Algorithm Design and Analysis

CSE 565

LECTURE 41

Approximation Algorithms

- The Pricing Method
  - Vertex Cover

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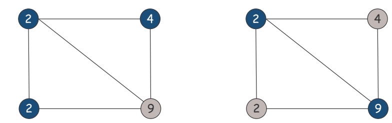
Weighted Vertex Cover

Weighted vertex cover. Given a graph G with vertex weights, find a vertex cover of minimum weight.

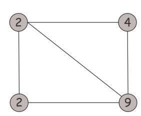


Pricing Method.

Idea. Each edge must be covered by some vertex. Edge e pays price $p_e \geq 0$ to get covered.

Fairness. Edges incident to vertex i should pay $\leq w_i$ in total.

$$\text{for each vertex } i: \sum_{e=(i,j)} p_e \leq w_i$$



Fairness Lemma. For any vertex cover S and any fair prices p_e: $\sum_e p_e \leq w(S)$.

Proof.

$$\sum_{e\in E} p_e \;\leq\; \sum_{i\in S} \sum_{e=(i,j)} p_e \;\leq\; \sum_{i\in S} w_i \;=\; w(S).$$

(first ≤: each edge e covered by at least one node in S; second ≤: sum fairness inequalities for each node in S) ▪

Pricing Method

Pricing method. Set prices and find vertex cover simultaneously.

```
Weighted-Vertex-Cover-Approx(G=(V,E), w) {
   foreach e in E
      pe ← 0
   foreach v in V                                              totali = wi
      totalv ← 0                                                    ↓

   while (∃ edge (i,j) such that neither i nor j is tight)
      select such an edge e
      \\give pe maximum increase without violating fairness
      increase ← max(wi-totali,wj-totalj)
      pe ← pe + increase
      totali ← totali + increase
      totalj ← totalj + increase

   S ← set of all tight nodes
   return S
}
```

Pricing Method

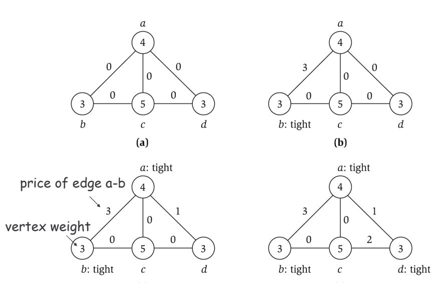


Figure 11.8

Pricing Method: Analysis

Theorem. Pricing method gives a 2-approximation.

Pf.

- Algorithm terminates since at least one new node becomes tight after each iteration of while loop.
- Let S = set of all tight nodes upon termination of algorithm. S is a vertex cover: if some edge i-j is uncovered, then neither i nor j is tight. But then while loop would not terminate.
- Let S* be an optimal vertex cover. We show $w(S) \leq 2w(S^*)$.

$$w(S) = \sum_{i\in S} w_i = \sum_{i\in S} \sum_{e=(i,j)} p_e \;\leq\; \sum_{i\in V} \sum_{e=(i,j)} p_e \;=\; 2\sum_{e\in E} p_e \;\leq\; 2w(S^*).$$

(all nodes in S are tight; $S \subseteq V$, prices ≥ 0; each edge counted twice; fairness lemma) ▪

Running time: O(m).

Weighted Vertex Cover

Theorem. 2-approximation algorithm for weighted vertex cover.

Theorem. [Dinur-Safra 2001] If $P \neq NP$, then no ρ-approximation for $\rho < 1.3607$, even with unit weights.

↖ $10\sqrt{5} - 21$

Open research problem. Close the gap.