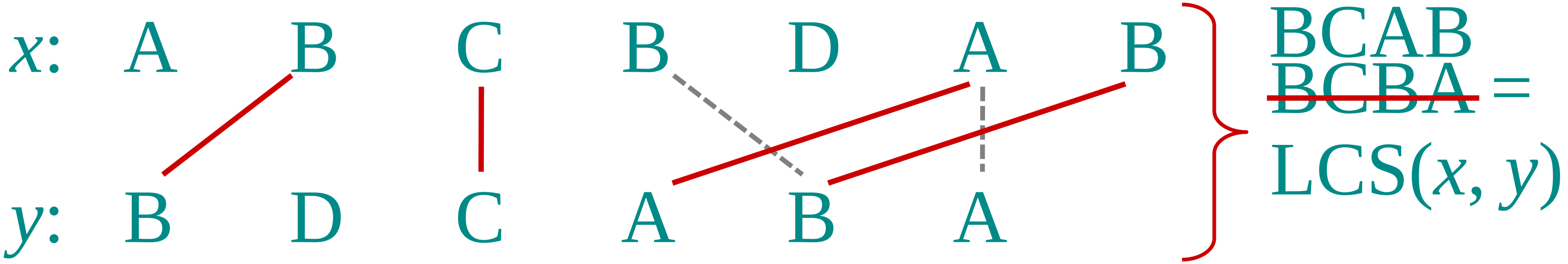
y: B D C A B A

BCBA = LCS(x, y)

Longest Common Subsequence (LCS)

- Given two sequences $x[1 \,.\,.\, m]$ and $y[1 \,.\,.\, n]$, find a longest subsequence common to them both.

"a" *not* "the"



x: A B C B D A B

y: B D C A B A

BCAB
~~BCBA~~ = LCS(x, y)

Motivation

- Approximate string matching [Levenshtein, 1966]
  - Search for “occurance”, get results for “occurrence”
- Computational biology [Needleman-Wunsch, 1970’s]
  - Simple measure of genome similarity

```
cgtacgtacgtacgtacgtacgtatcgtacgt
acgtacgtacgtacgtacgtacgtacgtacgt
```

Motivation

- Approximate string matching [Levenshtein, 1966]
  - Search for "occurance", get results for "occurrence"
- Computational biology [Needleman-Wunsch, 1970's]
  - Simple measure of genome similarity

```
 acgtacgtacgtacgtcgtacgtatcgtacgt
aacgtacgtacgtacgtcgtacgta cgtacgt
```

- n – length(LCS(x,y)) is called the "edit distance"

Brute-force LCS algorithm

Check every subsequence of $x[1 \,.\,.\, m]$ to see if it is also a subsequence of $y[1 \,.\,.\, n]$.

Analysis

- Checking $= O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst-case running time $= O(n2^m)$

$= \text{exponential time.}$

Dynamic programming algorithm

Simplification:

1. Look for the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

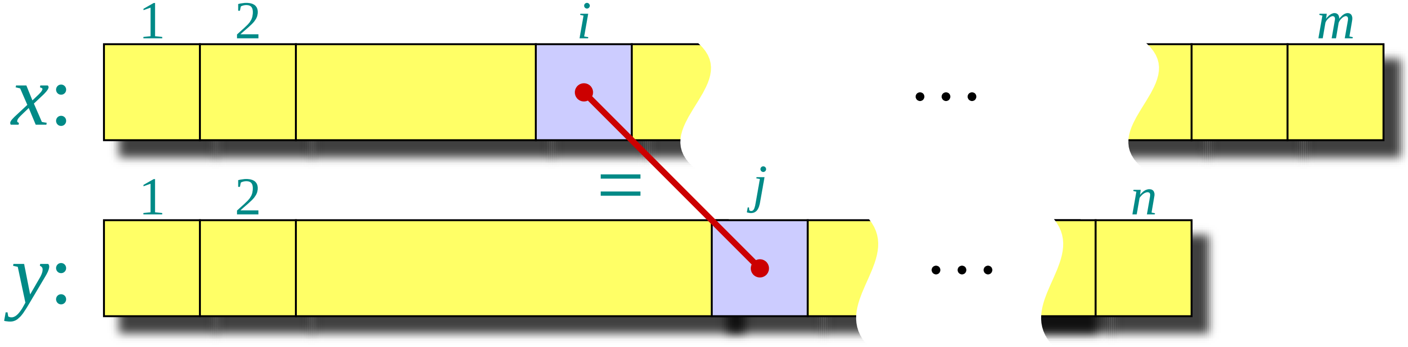
- Define $c[i, j] = |\mathrm{LCS}(x[1 \,.\,.\, i], y[1 \,.\,.\, j])|$.
- Then, $c[m, n] = |\mathrm{LCS}(x, y)|$.

