Intro to Theory of Computation

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LECTURE 4

Last time:

- Equivalence of NFAs and DFAs

Today:

- Closure properties of regular languages
- Equivalence of NFAs, DFAs and regular expressions

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Regular Operations on languages

Complement: $\neg A = \{ w \mid w \notin A \}$

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Concatenation: $A \circ B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

I-clicker problem (frequency: AC)

Let L be the set of words in English.

Then $L \cap L^R$ is

A. The set of English words in alphabetical order, followed the same words in reverse alphabetical order.

B. $\{w \mid w$ is an English word or an English word written backwords$\}$.

C. $\{w \mid w$ is an English word that is a palindrom$\}$.

D. None of the above.

Closure properties of the class of regular languages

THEOREM. The class of regular languages is **closed** under all 6 operations.

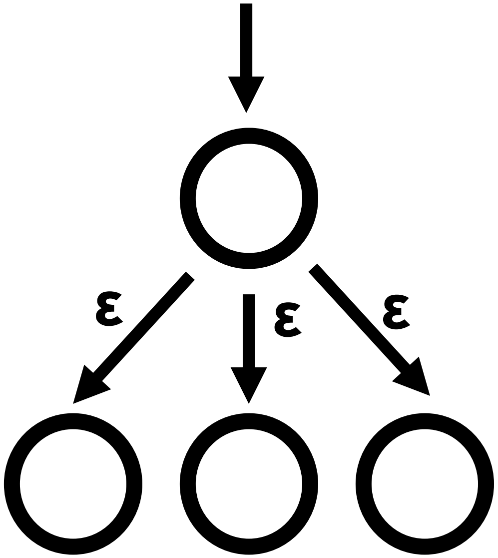
If A and B are regular, applying any of these operation yields a regular language.

Closure under reverse

Theorem. The reverse of a regular language is also regular

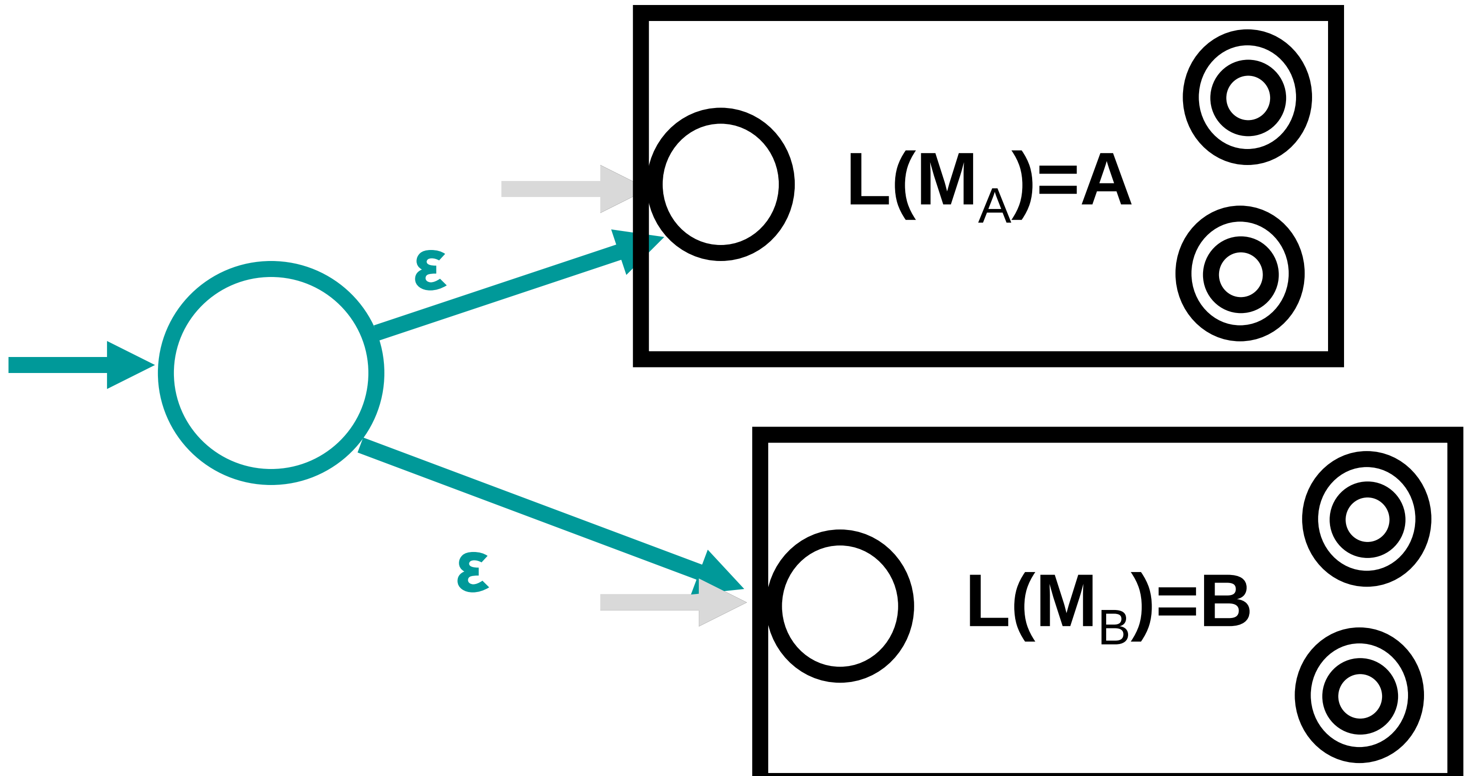
Proof: Let L be a regular language and M be a DFA that recognizes it. Construct an NFA M' recognizing L^R:

