Intro to Theory of Computation

LECTURE 29

CS 464

Last time

- Space complexity
- Savitch's theorem

Today

- The class PSPACE
- TQBF is PSPACE-complete
- Hierarchy theorems

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I-clicker question (frequency: AC)

Let $f(n)$ be a function, where $f(n) \geq n$.

If a language A is decided by a NTM that runs in space $f(n)$ then

A is decided by a TM that runs in space $f^2(n)$.

A. True

B. False

C. Unknown

D. None of the above

The class PSPACE

PSPACE is the class of languages decidable in polynomial space on a ***deterministic*** TM:

$$\boldsymbol{PSPACE} = \bigcup_k SPACE(n^k).$$

- NPSPACE – the same, but for NTMs.
- By Savitch's Theorem,

$$\boldsymbol{PSPACE = NPSPACE}$$

Relationships between classes

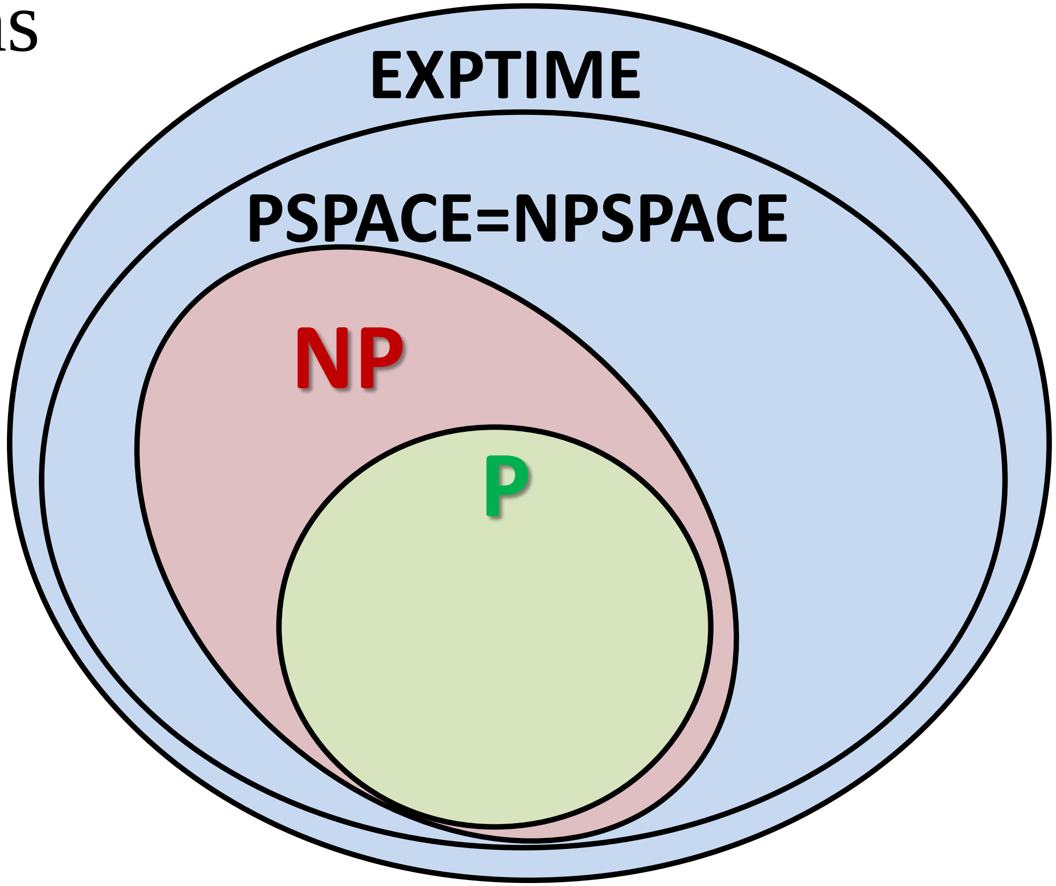
1. P ⊆ NP ⊆ PSPACE ⊆ EXPTIME

   Recall: a TM that runs in space $f(n)$ has $\leq f(n)2^{O(f(n))}$ configurations

2. P≠ EXPTIME

Which containments in **(1)** are proper?

