Intro to Theory of Computation

CS 464

LECTURE 6

Last time:

- NFAs to regular expressions
- Pumping lemma

Today:

- Proving a language is not regular
- Pushdown automata (PDAs)

Homework 2 due
Homework 3 out

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THE PUMPING LEMMA

Let L be a regular language with $|L| = \infty$

Then there exists a length p such that

if $w \in L$ and $|w| \geq p$ then

w can be split into three parts $w=xyz$ where:

1. **$|y| > 0$**
2. **$|xy| \leq p$**
3. **$xy^iz \in L$ for all $i \geq 0$**

GENERAL STRATEGY

Proof by contradiction: assume L is regular.

Then there is **a pumping length p.**

Find a string w $\in$ L with |w| $\geq$ p.

Show that no matter how you choose xyz, w cannot be pumped!

Conclude that L is not regular.

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Pumping lemma as a game

1. **YOU pick the language *L* to be proved nonregular.**
2. **ADVERSARY picks p, but doesn't reveal to YOU what p is; YOU must devise a play for all possible p's.**
3. **YOU pick w ∈ L, which should depend on p and which must be of length at least p.**
4. **ADVERSARY divides w into x, y, z, obeying PL conditions: y is not empty and falls within the first p characters of w. Again, ADVERSARY does not tell YOU what x, y, z are.**
5. **YOU win by picking i, which may be a function of $p, x, y,$ z, such that $xy^i z$ is not in L.**

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I-clicker problem (frequency: AC)

Prove $C = \{ 0^i1^j \mid i > j \geq 0\}$ is not regular.

Proof: Assume … pumping length p

What string can you choose in your next move?

A. 00011

B. 0^p1^p

C. $0^{p/2}1^{p/2-1}$

D. $0^{p+1}1^p$

E. More than one choice above works.

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USING THE **PUMPING LEMMA: PUMPING DOWN**

Prove $C = \{ 0^i 1^j \mid i > j \geq 0 \}$ is not regular.

Proof: **Assume … pumping length p**

Find a w ∈ C of length at least p

$0^{P+1}1^{P}$

Show that w cannot be pumped:

p+1 p

w= 00...0011...11

y must be in this part

xyyyz = 00...00011...11 > p+1

xz = 0...0011...11 ≤ p

CS 464 USING THE PUMPING LEMMA: PUMPING DOWN

Prove $C = \{ 0^i 1^j \mid i > j \geq 0\}$ is not regular.

Proof: Assume … pumping length p

Find a w ∈ C of length at least p

$0^{P+1}1^P$

If w = xyz with $|xy| \geq$ p then

$y = 0^J$ for some $J \geq 1$.

Then $xy^0z = xz = 0^{P+1-J}1^P \notin C$

Contradiction!

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WHAT THE PUMPING LEMMA **DOESN'T SAY - I**

For every p, w $\in$ L with |w| ≥ p

there exist xyz=w, where

1. **|y| > 0**
2. **|xy| ≤ p**
3. **$xy^iz \in$ L for all i ≥ 0**

Let L = $1\Sigma^*$. L is regular.
The string 10 cannot be pumped with p=1.

You need to show that no matter what p is chosen, L has a string w, |w| ≥ p, that cannot be pumped.

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WHAT THE PUMPING LEMMA **DOESN'T SAY - II**

For every xyz=w, where:

1. **$|y| > 0$**
2. **$|xy| \leq p$**
3. **$xy^iz \in L$ for all $i \geq 0$**

Example: Let L = 10*1*. Picking $w = 10^P1^P$; $x=\varepsilon$; $y=10$; $z=0^{P-1}1^P$ does not contradict pumping lemma!

(You *can* pick $x=1$, $y=0$, $z=0^{P-1}1^P$.)