- Define M' as M with the arrows reversed.
- Make the start state of M be the accept state in M'.
- Make a new start state that goes to all accept states of M by ε-transitions.



New construction for A ∪ B

Construct an NFA M:

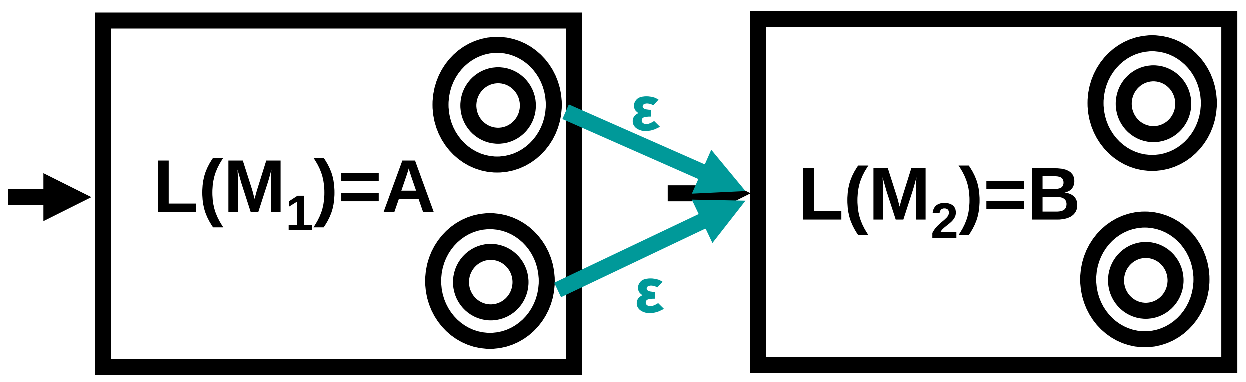


Concatenation operation

Concatenation: $A \circ B = \{ vw \mid v \in A \text{ and } w \in B \}$

Theorem. If A and B are regular, $A \circ B$ is also regular.

Proof: Given DFAs M_1 and M_2, construct NFA by connecting all accept states in M_1 to the start state in M_2.



