Last time
• Space complexity
• Savitch’s theorem

Today
• The class PSPACE
• TQBF is PSPACE-complete
• Hierarchy theorems
Let $f(n)$ be a function, where $f(n) \geq n$. If a language $A$ is decided by a NTM that runs in space $f(n)$ then $A$ is decided by a TM that runs in space $f^2(n)$.

A. True  
B. False  
C. Unknown  
D. None of the above
The class PSPACE

PSPACE is the class of languages decidable in polynomial space on a deterministic TM:

\[ PSPACE = \bigcup_{k} SPACE(n^k). \]

- NPSPACE – the same, but for NTMs.
- By Savitch’s Theorem,

\[ PSPACE = NPSPACE \]
1. \( P \subseteq NP \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \)

Recall: a TM that runs in space \( f(n) \) has \( \leq f(n)2^{O(f(n))} \) configurations

2. \( P \neq \text{EXPTIME} \)

Which containments in (1) are proper? Unknown!
I-clicker question (frequency: AC)

Relationships between the following classes are an open question

A. P vs NP
B. P vs PSPACE
C. NP vs EXPTIME
D. All of the above
E. All of the above and P vs EXPTIME
A language B is **PSPACE-complete** if

1. $B \in \text{PSPACE}$
2. B is **PSPACE-hard**, i.e.,
   every language in PSPACE is poly-time reducible to B.
The TQBF problem

- **Boolean variables**: variables that can take on values T/F (or 1/0)
- **Boolean operations**: $\lor$, $\land$, and $\neg$
- **Boolean formula**: expression with Boolean variables and ops
- **Quantified Boolean formula**: Boolean formula with quantifiers ($\forall$, $\exists$)
- **Fully Quantified Boolean formula**: all variables have quantifiers ($\forall$, $\exists$)

We only consider the form where all quantifiers appear in the beginning.

**Ex.**

\[
\forall x \exists y [(x \lor y) \land (\bar{x} \lor \bar{y})] \quad \text{True}
\]

\[
\exists y \forall x [(x \lor y) \land (\bar{x} \lor \bar{y})] \quad \text{False}
\]

- Each fully quantified Boolean formula is either true or false.
- The order of quantifiers matters!

\[\text{TQBF} = \{ \langle \phi \rangle \mid \phi \text{ is a true fully quantified Boolean formula} \}\]
TQBF is PSPACE-complete

1. TQBF is in PSPACE
2. TQBF is PSPACE-hard
Prove: TQBF ∈ PSPACE

\[ T = \text{`` On input } \langle \phi \rangle, \]

where \( \phi \) is a fully quantified Boolean formula:

1. If \( \phi \) has no quantifiers, it has only constants (and no variables). Evaluate \( \phi \).
   If true, accept; o.w., reject.
2. If \( \phi \) is of the form \( \exists x \psi \), recursively call \( T \)
on \( \psi \) with \( x = 0 \) and then on \( \psi \) with \( x = 1 \).
   If either call accepts, accept; o.w., reject.
3. If \( \phi \) is of the form \( \forall x \psi \), recursively call \( T \)
on \( \psi \) with \( x = 0 \) and then on \( \psi \) with \( x = 1 \).
   If both calls accepts, accept; o.w., reject.”

• If \( n \) is the input length, \( T \) uses space \( O(n) \).
If TQBF is in P then it implies that

A. P = NP
B. P = PSPACE
C. P = EXPTIME
D. (A) and (B) are true
E. (A), (B), (C) are true
Motivating question

Is there a decidable language which is provably not in P?
A function $f: \mathbb{N} \to \mathbb{N}$ is space constructible if the function that maps the string $1^n$ to the binary representation of $f(n)$ is computable in space $O(f(n))$.

- Fractional functions are rounded down.
- **Examples:** $\log_2 n, \sqrt{n}, n \log_2 n, n^2, 2^n$, all commonly occurring functions that are $\Omega(\log n)$.
Hierarchy Theorems

Corollary 1: For all $\alpha, \beta \geq 0$, where $\alpha < \beta$,
SPACE($n^\alpha$) is a strict subset of SPACE($n^\beta$).

Corollary 2: PSPACE is a strict subset of EXPSPACE, where
EXPSPACE=$\bigcup_k$ SPACE($2^{n^k}$)

Space Hierarchy Theorem
For any space constructible function $f$, a language exists that is decidable in $O(f(n))$ space, but not in $o(f(n))$ space.
A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is **time constructible** if the function that maps the string $1^n$ to the binary representation of $f(n)$ is computable in time $O(f(n))$.

- Fractional functions are rounded down.
- **Examples:** $n \log_2 n$, $n^2$, $2^n$, all commonly occurring functions that are $\Omega(n \log n)$.
Corollary 1: For all $\alpha, \beta \geq 0$, where $\alpha < \beta$, $\text{TIME}(n^\alpha)$ is a strict subset of $\text{TIME}(n^\beta)$.

Corollary 2: $\mathbf{P}$ is a strict subset of $\mathbf{EXPTIME}$.
A language $B$ is **EXPSPACE-complete** if

1. $B \in \text{EXPSPACE}$
2. $B$ is **EXPSPACE-hard**, i.e., every language in EXPSPACE is poly-time reducible to $B$. 

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**Diagram:**

- **P** (a set) is a subset of **EXPSPACE** (another set).
- An arrow points from each element in **P** to $B$, illustrating the reducibility.
The $EQ_{REX}^\uparrow$ problem

- **Regular operations**: $\cap$, $\circ$, $*$
- **Regular expressions**: use regular operations, $\emptyset$, $\varepsilon$, $\Sigma$, members of $\Sigma$
- We can test equivalence of two regular expressions in poly space (problem in the book).
- **Exponentiation operation**: $R^k = R \uparrow k = R \circ \cdots \circ R$ ($k$ times)

$EQ_{REX}^\uparrow = \{ \langle Q, R \rangle \mid Q$ and $R$ are equivalent regular expressions with exponentiation$\}$

$EQ_{REX}^\uparrow$ is EXPSPACE-complete.
recognizable

decidable

EXPSPACE

EXPTIME

PSPACE = NPSPACE

NP

coNP

P

CFL

regular

\( E Q_{REX} \uparrow \)