Intro to Theory of Computation

CS 464

LECTURE 15

Last time

- Decidable languages
- Designing deciders

Today

- Designing deciders
- Undecidable languages
- Diagonalization

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Recall

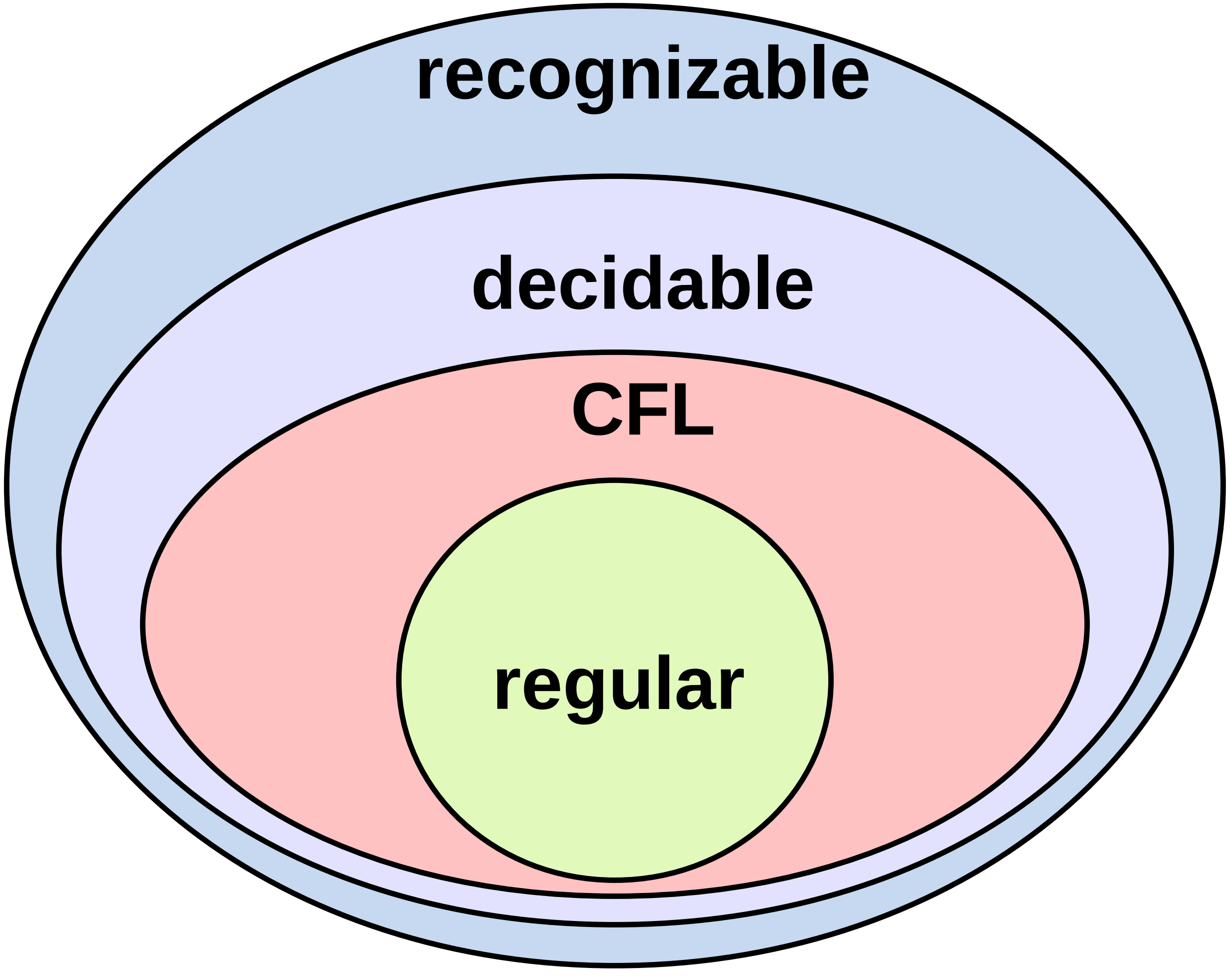
A_{DFA} = $\{\langle D, w\rangle$ | D is a DFA that accepts string w $\}$

E_{DFA} = $\{ \langle D\rangle$ | D is a DFA and L(D)=∅$\}$

EQ_{DFA} = $\{\langle D_1, D_2\rangle \mid D_1, D_2$ DFAs and $L(D_1) = L(D_2)\}$

A_{DFA}, E_{DFA}, EQ_{DFA}, A_{CFG}, E_{CFG} are decidable.