Unknown!



I-clicker question (frequency: AC)

Relationships between the following classes are an open question

A. **P vs NP**

B. **P vs PSPACE**

C. **NP vs EXPTIME**

D. **All of the above**

E. **All of the above and P vs EXPTIME**

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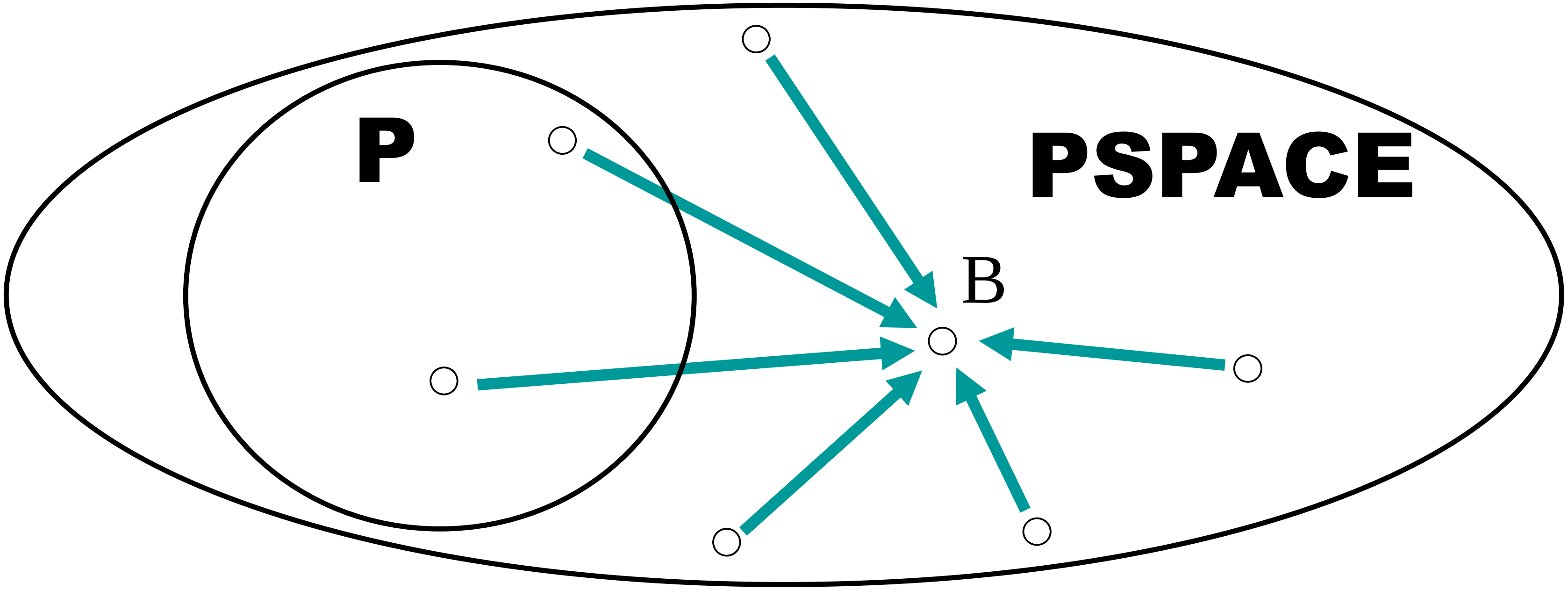
Hardest problems in PSPACE

A language B is **PSPACE-complete** if

1. B∈ **PSPACE**
2. B is **PSPACE-hard**,i.e.,
   every language in PSPACE is poly-time reducible to B.



The TQBF problem

- **Boolean variables:** variables that can take on values T/F (or 1/0)
- **Boolean operations:** $\vee$, $\wedge$, and $\neg$
- **Boolean formula:** expression with Boolean variables and ops
- **Quantified Boolean formula:** Boolean formula with quantifiers ($\forall, \exists$)
- **Fully Quantified Boolean formula:** all variables have quantifiers ($\forall, \exists$)

We only consider the form where all quantifiers appear in the beginning.