CHOOSING **WISELY**

Let BALANCED = { w | w has an equal # of 1s and 0s }

Assume … there is a p

Find a w ∈ BALANCED of length at least p

~~$(01)^P$~~ 0^P1^P

Show that w cannot be pumped:

If w = xyz with |xy| ≤ p then
y = 0^J for some J>0.
Then xyyz = $0^{P+J}1^P \notin$ BALANCED

REUSING A PROOF

Pumping a language can be lots of work…

Let's try to reuse that work!

$$\{0^n1^n \mid n \geq 0\} = \text{BALANCED} \cap 0^*1^*$$

If BALANCED is regular then so is $\{0^n1^n \mid n \geq 0\}$

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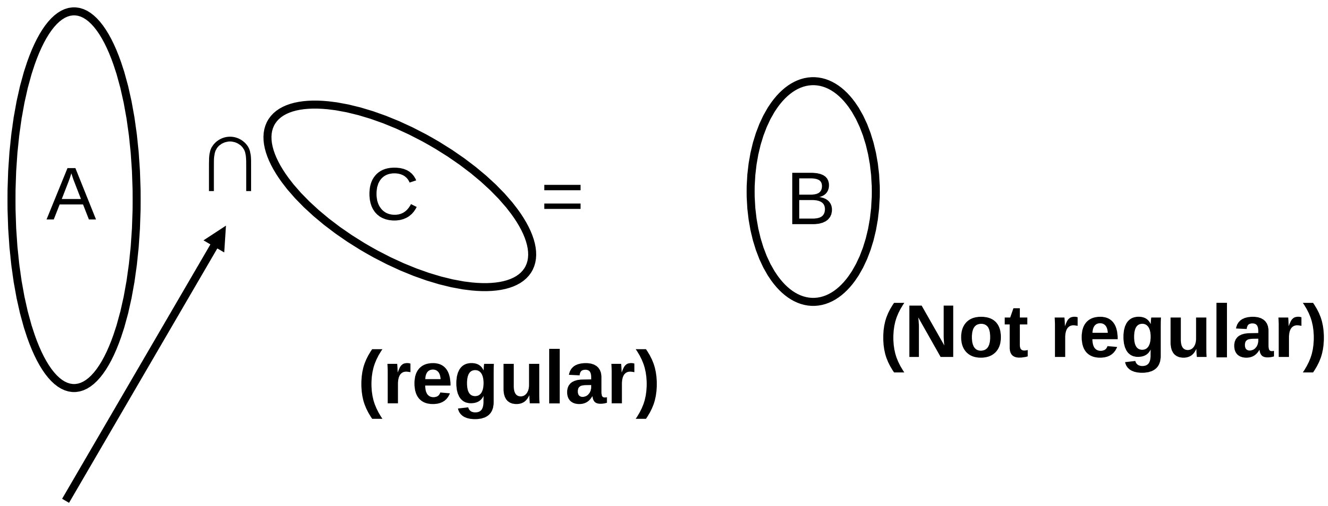
USING **CLOSURE**

Prove: A is not regular



any of $\{\circ, \cup, \cap\}$ or, for one language, $\{\neg, {}^{R}, *\}$

If A is regular, then A ∩ C (= B) is regular.

But B is not regular so neither is A.

USING **CLOSURE: example**

Prove $A = \{0^i1^j \mid i \neq j\}$ is not regular using $B = \{0^n1^n \mid n \geq 0\}$

$\neg A = B \cup \{$ strings that mix 0s and 1s $\}$

$\neg A \cap 0^*1^* = B$

PUMPING NON-REGULAR LANGUAGES

Let $F = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i=1 \Rightarrow j=k\}$

F has pumping length 2!

$i = 0$

$i = 1$

$i = 2$

$i > 2$

HW3: prove that F is not regular.

A non-regular language may still satisfy the pumping lemma.

464 so far

MODEL OF A PROBLEM: LANGUAGE

MODEL OF A PROGRAM: DFA

EQUIVALENT MODELS: NFA, REGEXP

PROBLEMS THAT A DFA CAN'T SOLVE

ARE WE DONE?