Recursive formulation

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Base case: $c[i,j]=0$ if $i=0$ or $j=0$.

Case $x[i] = y[j]$:

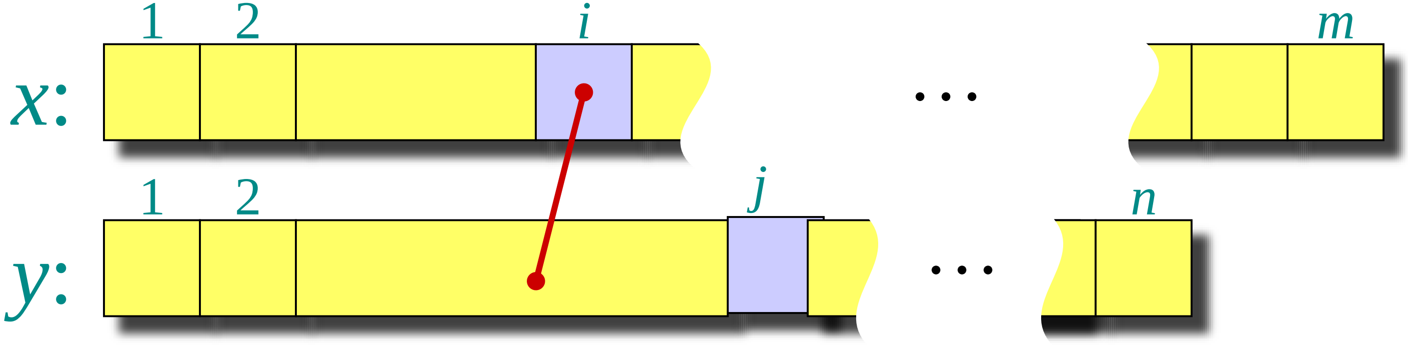


Without loss of generality, optimal solution matches x[i] to y[j]

Recursive formulation

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{c[i-1, j], c[i, j-1]\} & \text{otherwise.} \end{cases}$$

Case $x[i] \neq y[j]$: best matching might use x[i] or y[j] (or neither) but not both.



10/7/2016

Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

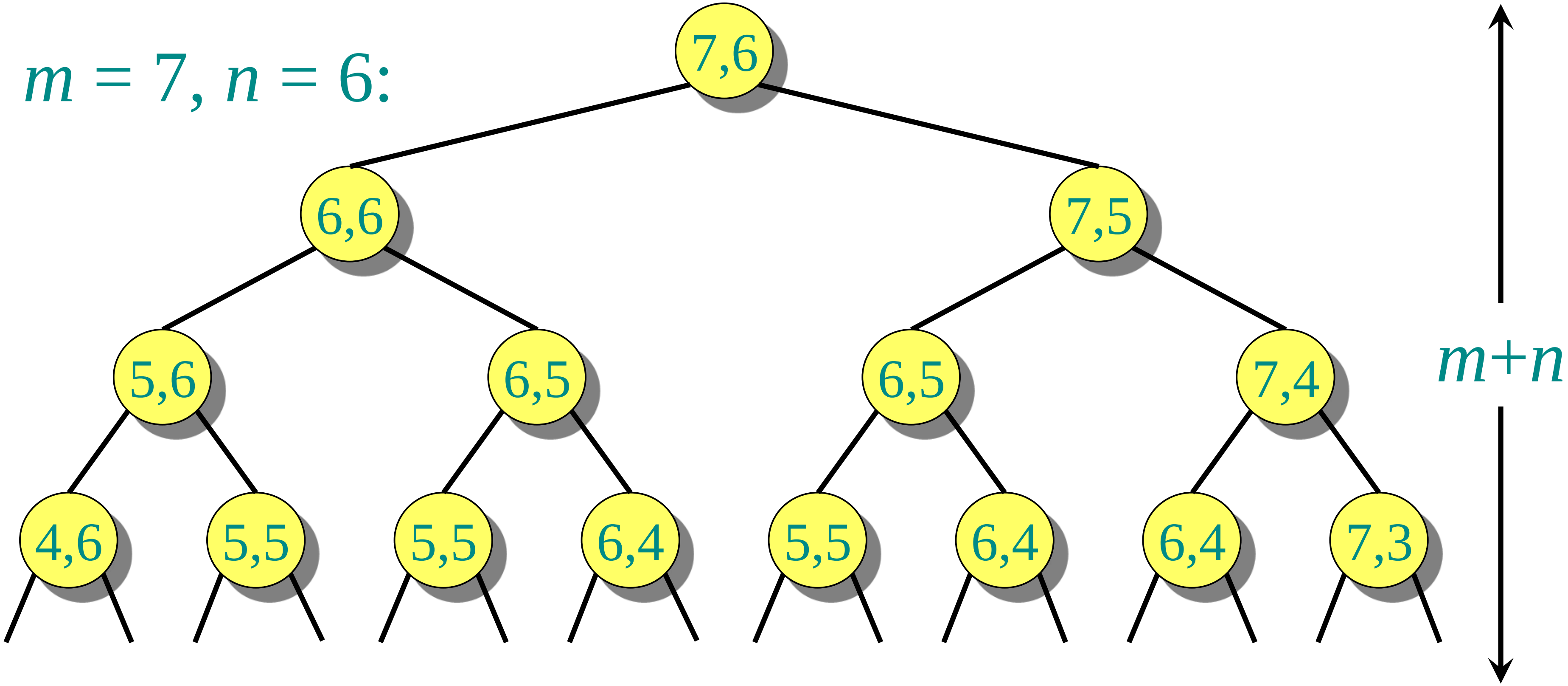
Recursive algorithm for LCS

```
LCS$(x, y, i, j)$    // base cases omitted
    if $x[i] = y[j]$
        then $c[i, j] \leftarrow$ LCS$(x, y, i-1, j-1) + 1$
        else $c[i, j] \leftarrow \max\{$LCS$(x, y, i-1, j)$,
                                  LCS$(x, y, i, j-1)\}$
    return $c[i, j]$
```

