

## 1. Examples

This document is an example of how to use L<sup>A</sup>T<sub>E</sub>X for writing homework solutions. Read the text, commented out by % signs, to get some explanations.

a) **This part includes a theorem with a proof and uses mathematical expressions.**

**Theorem 1.**

$$\sum_{i=1}^n i = \frac{n(n+1)}{2} \tag{1}$$

*Proof.* The proof is by induction.

*Base case:* Prove that the formula is true when  $n = 1$ . The LHS is  $\sum_{i=1}^1 i = 1$ , while the RHS is  $\frac{1(1+1)}{2} = 1$ . Hence, the base case holds.

*Induction step:* For each  $k \geq 1$ , assume that (1) is true for  $n = k$ . We show that it is true for  $n = k + 1$ .

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$

where the second equality follows from induction hypothesis that  $\sum_{i=1}^k i = \frac{k(k+1)}{2}$ . The formula (1) is true for  $n = k + 1$ , which proves the theorem.  $\square$

b) **This part has a figure that displays a picture from an external file.**



Figure 1: Comparing two sets of jobs

c) **This part has an example of writing algorithm pseudocode.**

Assume that there are  $n$  jobs and the  $i^{\text{th}}$  job has a start time  $s(i)$  and a finish time  $f(i)$ . These jobs are sorted with respect to their finish time. For simplicity, we assume that the sorted jobs are numbered  $1, 2, \dots, n$  such that  $f(1) \leq f(2) \leq \dots \leq f(n)$ .

A set of jobs is *compatible* with a job  $j$  if none of the jobs in the set overlaps with  $j$ . The algorithm maintains  $A$ , a set of selected jobs, which is initially empty. Our intuitive approach is to grow  $A$  by choosing a compatible job with the earliest finish time at each step.

Let  $i_1, \dots, i_k$  be the set of jobs in  $A$  in the order they were added to  $A$ . Similarly, let the set of jobs in  $B$ , which selects jobs in some method other than greedy approach, be denoted by  $j_1, \dots, j_\ell$ . One interesting consequence is that the greedy rule *stays ahead*:  $f(i_m) \leq f(j_m)$  for  $1 \leq m \leq \min(k, \ell)$ .

**Algorithm 1:** Earliest-Finish-Time( $L$ ).**input** : a list  $L$  of  $n$  jobs.**output**: a maximum set of mutually compatible jobs.

- 1 Sort jobs by finish times so that  $f(1) \leq f(2) \leq \dots \leq f(n)$ .
- 2 Maintain a set  $A$  which is initially empty.
- 3 **for**  $i = 1$  **to**  $n$  **do**
- 4     If the job  $i$  is compatible with  $A$ , then include  $i$  to  $A$ .
- end**
- 5 Output  $A$ .

**Claim 2.** For all indices  $m \leq \min(k, \ell)$ ,  $f(i_m) \leq f(j_m)$ .

*Proof.* We prove by induction on the index  $m$ . For  $m = 1$ , the statement is true because the greedy approach selects the job with the earliest finish time. For  $m > 1$ , we will assume the statement is true for  $m = t - 1$  and prove it for  $m = t$ . The  $t^{\text{th}}$  job in  $B$  must start after  $f(j_{t-1})$  since this job is compatible with  $B$ . It means  $f(j_{t-1}) \leq s(j_t)$ . By combining the induction hypothesis  $f(i_{t-1}) \leq f_j(t - 1)$ , it also means  $f(i_{t-1}) \leq s(j_t)$ . So this job is compatible with  $A$  too. As the greedy algorithm selects a job with earliest finish time,  $f(i_t)$  is not larger than  $f(j_t)$ . This completes the induction step; therefore, the statement is true.  $\square$

**Proof of Correctness** Assume for contradiction that the greedy approach returns a non-optimal solution  $A$  while an optimal set  $\mathcal{O}$  has more jobs. Assume that  $|A| = k$  and  $|\mathcal{O}| = \ell$  with  $\ell > k$ . By Claim 2, we have  $f(i_k) \leq f(j_k)$ . Let us focus on the  $(k + 1)^{\text{th}}$  job  $x$  in  $\mathcal{O}$ . The job  $x$  starts after the job  $j_k$  ends and hence after the job  $i_k$  ends. But the greedy algorithm stops with  $i_k$  while  $x$  is compatible with  $A$  – a contradiction.

**Implementation** Once the input jobs are sorted, an array is enough for the set  $A$ . When a new job is checked for compatibility with  $A$ , it is enough to compare its start time with the last added job  $x$ 's finish time rather than all the jobs' finish times in  $A$  – the resource becomes free after  $f(x)$  and the input jobs are sorted.

**Time and Space Complexity** It takes  $\Theta(n \log n)$  time to sort the input jobs of size  $n$ . Creating an array of size  $n$  takes  $O(n)$  time. For each job, it takes  $O(1)$  time to check whether a job is compatible with the set  $A$ , and the array can be updated in constant time if we maintain an end-of-the-array pointer. These operations must be repeated for each job, so the For loop takes  $O(n)$  time. Hence, the total running time is  $O(n \log n)$ .

It takes  $O(n)$  space to store the input. An in-place sorting takes  $O(n)$  space. Finally, the set  $A$  can be implemented by an array of size  $n$ . Thus, the space complexity is  $O(n)$ .