Ex. $\forall x \exists y [(x \vee y) \wedge (\bar{x} \vee \bar{y})]$ **True**

$\exists y \forall x \, [(x \vee y) \wedge (\bar{x} \vee \bar{y})]$ **False**

- Each fully quantified Boolean formula is either true or false.
- The order of quantifiers matters!

TQBF = {$\langle \phi \rangle$ | ϕ is a true fully quantified Boolean formula}

TQBF is PSPACE-complete

1. **TQBF is in PSPACE**
2. **TQBF is PSPACE-hard**

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Prove: TQBF $\in$ PSPACE

T = `` On input $\langle \phi \rangle$,
where ϕ is a fully quantified Boolean formula:

1. If ϕ has no quantifiers, it has only constants (and no variables). Evaluate ϕ.
   If true, accept; o.w., reject.
2. If ϕ is of the form $\exists x\, \psi$, recursively call T on ψ with $x = 0$ and then on ψ with $x = 1$.
   If either call accepts, accept; o.w., reject.
3. If ϕ is of the form $\forall x\, \psi$, recursively call T on ψ with $x = 0$ and then on ψ with $x = 1$.
   If both calls accepts, accept; o.w., reject.”

- If n is the input length, T uses space $O(n)$.

I-clicker question (frequency: AC)

If TQBF is in P then it implies that

A. **P = NP**

B. **P = PSPACE**

C. **P = EXPTIME**

D. **(A) and (B) are true**

E. **(A), (B), (C) are true**

Motivating question

Is there a decidable language which is provably not in P?

Hierarchy Theorems

A function $f\colon \mathbb{N} \rightarrow \mathbb{N}$ is **space constructible** if the function that maps the string 1^n to the binary representation of $f(n)$ is computable in space $O\big(f(n)\big)$.

- Fractional functions are rounded down.
- Examples: $\log_2 n$, $\sqrt{n}$, $n \log_2 n$, n^2, 2^n,

all commonly occurring functions that are $\Omega(\log n)$

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Hierarchy Theorems

Space Hierarchy Theorem

For any space constructible function f,
a language exists that is decidable in $O\big(f(n)\big)$ space,
but not in $o(f(n))$ space.

- **Corollary 1:** For all $\alpha, \beta \geq 0$, where $\alpha < \beta$,
  SPACE(n^{α}) is a strict subset of SPACE(n^{β}).
- **Corollary 2:** PSPACE is a strict subset of EXPSPACE, where
  EXPSPACE=$\cup_k SPACE(2^{n^k})$

Hierarchy Theorems

A function $f\colon \mathbb{N} \to \mathbb{N}$ is **time constructible** if the function that maps the string 1^n to the binary representation of $f(n)$ is computable in time $O\big(f(n)\big)$.

- Fractional functions are rounded down.
- Examples: $n \log_2 n$, n^2, 2^n,

all commonly occurring functions that are $\Omega(n \log n)$

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Hierarchy Theorems

Time Hierarchy Theorem

For any time constructible function f,

a language exists that is decidable in $O\big(f(n)\big)$ time,

but not in $o\left(\frac{f(n)}{\log f(t)}\right)$ time.