Worse case: $x[i] \neq y[j]$, in which case the algorithm evaluates two subproblems, each with only one parameter decremented.

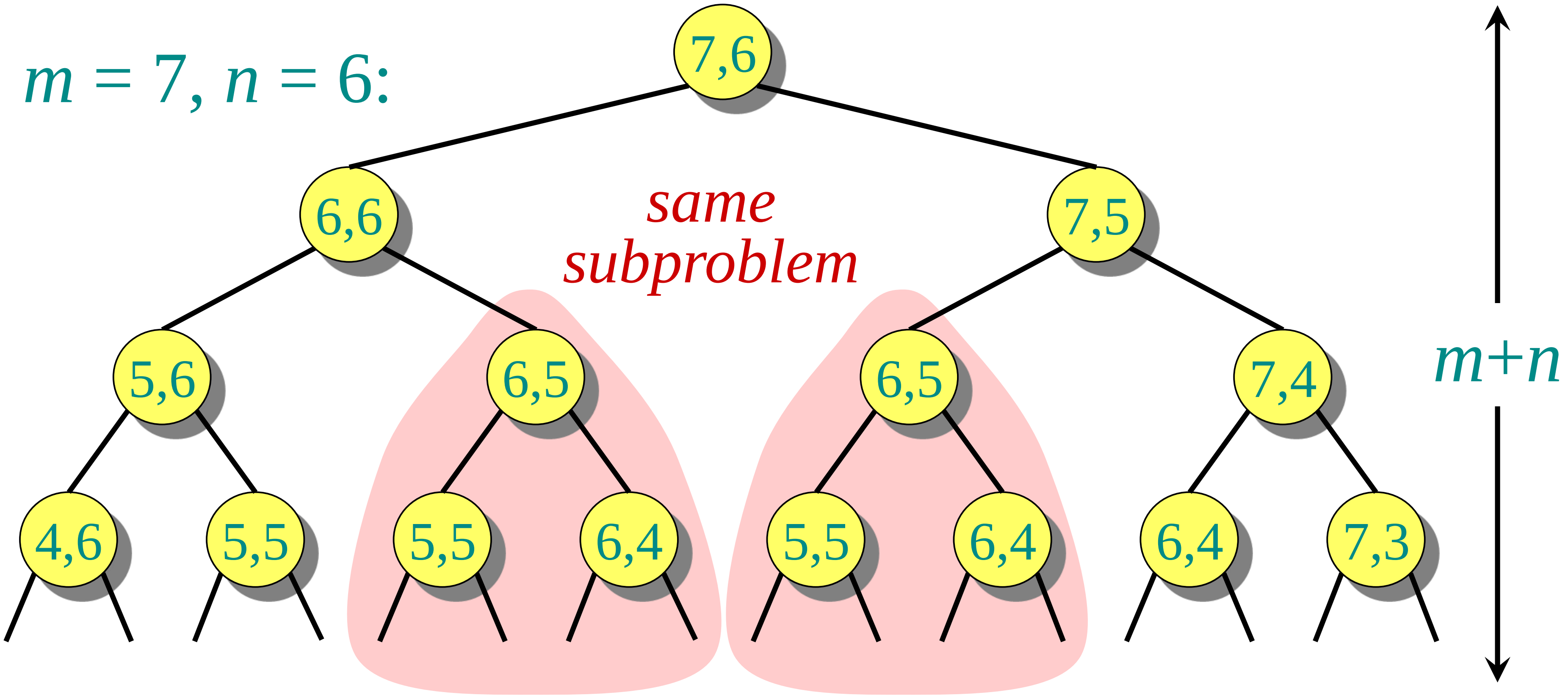
Recursion tree

$m = 7, n = 6$:



Height $= m + n \Rightarrow$ work potentially exponential.

Recursion tree



Height $= m + n \Rightarrow$ work potentially exponential, but we're solving subproblems already solved!

Dynamic-programming hallmark #2

Overlapping subproblems

A recursive solution contains a "small" number of distinct subproblems repeated many times.