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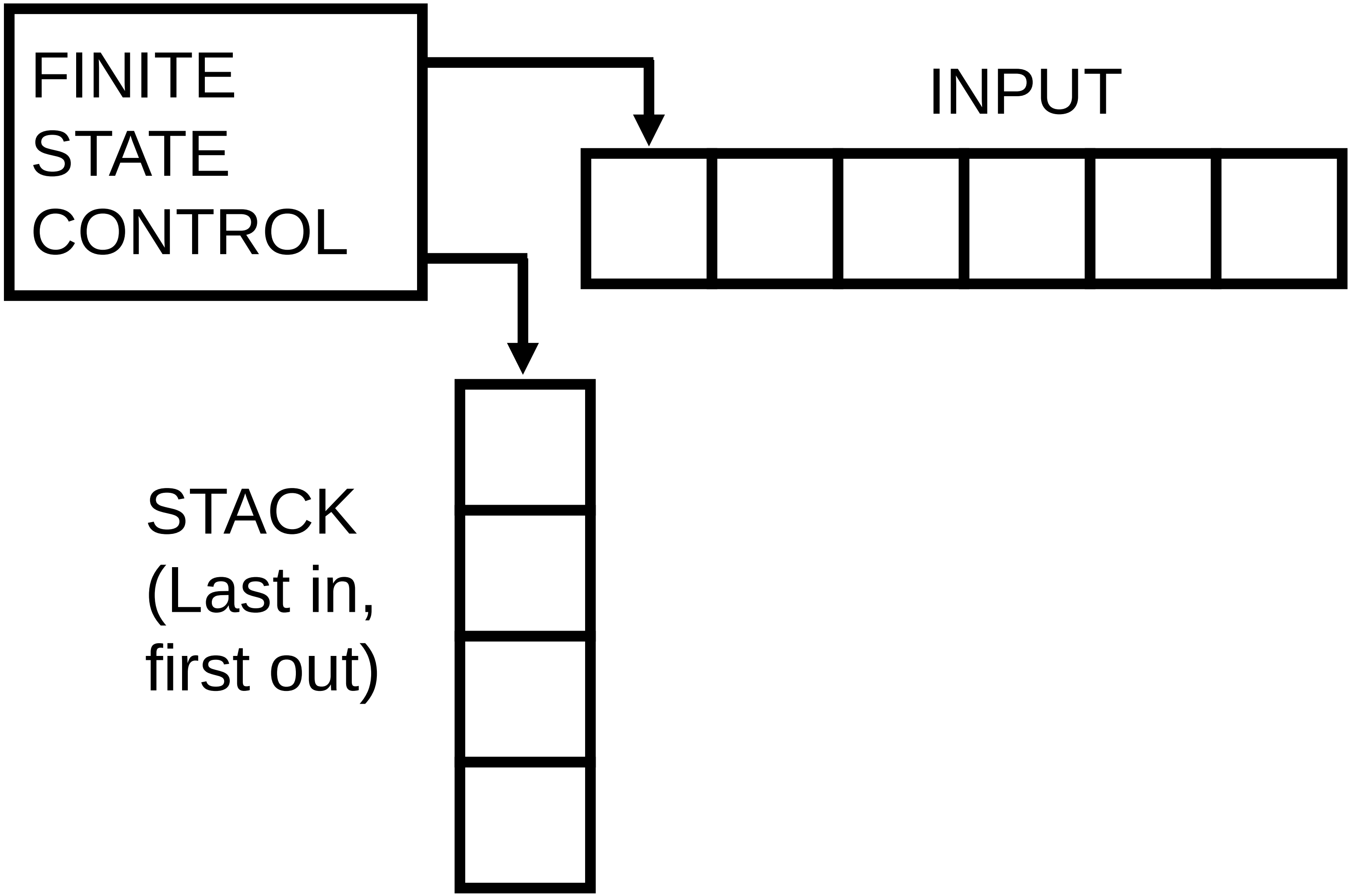
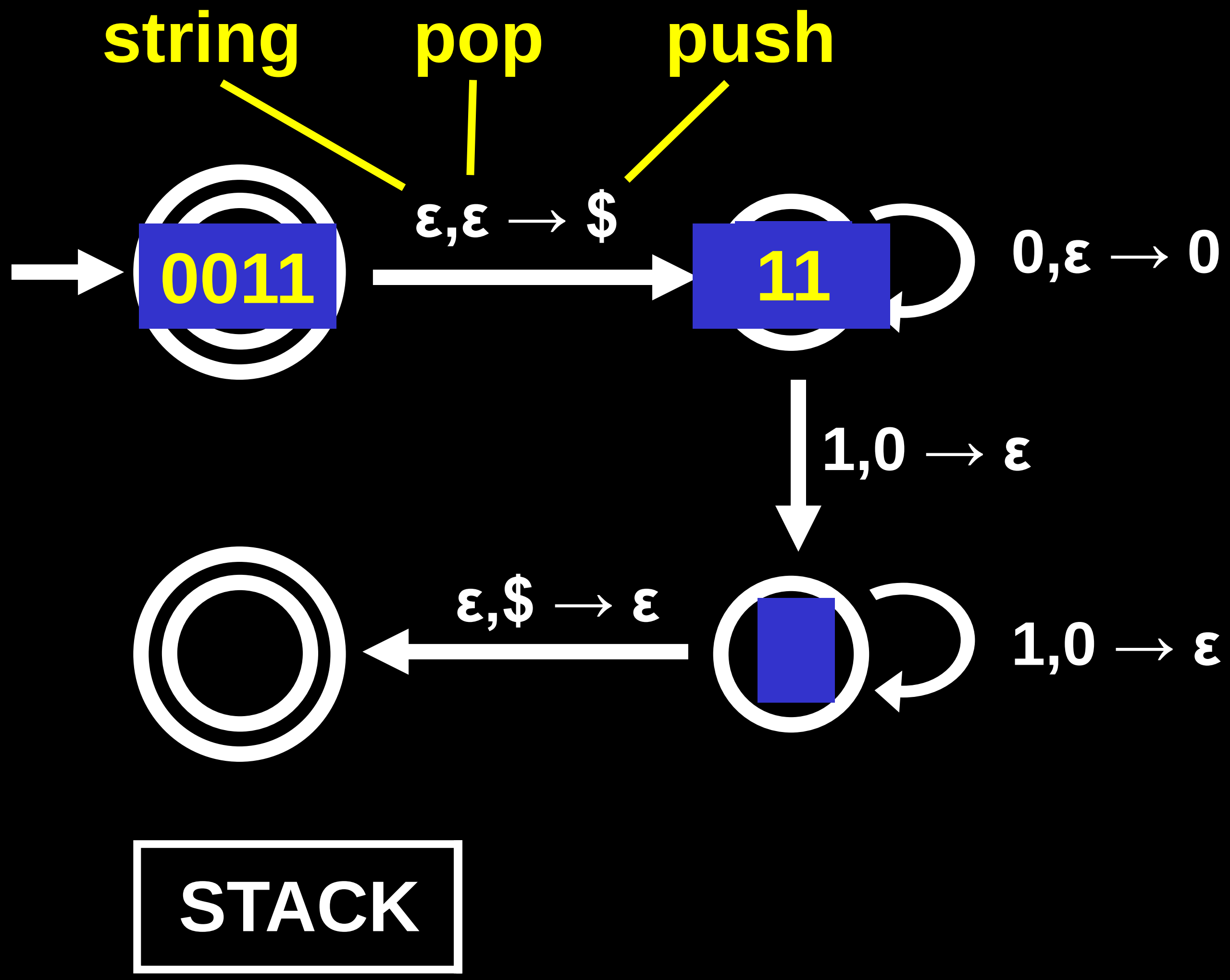