Classes of languages



Theorem. Every CFL is decidable.

Proof: Let G be a CFG for L.

Design a TM M_G that decides L.

- **Is it a good idea to convert G to an equivalent PDA P and have M_G simulate P?**

I-clicker problem (frequency: AC)

G is a CFG for L. Design a TM M_G that decides L.

Is it a good idea to convert G to an equivalent PDA P and have M_G simulate P?

A. Yes. Why not?

B. No, we can’t always convert G to an equivalent PDA.

C. No, P might loop on some inputs.

D. No, because we don’t have any input to run P on.

E. None of the above.

I-clicker problem (frequency: AC)

G is a CFG for L. Design a TM M_G that decides L.

A decider for which language is useful as a subroutine?

A. for $\mathbf{A_{DFA}}$

B. for $\mathbf{E_{DFA}}$

C. for $\mathbf{EQ_{DFA}}$

D. for $\mathbf{A_{CFG}}$

E. for $\mathbf{E_{CFG}}$

Theorem. Every CFL is decidable.

Proof: Let G be a CFG for L.

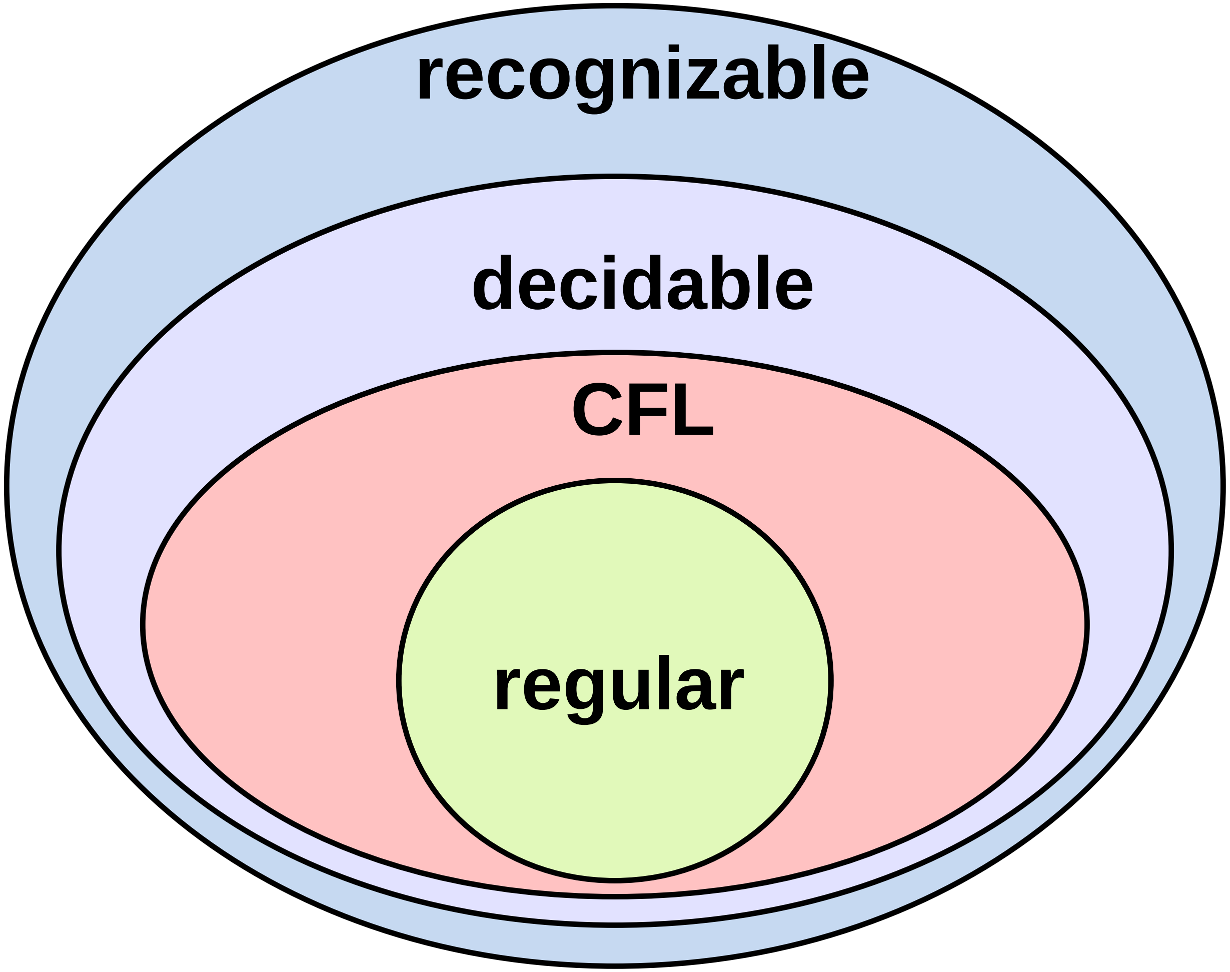
Design a TM M_G that decides L.