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

```
LCS$(x, y, i, j)$
    if $c[i, j]$ = NIL
        then if $x[i] = y[j]$
            then $c[i, j] \leftarrow$ LCS$(x, y, i-1, j-1) + 1$
            else $c[i, j] \leftarrow \max\{$LCS$(x, y, i-1, j)$,
                                   LCS$(x, y, i, j-1)\}$
    return $c[i, j]$
```

same as before

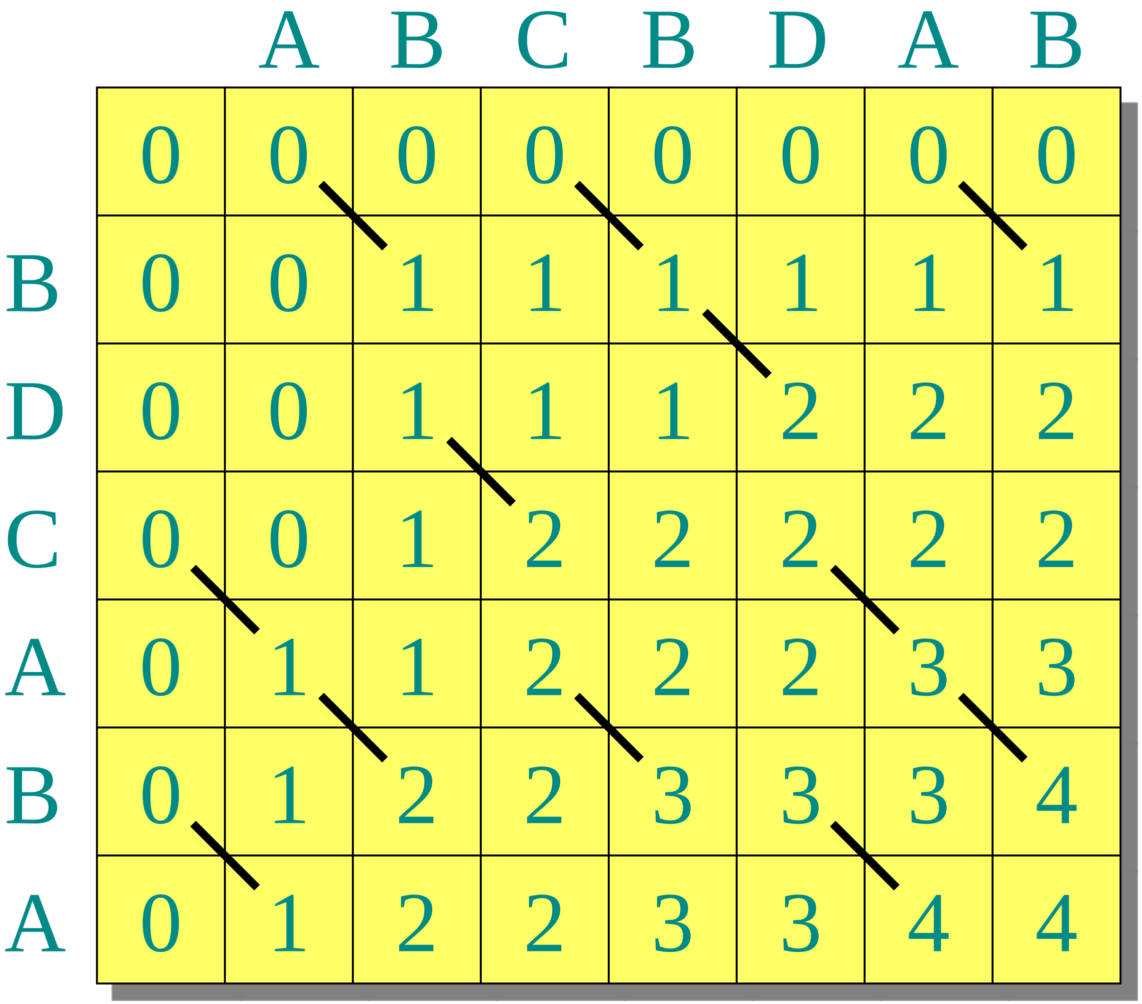
Time = $\Theta(mn)$ = constant work per table entry.