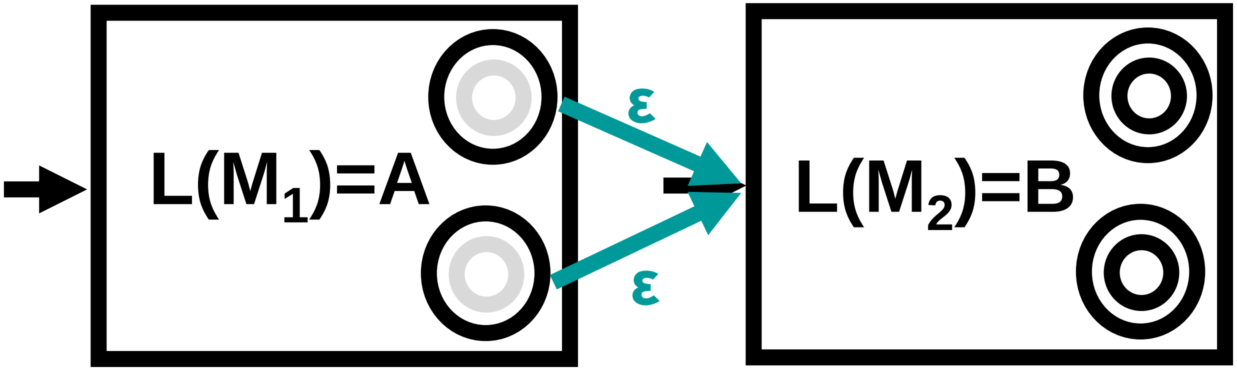
Concatenation operation

Concatenation: $A \circ B = \{ vw \mid v \in A \text{ and } w \in B \}$

Theorem. If A and B are regular, $A \circ B$ is also regular.

Proof: Given DFAs M_1 and M_2, construct NFA by connecting all accept states in M_1 to the start state in M_2.

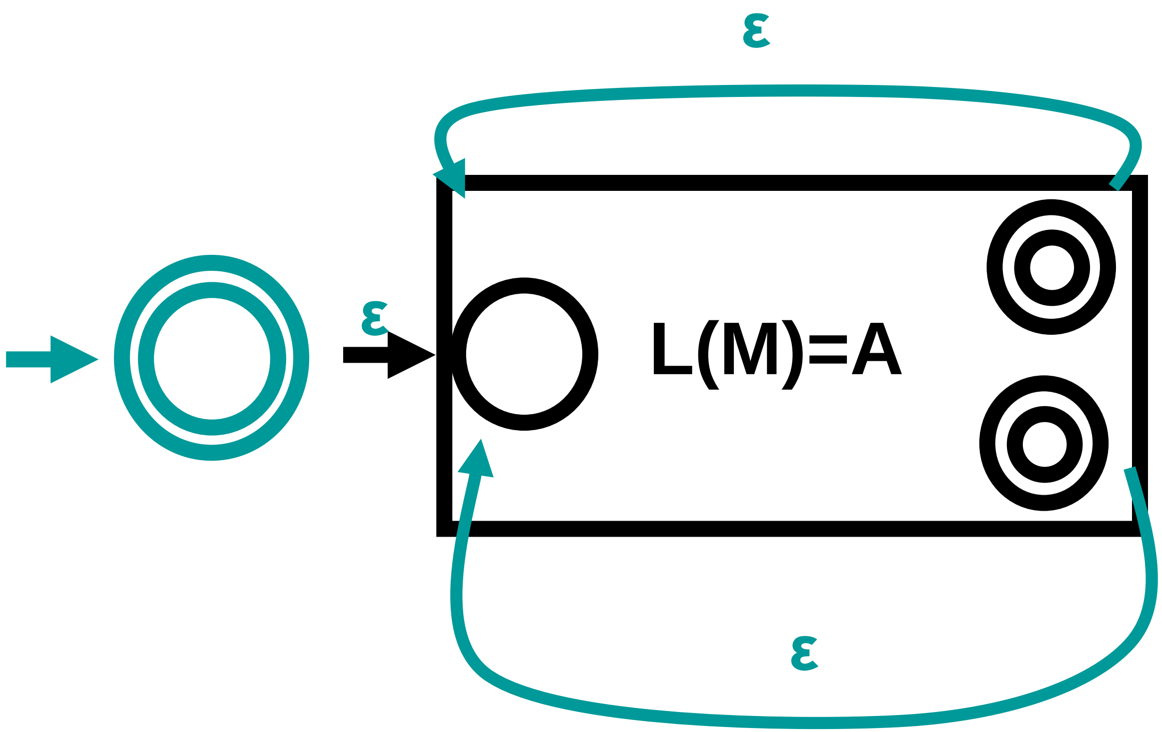
- Make all states in M_1 non-accepting.