- **Is it a good idea to convert G to an equivalent PDA P and have M_G simulate P?**

M = `` On input w:

1. **Run the decider for A_{CFG} on input <G,w>.**
2. **Accept if yes. O.w. reject."**

Classes of languages



Theorem. $\text{INFINITE}_{\text{DFA}}$ is decidable.

$\text{INFINITE}_{\text{DFA}}$ = { ⟨D⟩ | D is a DFA and L(D) is infinite}

Idea: **Let n be the number of states in D.**

L(D) is infinite iff D accepts a string of length $\geq n$.

Proof: **The following TM M decides** $\text{INFINITE}_{\text{DFA}}$.

M = `` On input $\langle D \rangle$, where D is a DFA:

1. **Let n be the number of states in D.**
2. **Let C be a DFA for {w | |w| ≥ n}.**
3. **Build a DFA B for L(C) ∩ L(D).**
4. **Run a decider for** E_{DFA} **on $\langle B \rangle$.**
5. **Accept if it rejects. O.w. reject.”**

Problems in language theory

A_{DFA} **decidable**	A_{CFG} **decidable**	A_{TM} **?**
E_{DFA} **decidable**	E_{CFG} **decidable**	E_{TM} **?**
EQ_{DFA} **decidable**	EQ_{CFG} **?**	EQ_{TM} **?**

Undecidability

We will prove that there are some undecidable languages:

- i.e., problems a computer cannot solve no matter how long it computes

The proof idea is "simple:"

There are more languages than there are Turing Machines.

A language L is undecidable if

there is no TM that decides L.

If L is undecidable, then every TM must either:

1. **Accept (infinitely many) strings s ∉ L.**
2. **Reject (infinitely many) strings s ∈ L.**
3. **Loop forever on (infinitely many) strings.**

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I-clicker POP **QUIZ** (frequency: AC)

Let $\mathbb{N}$ = {1,2,…} be the natural numbers.

Let E = {2,4,6,…} be the even natural numbers.

Let $\mathbb{Z}$ = {…,-2,-1,0,1,2,…} be the integers.

Which one is largest?