Space = $\Theta(mn)$.

Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.



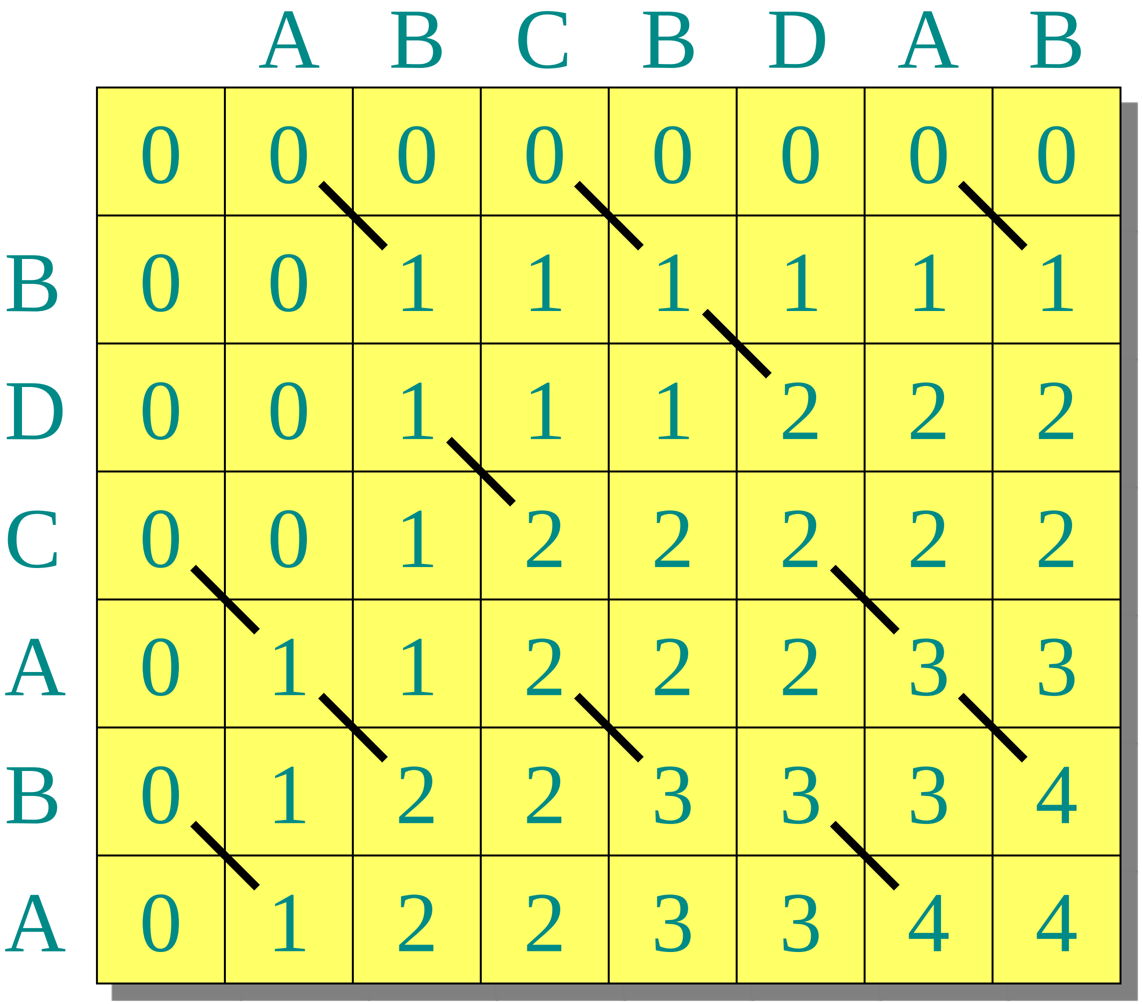
		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.



