LECTURE 18

Last time
• $A_{TM}$ is unrecognizable
• Reductions

Today
• Reductions
• Mapping reductions

Homework 7 due
Homework 8 out

Sofya Raskhodnikova
# Problems in language theory

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We want to prove that language $L$ is undecidable.

**Idea:** Use a proof by contradiction.

1. Suppose to the contrary that $L$ is decidable.
2. Use a decider for $L$ as a subroutine to construct a decider for $A_{TM}$.
3. But $A_{TM}$ is undecidable. Contradiction!
To prove that $E_{TM}$ is undecidable

A. we assumed $E_{TM}$ had a decider and used it to construct a decider for $A_{TM}$

B. we assumed $A_{TM}$ had a decider and used it to construct a decider for $E_{TM}$

C. we constructed a TM S that on input $<M,w>$ decides whether $M$ accepts $w$, assuming the existence of a TM R that decides on input $<M'>$ whether the language of $<M'>$ is empty

D. There is more than one correct answer.

E. None of the above.
Prove that $\text{CFL}_{\text{TM}}$ is undecidable

$\text{CFL}_{\text{TM}} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is context-free} \}$

Proof: Suppose to the contrary that $\text{CFL}_{\text{TM}}$ is decidable, and let $R$ be a TM that decides it.

We construct TM $S$ that decides $A_{\text{TM}}$.

$S = \langle \text{On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:} \rangle$

1. Construct TM $M'$.
   
   $M' = \langle \text{On input } x, \text{ reject.} \rangle$ 
   1. If $x$ is not of the form $0^n1^n2^n$, reject.
   2. Run TM $M$ on input $w$.
   3. If it accepts, accept."

2. Run TM $R$ on input $\langle M' \rangle$.

3. If it rejects, accept. O.w. reject."
Prove that $\text{EQ}_{\text{TM}}$ is undecidable

$\text{EQ}_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \}$

Proof: Suppose to the contrary that $\text{EQ}_{\text{TM}}$ is decidable, and let $R$ be a TM that decides it.

We construct TM $S$ that decides $A_{\text{TM}}$.

$S = \text{``On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:}''$


   $M' = \text{``On input } x, \text{ ignore the input.}''$
   1. Ignore the input.
   2. Run TM $M$ on input $w$.
   3. If it accepts, accept."

   $M'' = \text{``Accept.''}$

2. Run TM $R$ on input $\langle M', M'' \rangle$.

3. If it accepts, accept. O.w. reject."
Proof 2 that $\text{EQ}_{\text{TM}}$ is undecidable

$\text{EQ}_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2)\}$

Proof: Suppose to the contrary that $\text{EQ}_{\text{TM}}$ is decidable, and let $R$ be a TM that decides it.

We construct TM $S$ that decides $E_{\text{TM}}$. What do we change?

$S = $``On input $\langle M \rangle$, where $M$ is a TM and $w$ is a string:

1. Construct TM $M'$.

   $M' = $``Reject.''
   1. Ignore the input.
   2. Run TM $M$ on input $w$.
   3. If it accepts, accept.''

2. Run TM $R$ on input $\langle M, M' \rangle$.

3. If it accepts, accept. O.w. reject.'’
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Proving undecidability and unrecognizability

Mapping Reductions
A function $f: \Sigma^* \rightarrow \Sigma^*$ is **computable** if some TM $M$, on every input $w$, halts with only $f(w)$ on its tape.

**Example 1:** $f(\langle x, y \rangle) = x + y$.

**Example 2:** $f(\langle M, w \rangle) = \langle M' \rangle$, where $M$ is a TM and $w$ is a string, and $M'$ is a TM that ignore its input and runs $M$ on $w$. 
Given languages A and B, \( A \leq_m B \) if there is a computable function \( f \), such that for all strings \( w \), \( w \in A \) iff \( f(w) \in B \).
Theorem. If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.

Proof: Let $M$ be a decider for $B$ and $f$ be a mapping reduction from $A$ to $B$. Construct a decider for $A$:
``On input $w$:
1. Compute $f(w)$.
2. Run $M$ on $f(w)$.
3. If it accepts, accept. O.w. reject.”

Using mapping reductions to prove **undecidability**

**Theorem.** If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.

**Corollary.** If $A \leq_m B$ and $A$ is **undecidable**, then $B$ is **undecidable**.

**Example:** If $A_{TM} \leq_m B$, then $B$ is **undecidable**.
Using mapping reductions to prove recognizability

**Theorem.** If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

**Proof:** Let $M$ be a TM that recognizes $B$ and $f$ be a mapping reduction from $A$ to $B$. Construct a TM that recognizes $A$:
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On input $w$:
1. Compute $f(w)$.
2. Run $M$ on $f(w)$.
3. If it accepts, accept. O.w. reject.
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Using mapping reductions to prove unrecognizability

**Theorem.** If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

**Corollary.** If $A \leq_m B$ and $A$ is unrecognizable, then $B$ is unrecognizable.

**Example:** If $A_{TM} \leq_m B$, then $B$ is unrecognizable.