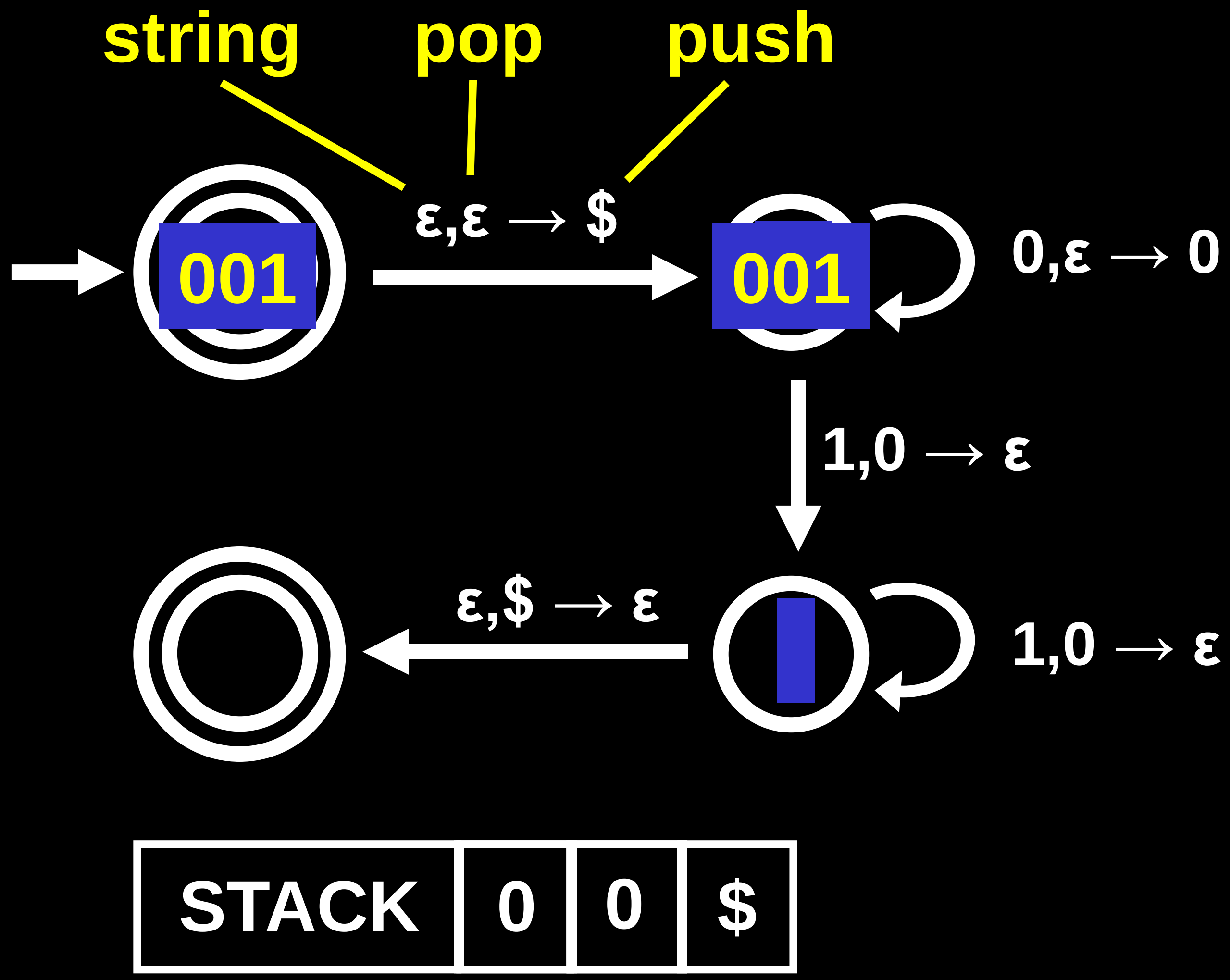
- $\Sigma = \{0, 1\}$, $L = \{ 0^n 1^n \mid n \geq 0 \}$
- $\Sigma = \{a, b, c, \ldots, z\}$, $L = \{ w \mid w = w^R \}$
- $\Sigma = \{ (,) \}$, L = { balanced strings of parens }