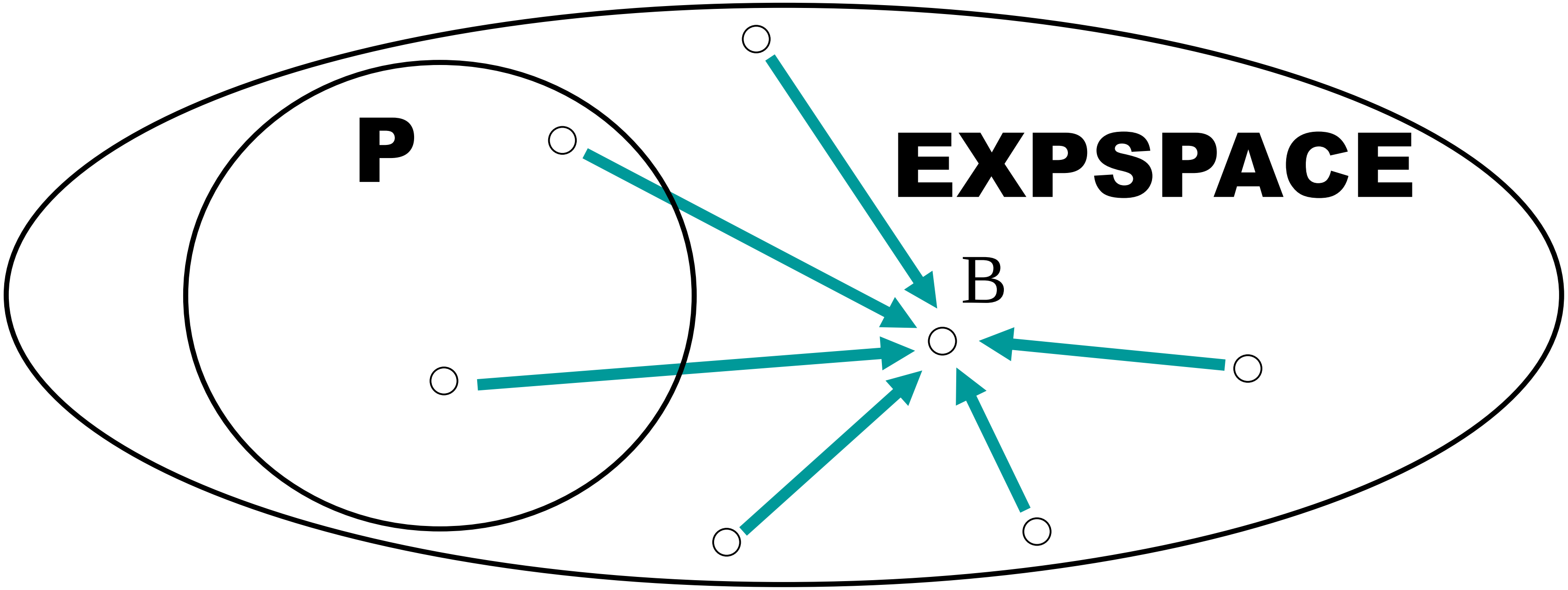
- **Corollary 1:** For all $\alpha, \beta \geq 0$, where $\alpha < \beta$,

  TIME(n^{α}) is a strict subset of TIME(n^{β}).
- **Corollary 2:** P is a strict subset of EXPTIME