		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

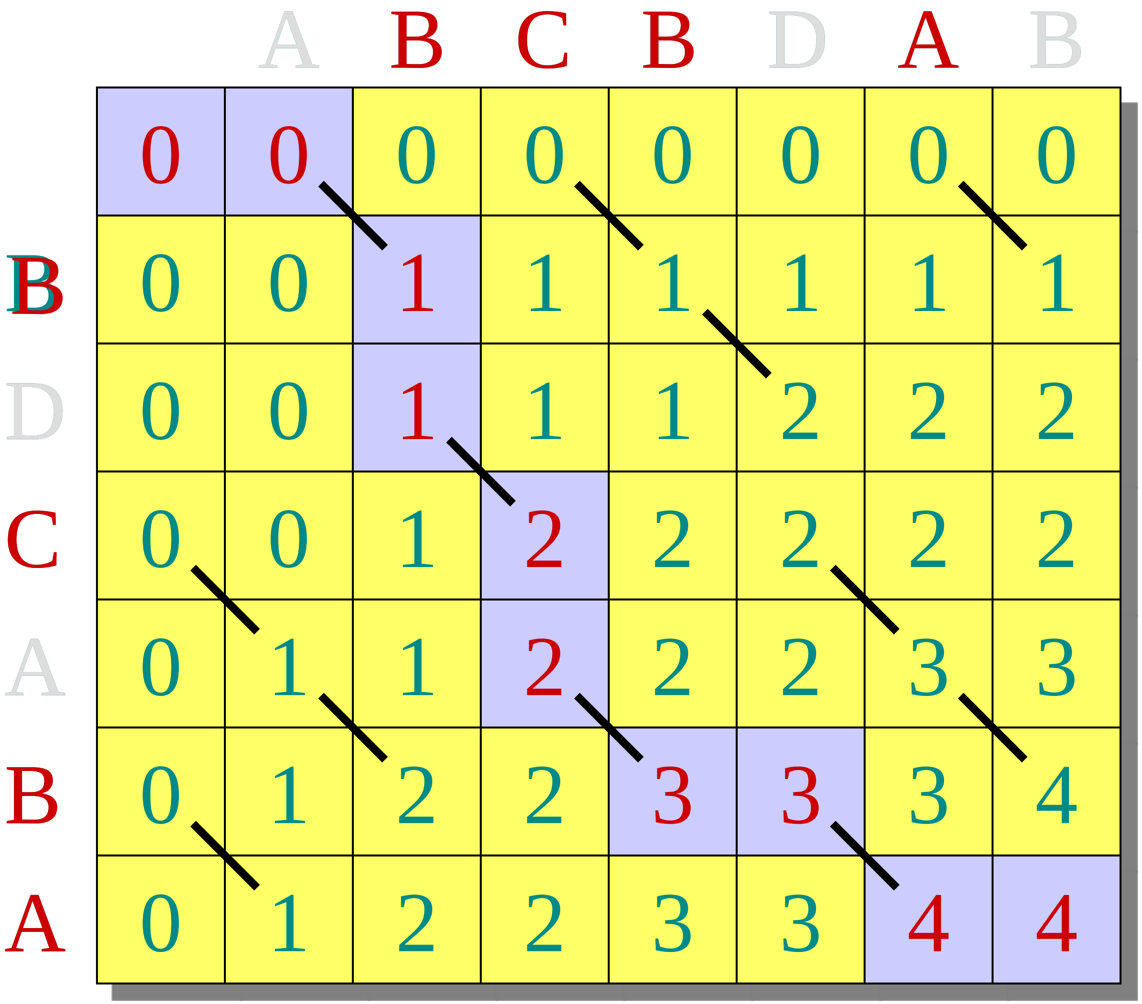
Dynamic-programming algorithm

IDEA:

Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.



		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Dynamic-programming algorithm