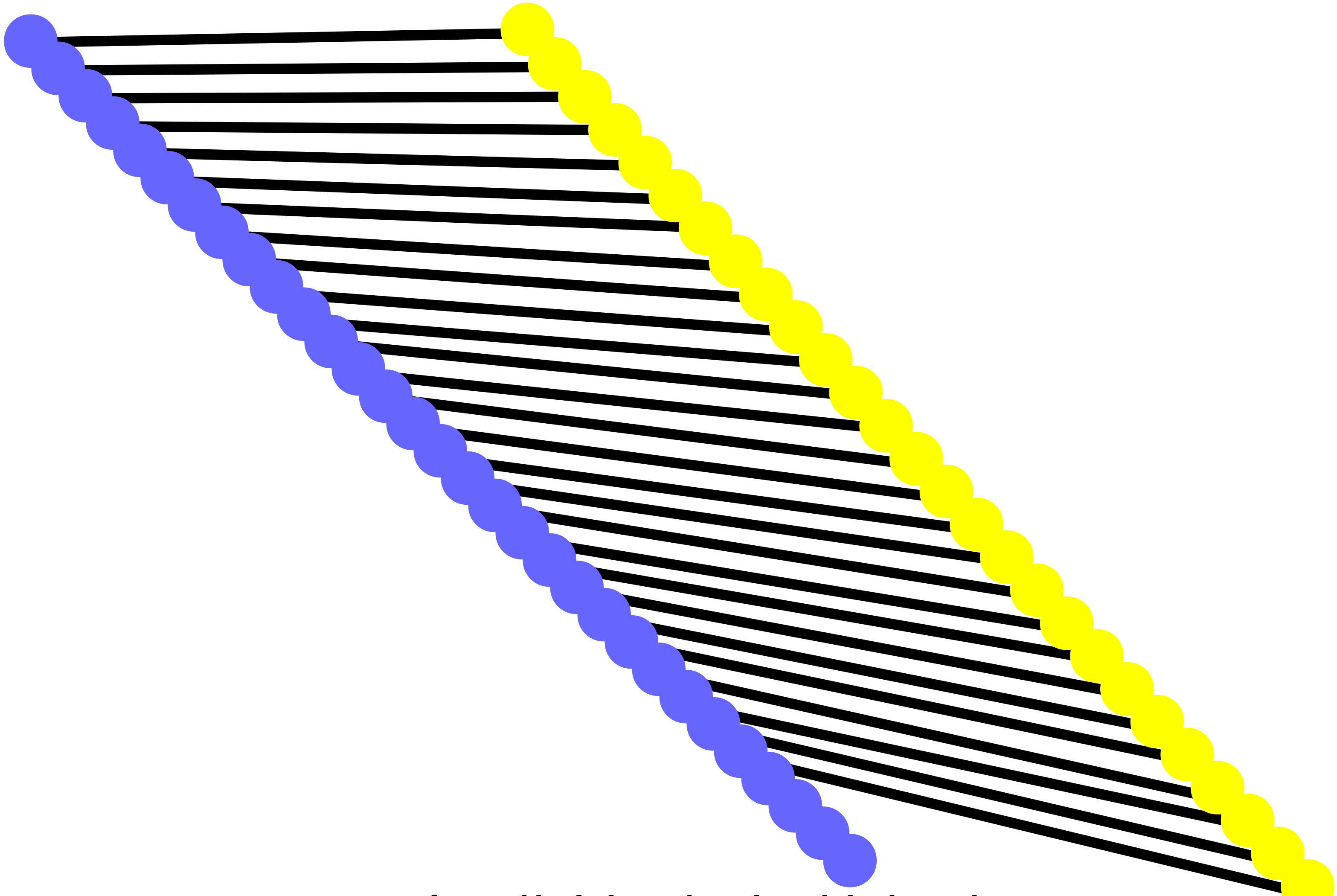
A. $\mathbb{N}$

B. E

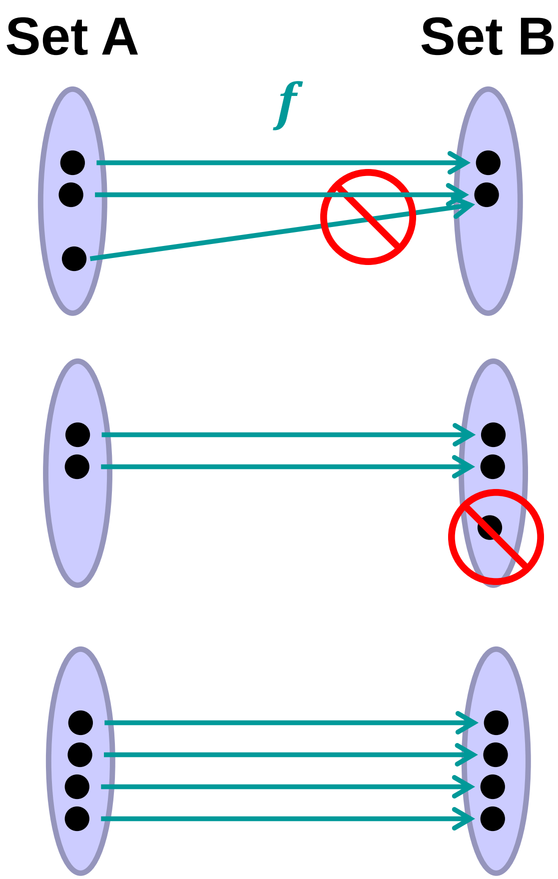
C. $\mathbb{Z}$

D. the same size.

Are there more blue or yellow dots?



Set Theory 101



A function $f: A \to B$ is

- **1-*to*-1** (or ***injective***) if

  $f(a) \neq f(b)$ for $a \neq b$.

- ***onto*** (or ***surjective***) if for all $b \in B$, some $a \in A$ maps to b: $f(a) = b$.

- ***correspondence*** (or ***bijective***) if

  it is 1-to-1 and onto, i.e.,

  each $a \in A$ maps to a unique $b \in B$, and each $b \in B$ has a unique $a \in A$ mapping to it.

How to compare sizes of infinite sets?