Hardest problems in EXPSPACE

A language B is **EXPSPACE-complete** if

1. B∈ **EXPSPACE**
2. B is **EXPSPACE-hard**,i.e., every language in EXPSPACE is poly-time reducible to B.



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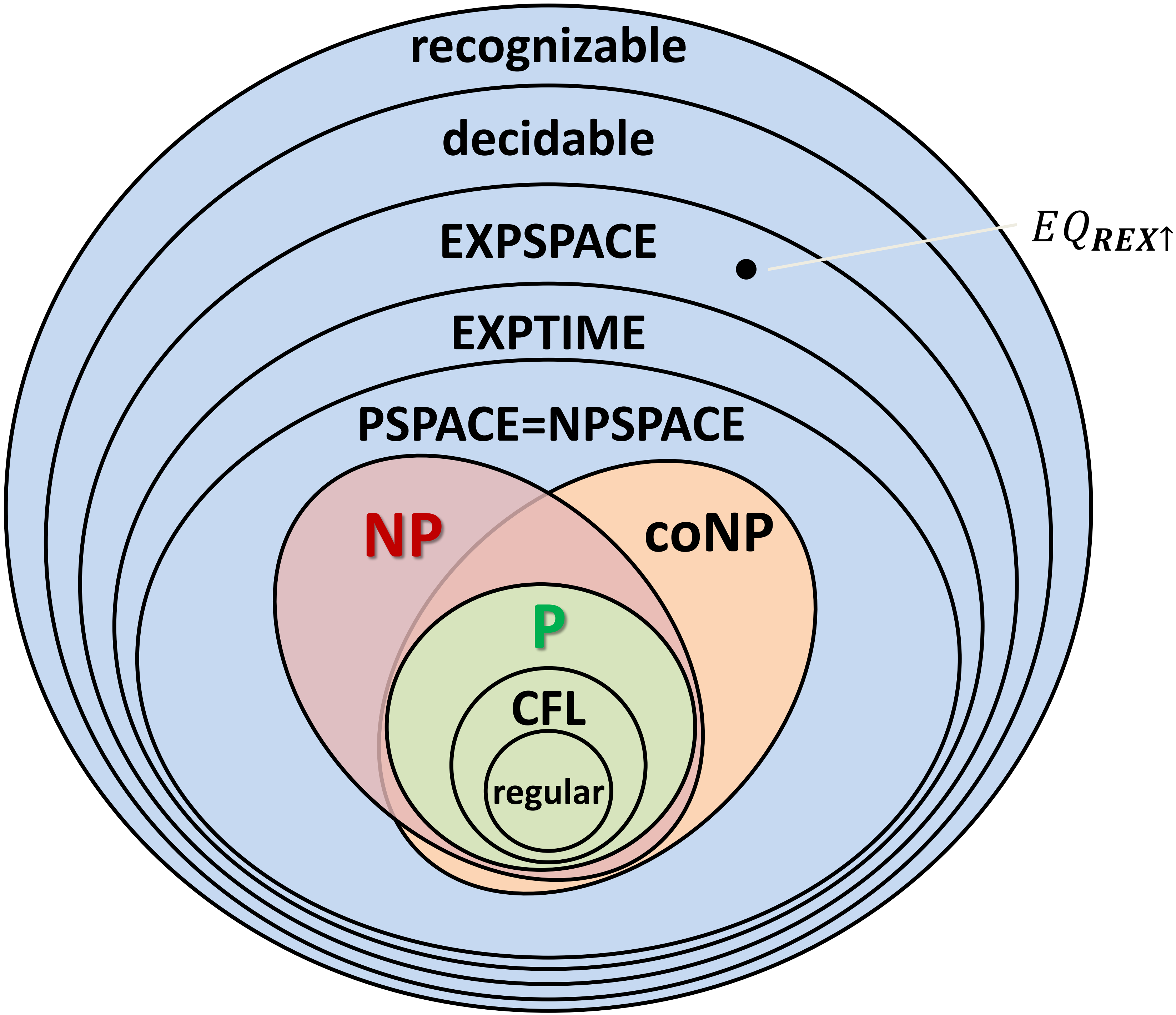
The $EQ_{\boldsymbol{REX}\uparrow}$ problem

- **Regular operations:** $\cap$, $\circ$, $*$
- **Regular expressions: use regular operations, $\emptyset, \boldsymbol{\varepsilon}, \boldsymbol{\Sigma}$, members of $\boldsymbol{\Sigma}$**
- **We can test equivalence of two regular expressions in poly space (problem in the book).**
- **Exponentiation operation:** $R^k = R \uparrow k = R \circ \cdots \circ R$ (k times)

$EQ_{\boldsymbol{REX}\uparrow}$ **= {$\langle \boldsymbol{Q}, \boldsymbol{R} \rangle$ | $\boldsymbol{Q}$ and R are equivalent regular expressions with exponentiation}**

$EQ_{\boldsymbol{REX}\uparrow}$ **is EXPSPACE-complete.**


recognizable
decidable
EXPSPACE
$EQ_{\mathbf{REX}\uparrow}$
EXPTIME
PSPACE=NPSPACE
NP
coNP
P
CFL
regular