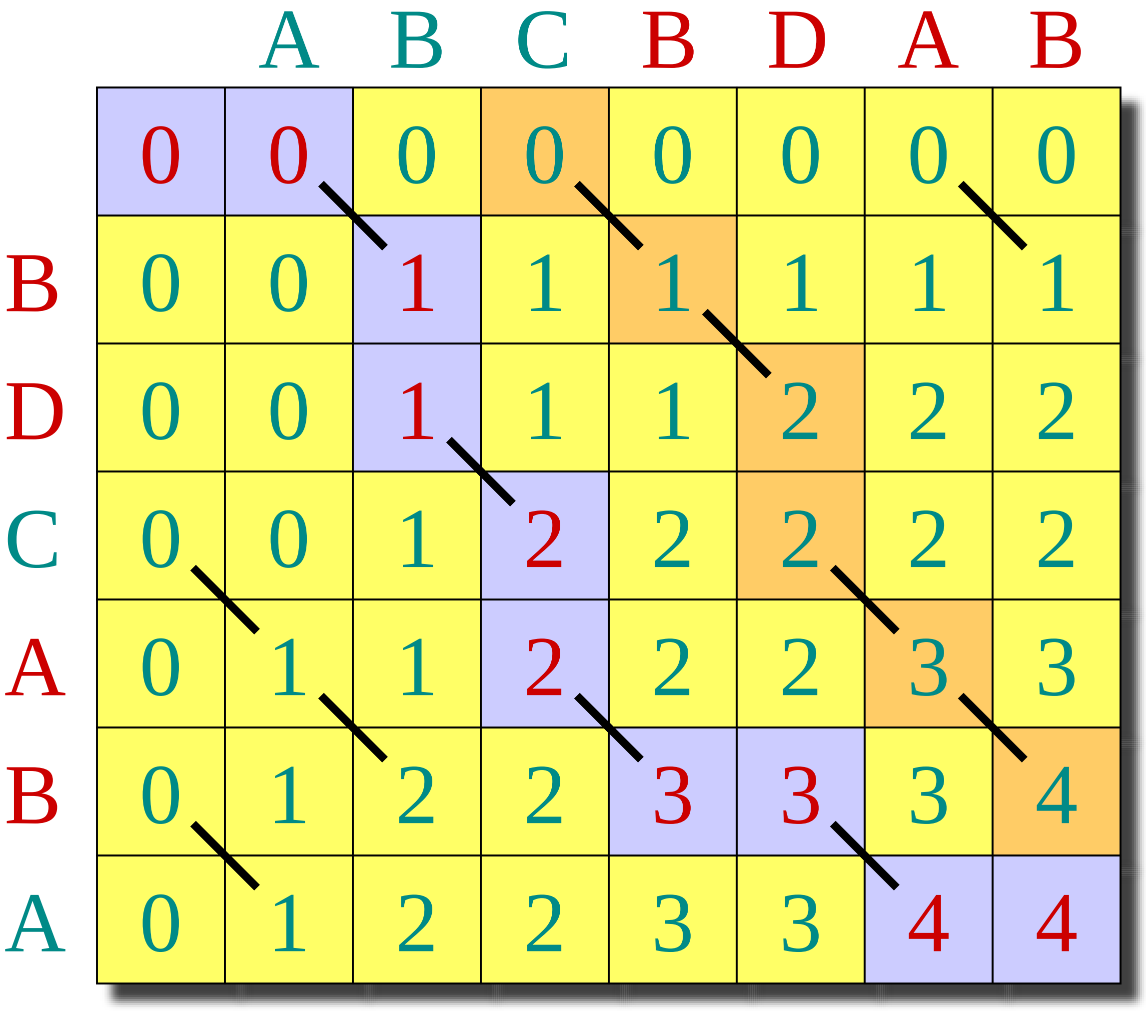
IDEA:

Compute the table bottom-up.

Time = Θ(mn).

Reconstruct LCS by tracing backwards.

Multiple solutions are possible.



		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Dynamic-programming algorithm