- Two sets are **the same size** if there is a bijection between them.
- A set is **countable** if it is
  - finite or
  - it has the same size as ℕ, the set of natural numbers

Examples of countable sets

Ø, {0}, {0,1}, {0,1, …, 255}

E = {2,4,6,…}

O = {1,3,5,7,…}

SQUARES = {1,4,9,16,25…}

POWERS = {1,2,4,8,16,32…}

|POWERS| = |SQUARES| = |E| = |O| = |N|

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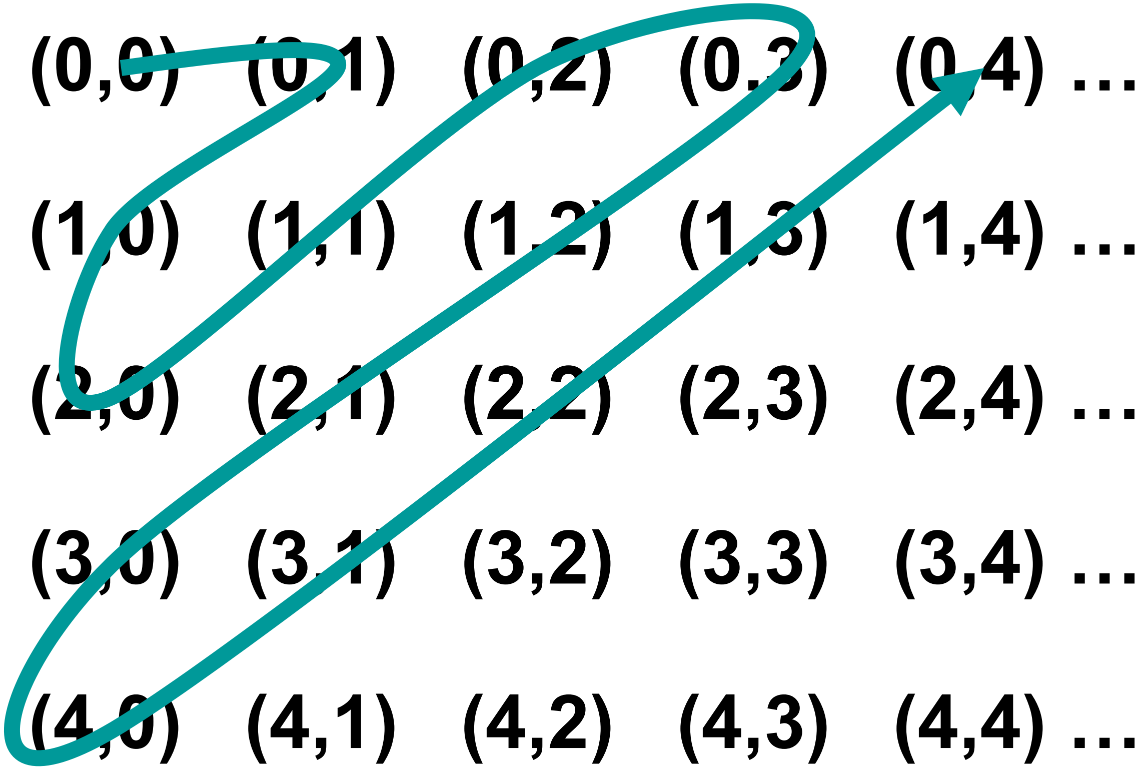
There is a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.



$\{0,1\}^*$ is countable

$\{ \langle M \rangle \mid M \text{ is a TM} \}$ is countable

$Q^+ = \{ p/q \mid p,q \in Z^+\}$ is countable!

Is any set *uncountable?*

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Theorem. There is no bijection from the positive integers to the real interval (0,1)

Proof: Suppose f is such a function:

n	f(n)
1	0.28347279…
2	0.88388384…
3	0.77635284…
4	0.11111111…
5	0.12345678…
:	:

Construct $b \in (0,1)$ that does not appear in the table.

b=0.$d_1 d_2 d_3 \ldots$, $where\ d_i \neq digit\ i\ of\ f(i)$.

Diagonalization

The process of constructing a counterexample by "contradicting the diagonal" is called
DIAGONALIZATION

Let L be any set and P(L) be the power set of L

Theorem: There is no onto map from L to P(L)

Proof: Assume, for a contradiction, that there is an onto map $f : L \to P(L)$

We construct a set S that cannot be the output, f(y) for any $y \in L$.

$$S = \{ x \in L \mid x \notin f(x) \}$$

If S = f(y) then $y \in S$ if and only if $y \notin S$

How is that diagonalization?

x	$y_1 \in f(x)$?	$y_2 \in f(x)$?	$y_3 \in f(x)$?	$y_4 \in f(x)$?	…
y_1	Y	N	Y	Y	
y_2	N	Y	N	Y	
y_3	N	N	N	N	
y_4	Y	N	N	Y	
$\vdots$					$\ddots$

$(y_i \in S) = Y$ iff $(y_i \in f(y_i)) = N$

For all sets L,
P(L) has more elements than L