Intro to Theory of Computation

Lecture 16

Last time
• Designing deciders.
• Countable/uncountable sets.
• Diagonalization

Today
• Undecidable/unrecognizable languages
• $A_{TM}$ is undecidable

Homework 6 due
Homework 7 out
How to compare sizes of infinite sets?

• Two sets are **the same size** if there is a bijection between them.

• A set is **countable** if it is
  – finite or
  – it has the same size as \( \mathbb{N} \), the set of natural numbers
Theorem. There is no bijection from the positive integers to the real interval (0,1).

Proof: Suppose f is such a function:

<table>
<thead>
<tr>
<th>n</th>
<th>f(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.28347279…</td>
</tr>
<tr>
<td>2</td>
<td>0.88388384…</td>
</tr>
<tr>
<td>3</td>
<td>0.77635284…</td>
</tr>
<tr>
<td>4</td>
<td>0.11111111…</td>
</tr>
<tr>
<td>5</td>
<td>0.12345678…</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

Construct b ∈ (0,1) that does not appear in the table.

\[ b = 0.d_1d_2d_3 ... , \text{where } d_i \neq \text{digit } i \text{ of } f(i). \]
The process of constructing a counterexample by “contradicting the diagonal” is called **DIAGONALIZATION**
What if we try this argument on $\mathbb{Q}$ instead of $\mathbb{R}$?

Proof: Suppose $f$ is such a function:

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</table>

Construct $b \in (0, 1)$ that does not appear in the table.

$b=0.d_1d_2d_3…$, where $d_i \neq \text{digit } i \text{ of } f(i)$. 
What if we try Cantor’s diagonalization argument on $\mathbb{Q}$ instead of $\mathbb{R}$?

A. It works.

B. It fails because there are some rational numbers that cannot be represented in decimal point notation.

C. It fails because the $i$-th number might have no digit in the $i$-th position after the decimal point.

D. It fails because the constructed number is not rational.

E. None of the above.
Let $L$ be any set and $P(L)$ be the power set of $L$.

**Theorem:** There is no bijection from $L$ to $P(L)$.

**Proof:** Assume, for a contradiction, that there is bijection $f : L \rightarrow P(L)$.

We construct a set $S$ that cannot be the output, $f(y)$, for any $y \in L$.

Let $S = \{ x \in L \mid x \notin f(x) \}$

If $S = f(y)$ then $y \in S$ if and only if $y \notin S$. 


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Sofya Raskhodnikova; based on slides by Nick Hopper
How is it diagonalization?

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 \in f(x)?$</th>
<th>$y_2 \in f(x)?$</th>
<th>$y_3 \in f(x)?$</th>
<th>$y_4 \in f(x)?$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$y_2$</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>$y_3$</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$y_4$</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

$(y_i \in S) = Y \text{ iff } (y_i \in f(y_i)) = N$
Let $L = \{0,1,2\}$. Then $P(L) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

Let $f(0) = \{1\}$, $f(1) = \emptyset$, $f(2) = \{0,2\}$. Then:

$\begin{array}{|c|c|c|c|}
\hline
x & 0 \in f(x) ? & 1 \in f(x) ? & 2 \in f(x) ? \\
\hline
0 & N & Y & N \\
\hline
1 & N & N & N \\
\hline
2 & Y & N & Y \\
\hline
\end{array}$

$S = \{0,1\}$
For all sets $L$, $P(L)$ has more elements than $L$
Not all languages over \( \{0,1\} \) are decidable

**TM Deciders**
- Strings of 0s and 1s

**Languages over \( \{0,1\} \)**
- Sets of strings of 0s and 1s

\[ L, P(L) \]
Not all languages over \{0,1\} are **recognizable**

**Turing Machines**

- Strings of 0s and 1s

**Languages over \{0,1\}**

- Sets of strings of 0s and 1s

\[ L \]

\[ P(L) \]
A specific undecidable language

$$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a TM, } w \text{ is a string, and } M \text{ accepts } w \}$$
Theorem. $A_{TM}$ is undecidable.

Proof: For contradiction, suppose a TM $H$ decides $A_{TM}$.

$$H(<M, w>) = \begin{cases} 
  \text{accept} & \text{if } M \text{ accepts } w \\
  \text{reject} & \text{if } M \text{ doesn't accept } w
\end{cases}$$

Idea: Use $H$ to check what TM $M$ does on its own description (and do the opposite).

TM $D = \``\text{ On input } <M> \text{ where } M \text{ is a TM: }$

1. Run $H$ on input $<M, <M>>$.  
2. Accept if it rejects. O.w. reject.”

D is a decider. What does it do on $<D>$?
Is it diagonalization again? Does $M$ accept $\langle M \rangle$?

<table>
<thead>
<tr>
<th>TMs</th>
<th>$\langle M_1 \rangle$</th>
<th>$\langle M_2 \rangle$</th>
<th>$\langle M_3 \rangle$</th>
<th>$\langle M_4 \rangle$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>$M_2$</td>
<td>N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>$M_3$</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>$M_4$</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
<td></td>
</tr>
<tr>
<td>:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>:</td>
</tr>
</tbody>
</table>

D accepts $\langle M_i \rangle$ iff entry $(i, j)$ is $N$.  

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Is it diagonalization again? Does M accept $\langle M \rangle$?

| TMs | $\langle M_1 \rangle$ | $\langle M_2 \rangle$ | $\langle M_3 \rangle$ | $\langle M_4 \rangle$ | ... | $\langle D \rangle$ | ...
|-----|----------------------|----------------------|----------------------|----------------------|-----|----------------------|-----
| $M_1$ | Y                    | N                    | Y                    | Y                    |     |                      |     
| $M_2$ | N                    | Y                    | N                    | Y                    |     |                      |     
| $M_3$ | N                    | N                    | N                    | N                    |     |                      |     
| $M_4$ | Y                    | N                    | N                    | Y                    |     |                      |     
| ...  |                      |                      |                      |                      |     |                      |     
| $D$  |                      |                      |                      |                      |     | ?                    |     
| ...  |                      |                      |                      |                      |     |                      |     

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Classes of languages

- \( L = \{0^n1^n0^n \mid n \geq 0\} \)
- \( L = \{0^n1^n \mid n \geq 0\} \)
- \( L = 1^* \)

\( A_{\text{TM}} \)