We can write a C or JAVA program for any of them!

PUSHDOWN AUTOMATA (PDA)




string
pop
push
ε,ε → $
0011
11
0,ε → 0
1,0 → ε
ε,$ → ε
1,0 → ε
STACK



PDA to recognize L = { 0^n1^n | n ≥ 0 }

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Formal Definition

A ***PDA*** is a 6-tuple $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$

Q is a finite set of states

Σ is the alphabet

Γ is the stack alphabet

$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P(Q \times \Gamma_\varepsilon)$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accept states

$P(Q \times \Gamma_\varepsilon)$ is the set of subsets of $Q \times \Gamma_\varepsilon$ and $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Note: A PDA is defined to be nondeterministic.

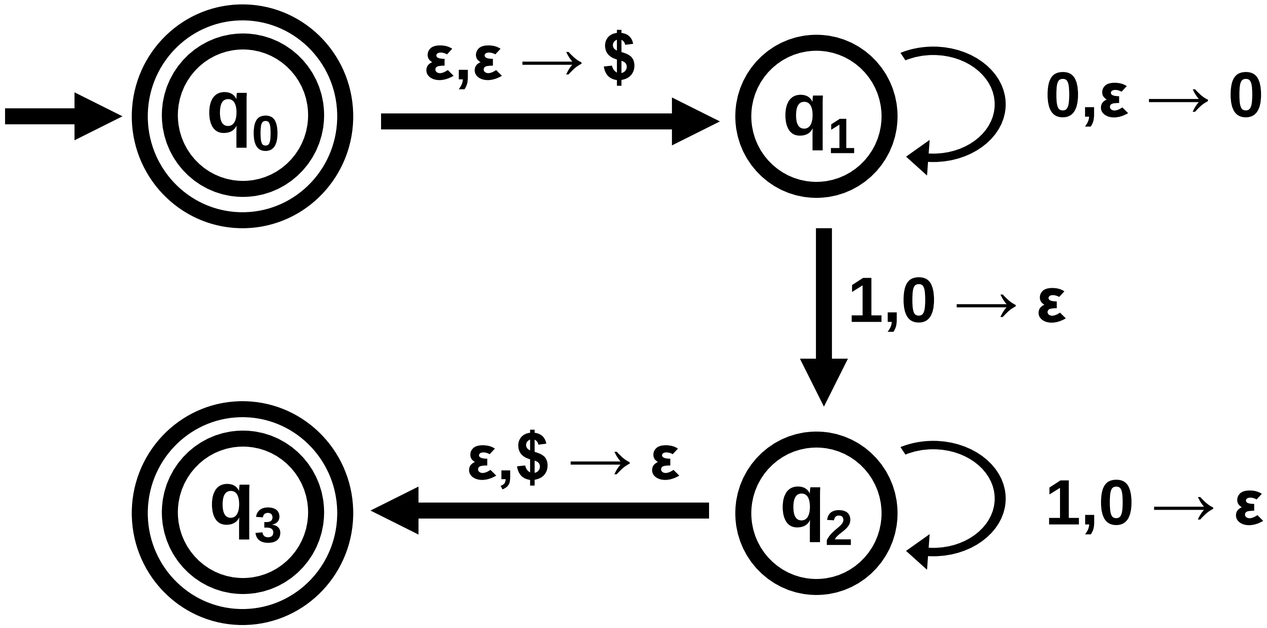
Formal Definition

A ***PDA*** is a 6-tuple $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$

- Q is a finite set of states
- Σ is the alphabet
- Γ is the stack alphabet
- $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow P(Q \times \Gamma_\varepsilon)$ **is the transition function**
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

A PDA starts with an *empty* stack.
It accepts a string if *at least one* of its computational branches reads *all the input* and gets into an *accept state* at the end of it.

An example PDA



$Q = \{q_0, q_1, q_2, q_3\}$ $\Sigma = \{0,1\}$ $\Gamma = \{\$,0\}$

$$\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \to P(Q \times \Gamma_\varepsilon)$$

