Algorithm Design and Analysis

CSE
565

LECTURE 4

Graphs

- Traversals
- DFS
- Acyclicity

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Depth-first search: alternate presentation

Traversals as generic templates

DFS and BFS are useful generic "templates" for graph algorithms

- Modifying BFS:
  - Bipartiteness (2-coloring)
  - Shortest paths (Dijsktra)
  - Minimum spanning trees (Prim)
- Modifying DFS
  - Finding cycles
  - Topological sort
  - Strongly connected components

DFS: setting up notation

- Maintain a global counter **time**
- Maintain for each vertex v
  - Two timestamps:
    - **v.d** = time first discovered
    - **v.f** = time when finished
  - "color": **v.color**
    - **white** = unexplored
    - **gray** = in process
    - **black** = finished
  - Parent **v.$\boldsymbol{\pi}$** in DFS tree

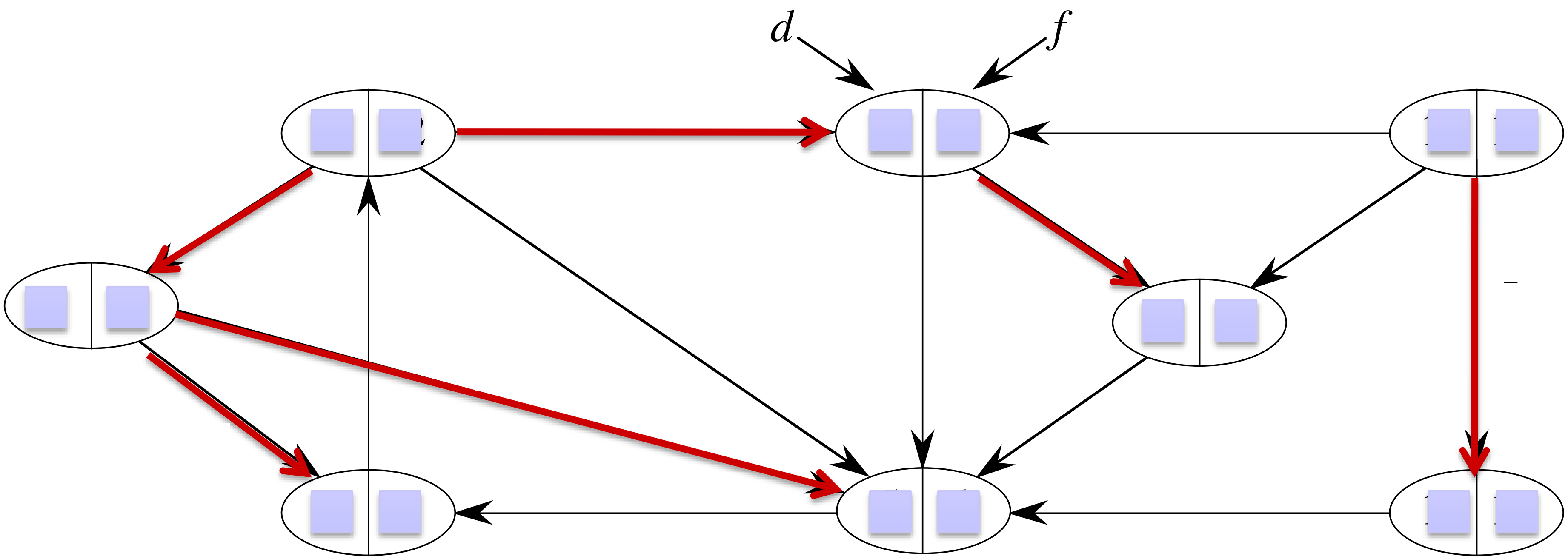
DFS pseudocode

```
DFS(G(V,E))
for each u ∈ V
     u.color ← WHITE
     u.π ← NIL
time ← 0
for each u ∈ V
     if u.color = WHITE
          DFS-Visit(G,u)
```

- **Note**: recursive function different from first call…

```
DFS-Visit(G,u)
time ← time + 1              // White vertex u is discovered
u.d ← time
u.color ← GRAY
for each v ∈ G.Adj[u]        // Explore edge (u,v)
     if v.color = WHITE
          v.π ← u
          DFS-Visit(G,v)
u.color ← BLACK              // Finish exploring u
time ← time + 1
u.f ← time
```

DFS example, animated



Parenthesis Theorem

- If we represent $v.d$ and $v.f$ as matching open and closed parentheses (or brackets), so that each vertex gets its own type (or color) of parentheses, then the history of discoveries and finishings is a properly nested expression of parentheses.
- Specifically, for all $u, v \in V$,