Star operation

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Theorem. **If A is regular, A* is also regular.**



The class of regular languages is closed under

Regular operations

Union: $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$

Concatenation: $A \circ B = \{ vw \mid v \in A \text{ and } w \in B \}$

Star: $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Other operations

Complement: $\neg A = \{ w \mid w \notin A \}$

Intersection: $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$

Reverse: $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$

Regular expressions

- **In a regular expression, we can use**
  - **Constants:** ε , $\varnothing$, a set Σ, members of Σ.
  - **Regular operations:** $*$, $\circ$, $\cup$
- **Examples**:
  - 0^*1^* = { w | w has a run of 0s followed by a run of 1s}
  - $(0 \cup 1)^*$ = the set of all strings over the alphabet $\Sigma=\{0,1\}$
  - $0^*1^*(\varepsilon \cup 0)$
- **L(R)** = the language regular expression R describes

Precedence

$$* \quad \circ \quad \cup$$

EXAMPLE

$$R_1{}^*R_2 \cup R_3 = ((R_1{}^*)R_2) \cup R_3$$

Regular expressions: examples

1) { w | w has exactly one character 1 }

$$0^*10^*$$

2) { w | w has length ≥ 3 and its 3rd symbol is 0 }

$$(0\cup 1)(0\cup 1)\ 0\ (0\cup 1)^*$$

3) { w | every odd position of w is a 1 }

$$(1(0\cup 1))^*\ (\varepsilon \cup 1)$$

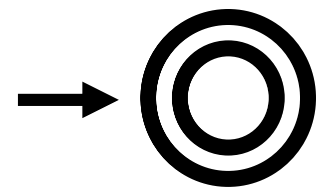
Regular expressions to NFAs

Theorem. **Every regular expression has an equivalent NFA.**

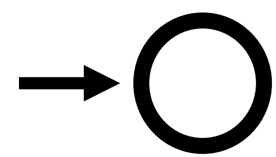
Proof: **Induction on the length of regular expression R.**

Base case: **length 1**

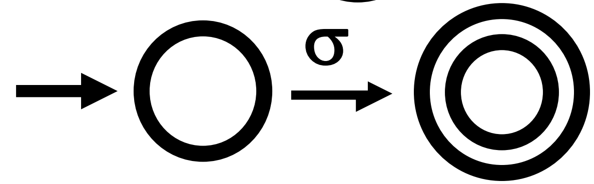
R = ε



R = ∅



R = σ



Inductive step: **follows from closure of the class of regular languages under the regular operations.**

I-clicker problem (frequency: AC)

What should the induction hypothesis be?

A. Suppose some regular expression of length k can be converted an NFA, for some $k \in \mathbb{N}$.

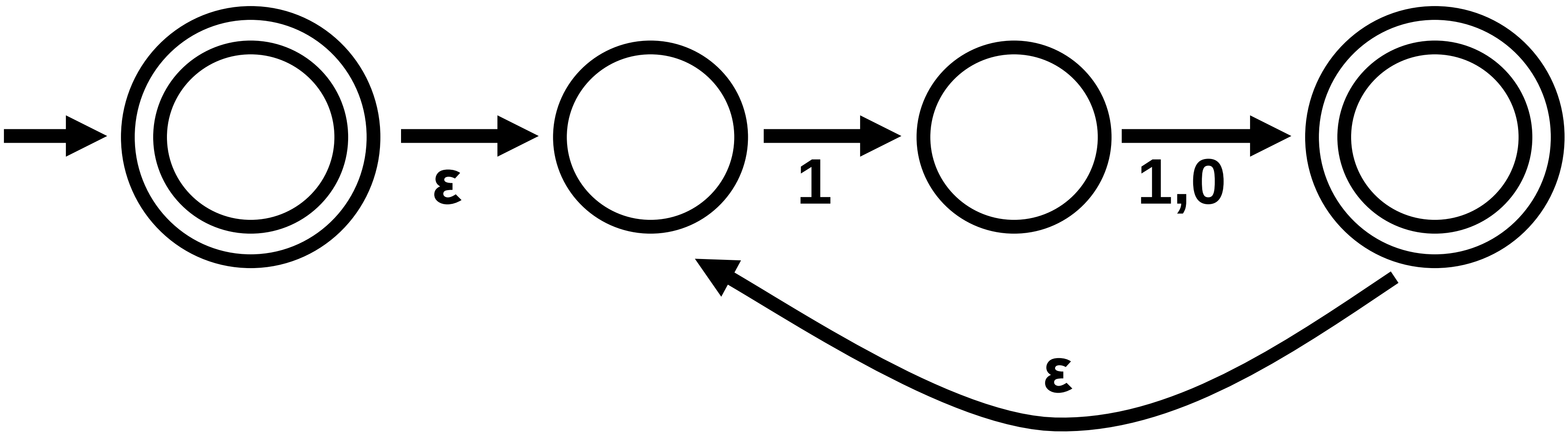
B. Suppose all regular expressions of length k can be converted an NFA, for some $k \in \mathbb{N}$.

