Intro to Theory of Computation

LECTURE 18

CS
464

Last time

- A_{TM} is unrecognizable
- Reductions

Today

- Reductions
- Mapping reductions

Homework 7 due
Homework 8 out

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Problems in language theory

A_{DFA} decidable	A_{CFG} decidable	A_{TM} undecidable
E_{DFA} decidable	E_{CFG} decidable	E_{TM} undecidable
EQ_{DFA} decidable	EQ_{CFG} ?	EQ_{TM} ?

Using reductions to prove undecidability

We want to prove that language L is undecidable.

Idea: Use a proof by contradiction.

1. Suppose to the contrary that L is decidable.
2. Use a decider for L as a subroutine to construct a decider for A_{TM}.
3. But A_{TM} is undecidable. Contradiction!

I-clicker question (frequency: AC)

To prove that E_{TM} is undecidable

A. we assumed E_{TM} had a decider and used it to construct a decider for A_{TM}

B. we assumed A_{TM} had a decider and used it to construct a decider for E_{TM}

C. we constructed a TM S that on input <M,w> decides whether M accepts w, assuming the existence of a TM R that decides on input <M'> whether the language of <M'> is empty

D. There is more than one correct answer.

E. None of the above.

Prove that CFL_{TM} is undecidable

$\text{CFL}_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is context-free}\}$

Proof: Suppose to the contrary that CFL_{TM} is decidable, and let R be a TM that decides it.

We construct TM S that decides A_{TM}.

S = `` On input $\langle M, w\rangle$, where M is a TM and *w* is a string:

1. Construct TM M'.

   M' = `` On input *x*,
   1. If x is not of the form $0^n1^n2^n$, reject.
   2. Run TM *M* on input w.
   3. If it accepts, accept."

2. Run TM *R* on input <M'>.
3. If $\boxed{\text{it rejects}}$, accept. O.w. reject."

Prove that EQ_{TM} is undecidable

$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1, M_2$ are TMs, $L(M_1) = L(M_2)\}$

Proof: Suppose to the contrary that EQ_{TM} is decidable, and let R be a TM that decides it.

We construct TM S that decides A_{TM}.

S = `` On input $\langle M, w \rangle$, where M is a TM and w is a string:

1. Construct TMs M', M''.

   M' = `` On input *x*,
   1. Ignore the input.
   2. Run TM M on input w.
   3. If it accepts, accept.”

   M'' = ``Accept.”

2. Run TM *R* on input <M', M''>.
3. If it accepts, accept. O.w. reject.”

Proof 2 that EQ_{TM} is undecidable

$EQ_{TM} = \{\langle M_1, M_2 \rangle \mid M_1, M_2$ are TMs, $L(M_1) = L(M_2)\}$

Proof: Suppose to the contrary that EQ_{TM} is decidable, and let R be a TM that decides it.

We construct TM S that decides E_{TM}. **What do we change?**

S = `` On input $\langle \boxed{M} \rangle$, where M is a TM ~~and w is a string~~:

1. Construct $\boxed{\text{TM } M'}$.

   M' = ``Reject.”

   M' = ``Accept.”
   1. Ignore the input.
   2. Run TM M on input w.
   3. If it accepts, accept.”

   M'' = ``Accept.”

2. Run TM R on input $\boxed{< M, M' >}$
3. If $\boxed{\text{it accepts,}}$ accept. O.w. reject.”

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Problems in language theory

A_{DFA} decidable	A_{CFG} decidable	A_{TM} undecidable
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Mapping Reductions

Computable functions

A function $f: \Sigma^* \rightarrow \Sigma^*$ is computable if some TM M, on every input w, halts with only $f(w)$ on its tape.

Example 1: $f(\langle x, y \rangle) = x + y$.

Example 2: $f(\langle M, w \rangle) = \langle M' \rangle$, where M is a TM and w is a string, and M' is a TM that ignore its input and runs M on w.

Mapping reductions

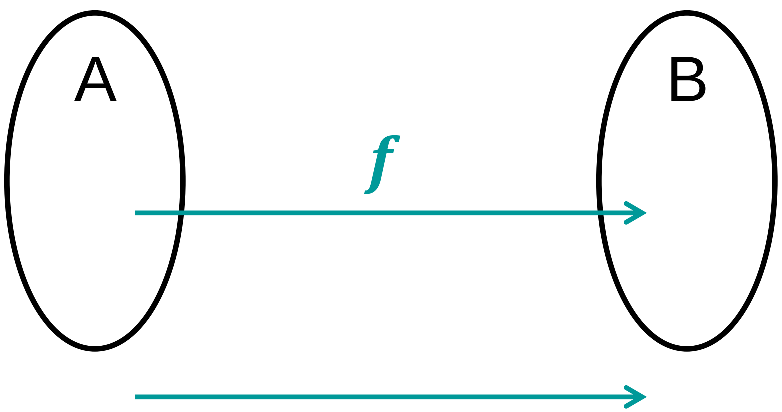
Given languages A and B,

$A \leq_m B$

if there is a computable function f,

such that for all strings w,

$w \in A$ iff $f(w) \in B$.



Using mapping reductions to prove decidability

Theorem. If $A \leq_m B$ and B is decidable, then A is decidable.

Proof: Let M be a decider for B and f be a mapping reduction from A to B.

Construct a decider for A:

``On input w:

1. Compute f(w).
2. Run M on f(w).
3. If it accepts, accept. O.w. reject."

Using mapping reductions to prove undecidability

Theorem. If $A \le_m B$ and B is decidable, then A is decidable.

Corollary. If $A \le_m B$ and A is undecidable, then B is undecidable.

Example: If $A_{TM} \le_m B$, then B is undecidable.

Using mapping reductions to prove recognizability

Theorem. If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Proof: Let M be a TM that recognizes B and f be a mapping reduction from A to B.
Construct a TM that recognizes A:
``On input w:
1. Compute f(w).
2. Run M on f(w).
3. If it accepts, accept. O.w. reject."

Using mapping reductions to prove unrecognizability

Theorem. If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Corollary. If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable.

Example: If $\overline{A_{TM}} \leq_m B$, then B is unrecognizable.