# *Intro to Theory of Computation*

## LECTURE 16

CS 464

**Last time**

- Designing deciders.
- Countable/uncountable sets.
- Diagonalization

**Today**

- Undecidable/unrecognizable languages
- $A_{TM}$ is undecidable

**Homework 6 due**
**Homework 7 out**

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# How to compare sizes of infinite sets?

- Two sets are **the same size** if there is a bijection between them.
- A set is **countable** if it is
  - finite or
  - it has the same size as $\mathbb{N}$, the set of natural numbers

# **Theorem.** There is no bijection from the positive integers to the real interval (0,1)

**Proof: Suppose f is such a function:**

| n | f(n) |
|---|---|
| 1 | 0.**2**8347279… |
| 2 | 0.8**8**388384… |
| 3 | 0.77**6**35284… |
| 4 | 0.111**1**1111… |
| 5 | 0.1234**5**678… |
| : | : |

**Construct $b \in (0,1)$ that does not appear in the table.**

**b=0.$d_1 d_2 d_3 \ldots, \textit{where } d_i \neq \textit{digit } i \textit{ of } f(i)$.**

# Diagonalization

**The process of constructing a counterexample by "contradicting the diagonal" is called DIAGONALIZATION**

# What if we try this argument on $\mathbb{Q}$ instead of $\mathbb{R}$?

**Proof: Suppose f is such a function:**

| n | f(n) |
|---|---|
| 1 | 0.**2**8347279… |
| 2 | 0.8**8**388384… |
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**Construct $b \in (0,1)$ that does not appear in the table.**

**b=0.$d_1 d_2 d_3 \ldots$, $where\ d_i \neq digit\ i\ of\ f(i)$.**

# I-clicker question (frequency: AC)

## What if we try Cantor's diagonalization argument on $\mathbb{Q}$ instead of $\mathbb{R}$?

A. It works.

B. It fails because there are some rational numbers that cannot be represented in decimal point notation.

C. It fails because the $i$-th number might have no digit in the $i$-th position after the decimal point.

D. It fails because the constructed number is not rational.

E. None of the above.

# Let L be any set and P(L) be the power set of L

**Theorem: There is no bijection from L to P(L)**

**Proof:** **Assume, for a contradiction, that there is bijection $f : L \to P(L)$**

**We construct a set S that cannot be the output, f(y), for any $y \in L$.**

$$\text{Let } S = \{ x \in L \mid x \notin f(x) \}$$

**If $S = f(y)$ then $y \in S$ if and only if $y \notin S$**

# How is it diagonalization?

| x | $y_1 \in f(x)$? | $y_2 \in f(x)$? | $y_3 \in f(x)$? | $y_4 \in f(x)$? | … |
|---|---|---|---|---|---|
| $y_1$ | Y | N | Y | Y | |
| $y_2$ | N | Y | N | Y | |
| $y_3$ | N | N | N | N | |
| $y_4$ | Y | N | N | Y | |
| ⋮ | | | | | ⋱ |

$(y_i \in S) = Y \text{ iff } (y_i \in f(y_i)) = N$

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# EXAMPLE

Let L = {0,1,2}. Then P(L) =
{Ø, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}}

Let f(0) = {1}, f(1) = Ø, f(2) = {0,2}. Then:

| x | 0 ∈ f(x)? | 1 ∈ f(x)? | 2 ∈ f(x)? |
|---|---|---|---|
| 0 | N | Y | N |
| 1 | N | N | N |
| 2 | Y | N | Y |

S = {0,1}

**For all sets L,**
**P(L) has more elements than L**

# Not all languages over {0,1} are decidable

**TM Deciders**

**Strings of 0s and 1s**

**L**

**Languages over {0,1}**

**Sets of strings of 0s and 1s**

**P(L)**

# Not all languages over {0,1} are recognizable

**Turing Machines**

**Strings of 0s and 1s**

**L**

**Languages over {0,1}**

**Sets of strings of 0s and 1s**

**P(L)**

# A specific undecidable language

**$A_{TM}$ = {$\langle M, w \rangle$ | M is a TM, w is a string, and M accepts w }**

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# Theorem. $A_{TM}$ is undecidable.

**Proof: For contradiction, suppose a TM H decides $A_{TM}$.**

$$H(\langle M, w\rangle) = \begin{cases} \text{accept} & \text{if } M \text{ accepts } w \\ \text{reject} & \text{if } M \text{ doesn't accept } w \end{cases}$$

**Idea: Use $H$ to check what TM $M$ does on its own description (and do the opposite).**

**TM D = `` On input $\langle M \rangle$ where $M$ is a TM:**

1. **Run $H$ on input <M, $\langle M \rangle$>.**
2. **Accept if it rejects. O.w. reject.”**

**D is a decider. What does it do on $\langle D \rangle$?**

# Is it diagonalization again? Does M accept $\langle M \rangle$?

| TMs | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\langle M_4 \rangle$ | … |
|---|---|---|---|---|---|
| $M_1$ | Y | N | Y | Y | |
| $M_2$ | N | Y | N | Y | |
| $M_3$ | N | N | N | N | |
| $M_4$ | Y | N | N | Y | |
| ⋮ | | | | | ⋱ |

**D accepts $\langle M_i \rangle$ iff entry $(i, j)$ is N .**

# Is it diagonalization again? Does M accept $\langle M \rangle$?

| TMs | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\langle M_4 \rangle$ | … | $\langle D \rangle$ | … |
|---|---|---|---|---|---|---|---|
| $M_1$ | Y | N | Y | Y | | | |
| $M_2$ | N | Y | N | Y | | | |
| $M_3$ | N | N | N | N | | | |
| $M_4$ | Y | N | N | Y | | | |
| ⋮ | | | | | ⋱ | | |
| $D$ | | | | | | ? | |
| ⋮ | | | | | | | ⋱ |

# Classes of languages