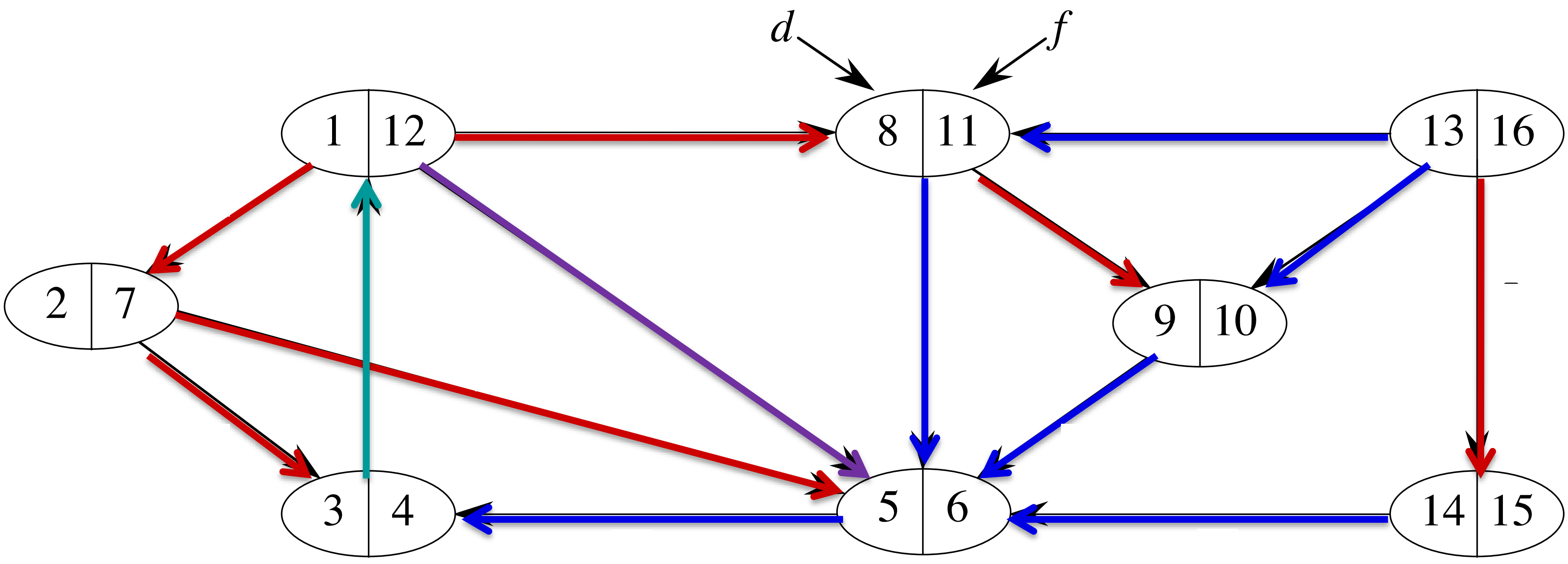
  the intervals $[u.d, u.f]$ and $[v.d, v.f]$ either
  - entirely disjoint (and u, v are not descendant/ancestor)
  - or one of the intervals is entirely contained in the other (the interval of the descendant is contained in the interval of the ancestor)

DFS Edge Types

- Tree
- Forward
- Backward
- Cross

- Classifying edges according to type gives info about graph structure

DFS example



- T = tree edge
- F = forward edge (to a *descendant* in DFS forest)
- B= back edge (to an *ancestor* in DFS forest)
- C = cross edge (goes to a vertex that is neither ancestor nor descendant)

Modifying DFS to classify edges

- We have enough information to classify edges as DFS explores them

When (u, v) is first explored:

- If v is WHITE, then (u, v) is a tree edge
- If v is GRAY, then (u, v) is a back edge
- If v is BLACK, then (u, v) is a forward or cross edge

  Exercise: Show that in this case,

  if $u.d < v.d$ then (u, v) is a forward edge;

  if $u.d > v.d$ then (u, v) is a cross edge.

Running time with adjacency lists

```
DFS(G(V,E))
for each u ∈ V
     u.color ← WHITE
     u.π ← NIL
time ← 0
for each u ∈ V
     if u.color = WHITE
          DFS-Visit(G,u)
```

- **Outer code** runs once, takes time $\boldsymbol{O(n)}$ (not counting time for recursive calls)
- **Recursive calls:**
  - Run once per vertex
  - time = $\boldsymbol{O}(degree(\boldsymbol{v}))$

```
DFS-Visit(G,u)
time ← time + 1                  // u is discovered
u.d ← time
u.color ← GRAY
for each v ∈ G.Adj[u]            // Explore edge (u,v)
     if v.color = WHITE
          v.π ← u
          DFS-Visit(G,v)
u.color ← BLACK                  // Finish exploring u
time ← time + 1
u.f ← time
```

- $\Sigma_v degree(v) = m$ or $2m$
- Total: $\boldsymbol{O(m+n)}$

Review Questions

- Suppose we run DFS on a directed graph G.
- True or false?
  1. G has a cycle if and only if there exists a back edge
  2. G has a cycle if and only if there exists any non-tree edges
  3. G has a cycle if and only if there exists a forward edge

Exercise: Write a proof of the true statements; give counterexamples for false ones.