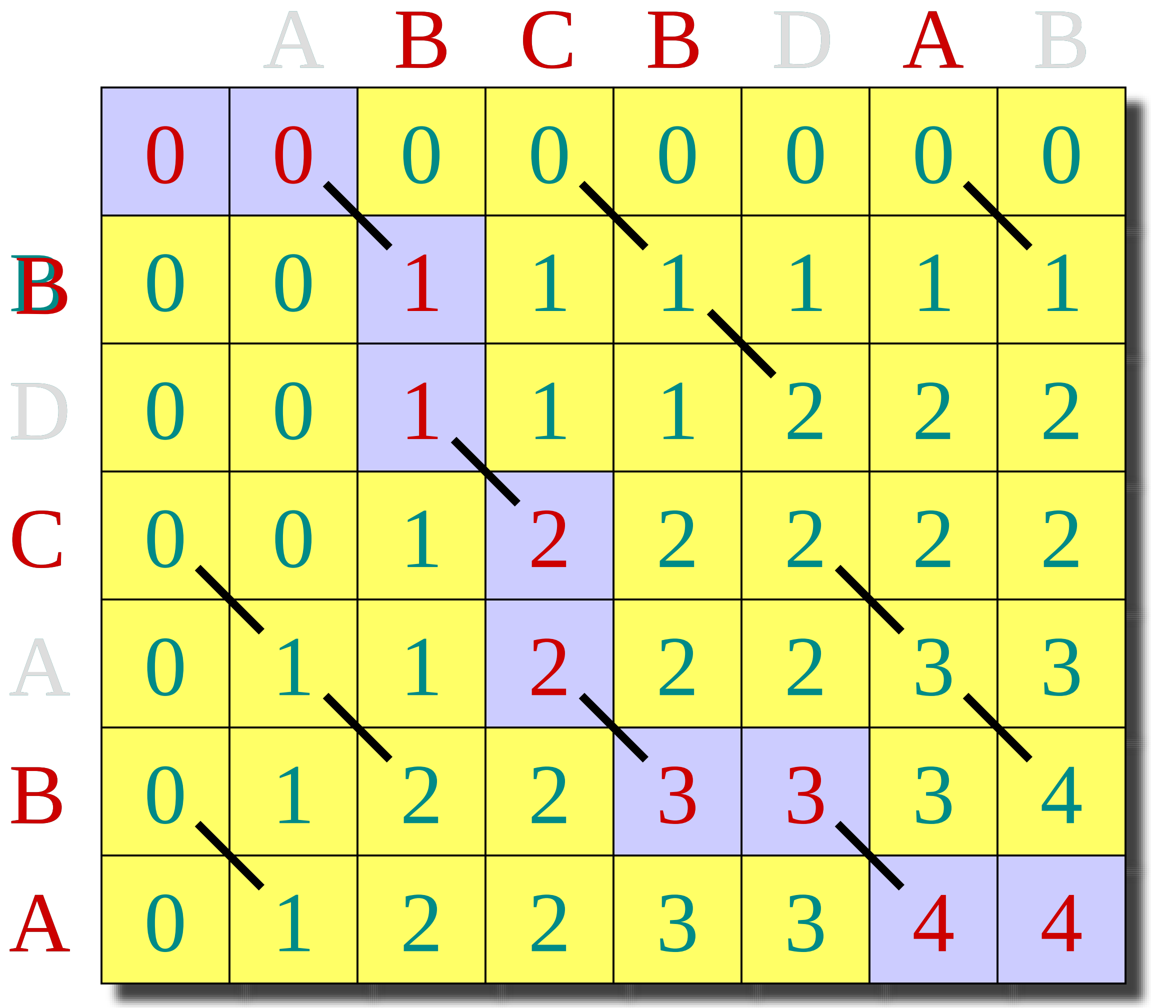
IDEA:
Compute the table bottom-up.

Time = $\Theta(mn)$.

Reconstruct LCS by tracing backwards.

Space = $\Theta(mn)$.

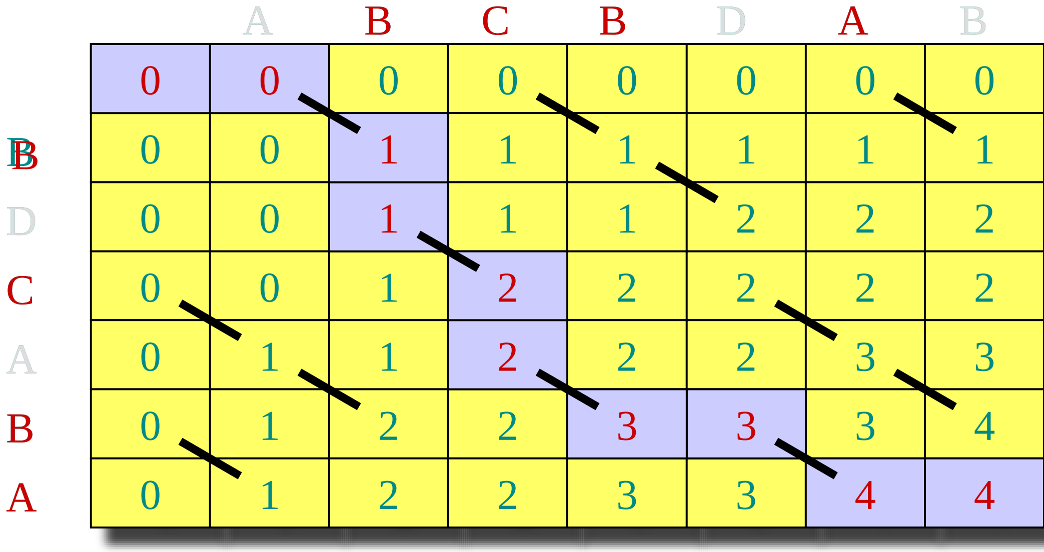
Section 6.7:
$O(\min\{m, n\})$



		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Saving space

- Why is space important?
  - Cost of storage
  - Cache misses
- To compute the **cost** of the optimal solution
  - Top to bottom
  - Remember only the previous row
  - O(m) space in addition to input



		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4