$\delta(q_1,1,0) = \{ (q_2,\varepsilon) \}$ $\delta(q_2,1,1) = \varnothing$

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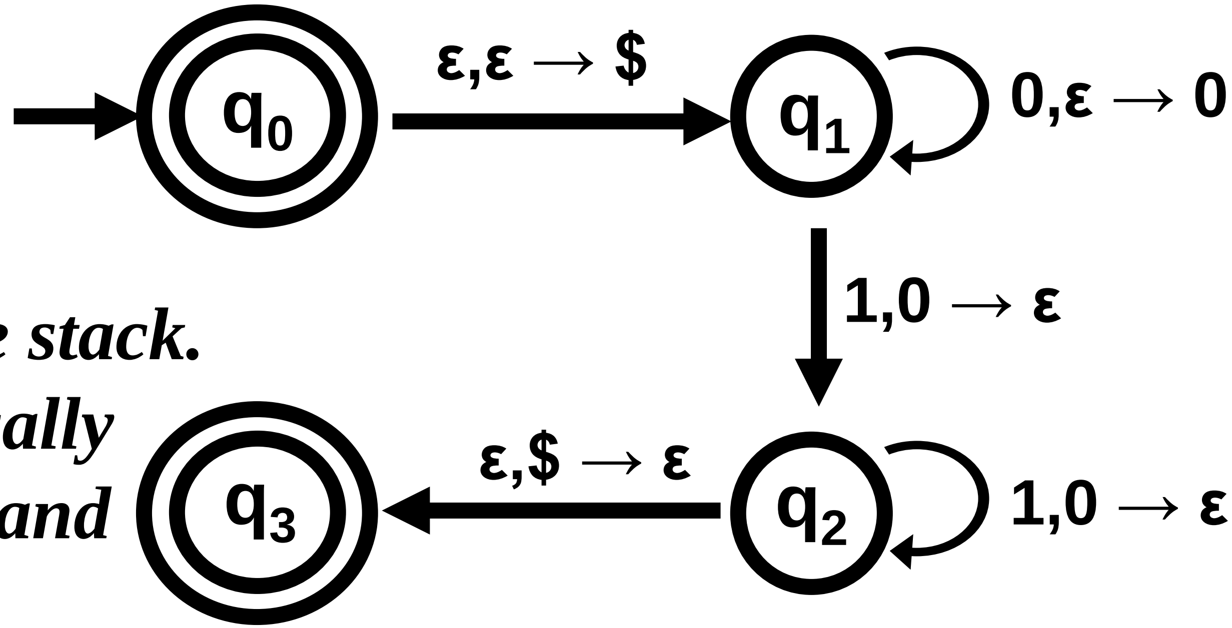
PDA: algorithmic description