C. Suppose all regular expressions of length at most k can be converted an NFA, for some $k \in \mathbb{N}$.

D. None of the above.

Regular expression to NFA

Transform $(1(0 \cup 1))^*$ to an NFA

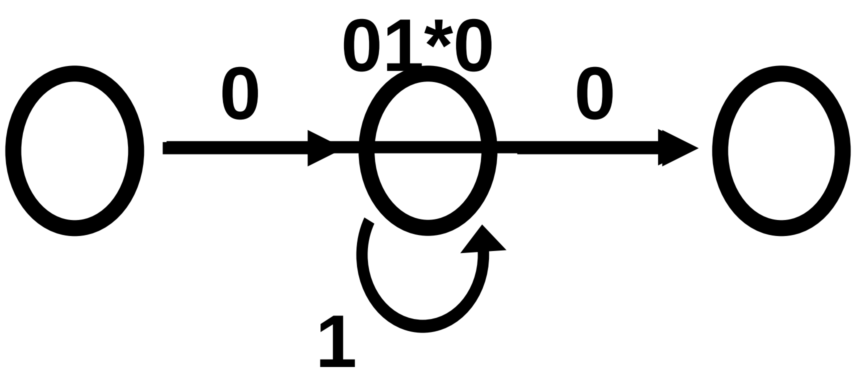


NFAs to regular expressions

Theorem. **Every NFA has an equivalent regular expression.**

Proof idea:

Transform NFA to a regular expression by removing states and relabeling the arrows with regular expressions.



Generalized NFAs

- **Each transition is labeled with a regular expression**
- Unique and distinct start and accept states
- No transitions **to** the start state
- No transitions **from** the accept state

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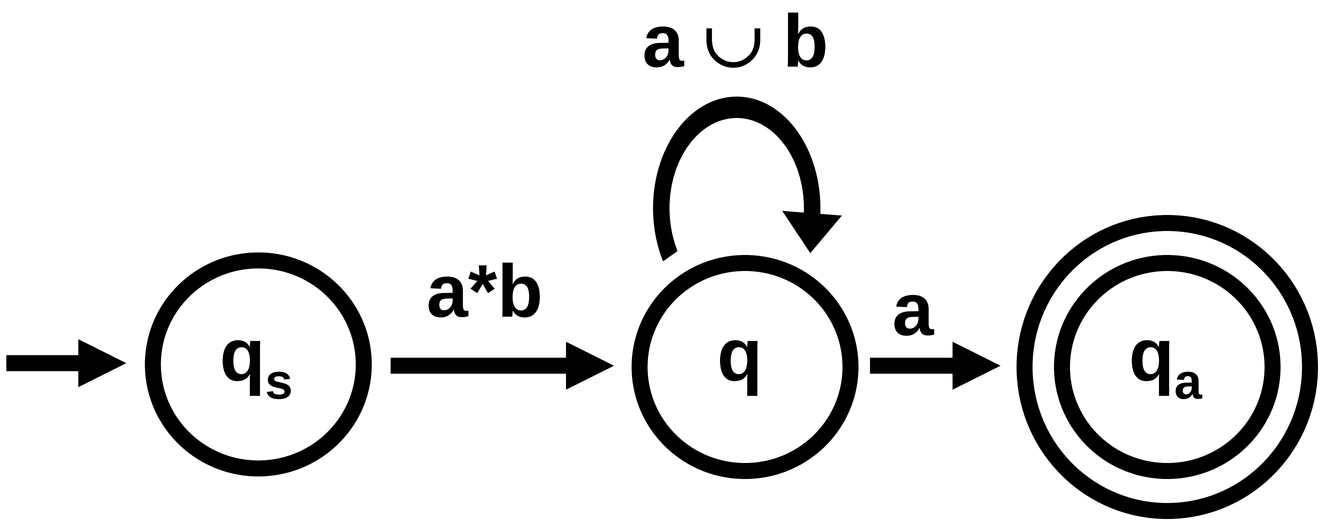
Generalized NFAs



G accepts w if it finds $q_0 q_1 \ldots q_k$, $w_1 \ldots w_k$:

w_i is generated by $R(q_i, q_{i+1})$

$w = w_1 w_2 \ldots w_k$

$q_0 = q_s$, $q_k = q_a$

$R(q_s, q) = a^*b$

$R(q_a, q) = \emptyset$

$R(q, q_s) = \emptyset$