Homework 6 – Due Thursday, March 3, 2016 before the lecture

Please refer to the general information handout for the full homework policy and options. *Your solution to each problem should be handed in on a separate sheet of paper.*

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Problems Please practice on exercises and solved problems in Chapter 4. The material they cover may appear on exams.

1. (**Decidable languages, part I**)

   (a) (COMPLEMENT$_{\mathsf{DFA}}$) Consider the problem of determining if the languages of two given DFAs are complements of each other.

      i. Formulate this problem as a language COMPLEMENT$_{\mathsf{DFA}}$ and show that it is decidable.

      ii. Why does a similar approach fail to show that COMPLEMENT$_{\mathsf{PDA}}$ is decidable? (This language is defined analogously to COMPLEMENT$_{\mathsf{DFA}}$, with PDAs instead of DFAs as inputs.)

   (b) Let $A = \{\langle G, R\rangle |\ G$ is a context-free grammar, R is a regular expression, and $L(G) \subseteq L(R)\}$.

      i. Give an example of G and R such that $\langle G, R\rangle \in A$.

      ii. Show that A is decidable.

2. (**Decidable languages, part II)**

   (a) (**Persistent variable)** A variable A in CFG G is *persistent* if it appears in every derivation of every string w in G. Given a CFG G and a variable A, consider the problem of testing whether A is persistent. Formulate this problem as a language and show that it is decidable.

   (b) (**Sortedness-obsessed NFA)** Consider strings over an alphabet $\Sigma = \{1, 2, \ldots, k\}$, where k is a natural number. A string w of length n over Σ is called *sorted* if $w = w_1 w_2 \ldots w_n$ where all characters $w_1, w_2, \ldots, w_n \in \Sigma$ and $w_1 \leq w_2 \leq \cdots \leq w_n$. For example, the string 1112778 is sorted, but the strings 5531 and 44427 are not. (Vacuously, the empty string is sorted.) We call an NFA *sortedness-obsessed* if every string it accepts is sorted. (But it does not have to accept all such strings). Consider the problem of determining whether a given NFA is sortedness-obsessed. Formulate this problem as a language and show that it is decidable.

      Hint: There are some solved problems in Chapter 4 that can help you.

3. (**Countable sets**)

   (a) Let $\mathcal{S}$ be the set of squares whose vertices have integer coordinates. For example, the square with vertices at $(1, 1), (1, 4), (-2, 4), (-2, 1)$ is in $\mathcal{S}$. Show that $\mathcal{S}$ is countable.

(b) A *polynomial in variable x* is an expression of the form $c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_dx^d$, where d is a non-negative integer and $c_0, \cdots, c_d$ are constants, called coefficients. Let $\mathcal{P}$ be the set of polynomials with integer coefficients. Show that $\mathcal{P}$ is countable.

(c) Let $\mathcal{F}$ be the set of all finite languages over alphabet $\{0, 1\}$. Show that $\mathcal{F}$ is countable.

4* (**Optional, no collaboration is allowed**) In this problem, you are asked to think about *LOSS* operations on languages. Each LOSS operation is specified by a set Σ of symbols. When the *LOSS of* Σ operation, denoted by $LOSS_\Sigma$ is applied to a string w, all characters in Σ disappear from w. For example, $LOSS_{\{\texttt{1},\texttt{3}\}}(\texttt{121023}) = \texttt{202}$ and $LOSS_{\{\texttt{1},\texttt{3}\}}(\texttt{241222}) = \texttt{24222}$, whereas $LOSS_{\{\texttt{1},\texttt{3}\}}(\texttt{24222}) = \texttt{24222}$. To apply $LOSS_\Sigma$ to a language, we apply it to every string in the language. For example, $LOSS_{\{\texttt{0},\texttt{1},\texttt{3}\}}(\texttt{0}^*\texttt{1}^*\texttt{2}^*\texttt{3}) = \texttt{2}^*$. More formally,

$$LOSS_\Sigma(L) = \{LOSS_\Sigma(w) \mid w \in L\}.$$

(a) Prove that the class of regular languages is closed under the LOSS operations.

(b) Prove that the class of decidable languages is not closed under the LOSS operations.