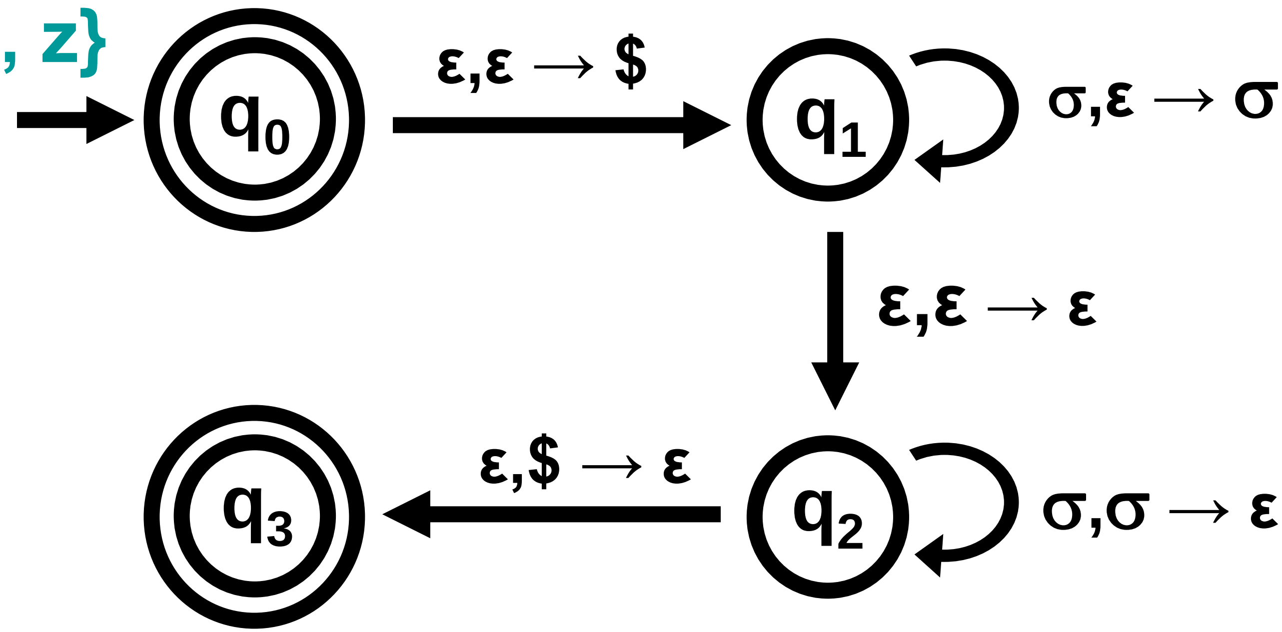


1. *Place the marker symbol $ onto the stack.*
2. *Nondeterministically keep reading a 0 and pushing it onto the stack or read a 1, pop a 0 and go to the next step.*
3. *Nondeterministically keep reading a 1 and popping a 0 or go to the next step.*
4. *If the top of the stack is $, enter the accept state. (Then PDA accepts if the input has been read).*

I-clicker problem (frequency: AC)

What strings are accepted by this PDA?

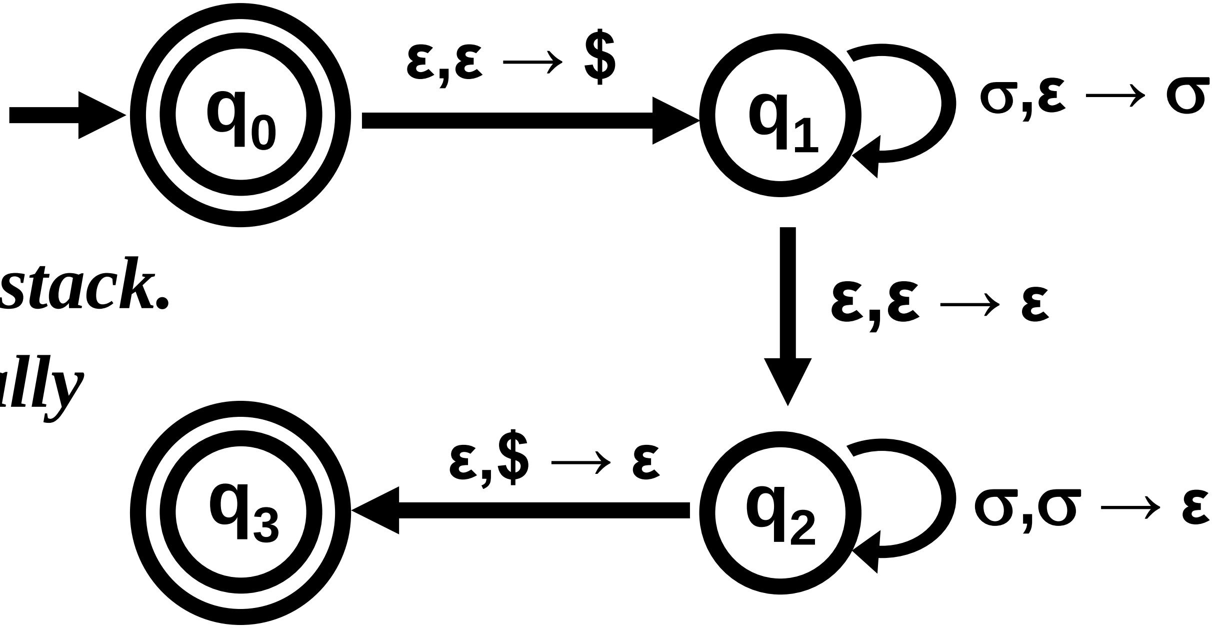
$\Sigma = \{a, b, c, \ldots, z\}$



A. Only *ε*
B. Palindromes
C. Even-length palindromes
D. All strings that start and end with the same letter
E. None of the above

Give an **ALGORITHMIC** description

$\Sigma = \{a, b, c, \ldots, z\}$



1. ***Place the marker symbol $ onto the stack.***
2. ***Nondeterministically keep reading a character and pushing it onto the stack or go to the next step.***
3. ***Nondeterministically keep reading and popping a matching character or go to the next step.***
4. ***If the top of the stack is $, enter the accept state.***

Build a PDA to recognize

L = { $a^i b^j c^k$ | i, j, k ≥ 0 **and** (i = j or i = k) }