CMPSC 464: *Intro to Theory of Computation* — Professor Sofya Raskhodnikova
Pennsylvania State University — February 4, 2016

Homework 4 – Due Thursday, February 11, 2016 before the lecture

Please refer to the general information handout for the full homework policy.

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the course staff if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Sipser, 2.30–2.32. Practice converting CFGs to PDAs and vice versa.

Problems

1. (**CFGs**) Give CFGs that generate the following languages. Your CFGs should have at most 2 variables. Unless specified otherwise, the alphabet is $\Sigma = \{0, 1\}$.

   (a) $L_1 = \{w \mid w \in \{\texttt{0,1}\}^*$ and w contains a number of 1s that is divisible by $3\}$.

   (b) Let $\Sigma_2 = \{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$.
   $L_2 = \big\{w \in \Sigma_2^* \mid$ the bottom row of w is the reverse of the top row of $w\big\}$.

   (c) $L_3 = \{w \mid w$ is a balanced string of parentheses and brackets$\}$. The alphabet here is $\Sigma = \{\texttt{(,),[,]}\}$.

   (d) L_4 is the collection of all strings that contain at least one 1 in their second half (if the string is of odd length, we exclude the middle symbol to construct the second "half"). In other words, $L_4 = \{uv \mid u \in \Sigma^*, v \in \Sigma^*\texttt{1}\Sigma^* \text{ and } |u| \geq |v|\}$.

   (e) For one of the parts above, prove that your CFG generates the right language. Do not forget to argue that: (1) each string in the language is generated and (2) each generated string is in the language.

2. (**CFGs and PDAs**)

   (a) (**CFG to PDA**) What language does the following CFG generate?

   $$\begin{aligned} R &\to XRX \mid S \\ S &\to \texttt{a}T\texttt{b} \mid \texttt{b}T\texttt{a} \\ T &\to XTX \mid \varepsilon \\ X &\to \texttt{a} \mid \texttt{b} \end{aligned}$$

   (b) Convert the CFG from part (a) to an equivalent PDA, using the procedure described in class and in the book.

   (c) (**Erroneous proof of closure under** $*$) Give a counterexample to show that the following construction fails to prove that the class of context-free languages is closed under star. Let A be a CFL that is generated by the CFG $G = (V, \Sigma, R, S)$. Add the new rule $S \to SS \mid \varepsilon$ and call the resulting grammar G'. This grammar is supposed to generate A^*.

(d) Prove that the class of context-free languages is closed under the union and reverse operations. (You should also be able to show, but do not hand in, that (1) it is closed under star and concatenation; (2) it is not closed under intersection and complement – see Sipser, exercise 2.2 for hints.)

3. (**Non-CFLs**) Prove that the following languages are not context-free.

(a) $A = \{0^n 1^n 0^n \mid n \geq 0\}$.

(b) Let $\Sigma_2 = \{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$.
$B = \big\{w \in \Sigma_2^* \mid$ the bottom row of w is the reverse of the top row of w **and** the number of $\mathtt{0}$s and $\mathtt{1}$s in w is the same$\big\}$.

(c) $C = \{\mathtt{a}^i \mathtt{b}^j \mathtt{c}^k \mathtt{d}^l \mid i, j, k, l \geq 0$ and if $i = 1$ then $j = k = l\}$. (*Careful:* C satisfies the pumping lemma for CFLs! Make sure you understand why, but you don't need to write it down.)