

# Algorithm Design and Analysis

CSE  
565

## LECTURE 31 Network Flow

- Application: Edge Disjoint Paths
- Extensions

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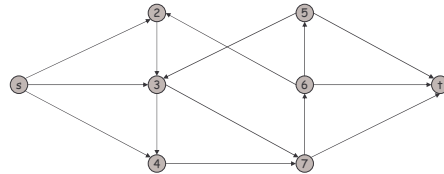
S. Raskhodnikova; based on slides by K. Wayne

### Edge Disjoint Paths

Problem. Given a digraph  $G = (V, E)$  and two nodes  $s$  and  $t$ , find the max number of edge-disjoint  $s$ - $t$  paths.

Def. Two paths are **edge-disjoint** if they have no edge in common.

Ex: communication networks.

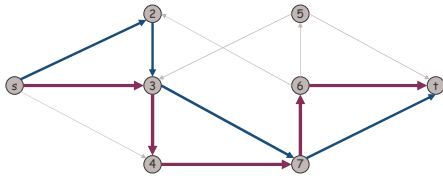


### Edge Disjoint Paths

Disjoint path problem. Given a digraph  $G = (V, E)$  and two nodes  $s$  and  $t$ , find the max number of edge-disjoint  $s$ - $t$  paths.

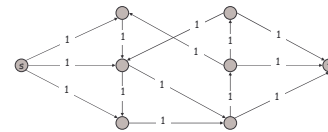
Def. Two paths are **edge-disjoint** if they have no edge in common.

Ex: communication networks.



### Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



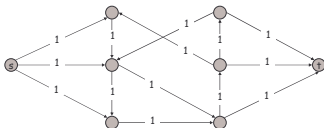
Theorem. Max number edge-disjoint  $s$ - $t$  paths equals max flow value.

Pf.  $\leq$

- Suppose there are  $k$  edge-disjoint paths  $P_1, \dots, P_k$ .
- Set  $f(e) = 1$  if  $e$  participates in some path  $P_i$ ; else set  $f(e) = 0$ .
- Since paths are edge-disjoint,  $f$  is a flow of value  $k$ .

### Edge Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Theorem. Max number edge-disjoint  $s$ - $t$  paths equals max flow value.

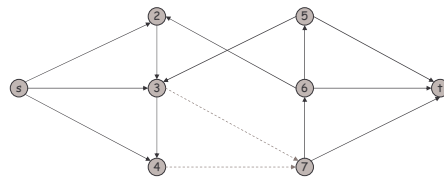
Pf.  $\geq$

- Suppose max flow value is  $k$ .
- Integrality theorem  $\Rightarrow$  there exists 0-1 flow  $f$  of value  $k$ .
- Consider edge  $(s, u)$  with  $f(s, u) = 1$ .
  - by conservation, there exists an edge  $(u, v)$  with  $f(u, v) = 1$
  - continue until reach  $t$ , always choosing a new edge
- Produces  $k$  (not necessarily simple) edge-disjoint paths.
  - can eliminate cycles to get simple paths if desired

### Network Connectivity

Network connectivity. Given a digraph  $G = (V, E)$  and two nodes  $s$  and  $t$ , find min number of edges whose removal disconnects  $t$  from  $s$ .

Def. A set of edges  $F \subseteq E$  **disconnects  $t$  from  $s$**  if each  $s$ - $t$  path uses at least one edge in  $F$ .

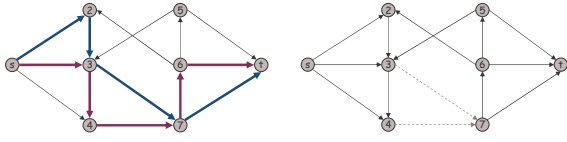


### Edge Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf.  $\leq$

- Suppose the removal of  $F \subseteq E$  disconnects t from s, and  $|F| = k$ .
- Each s-t paths uses at least one edge of F. Hence, the number of edge-disjoint paths is at most k. •

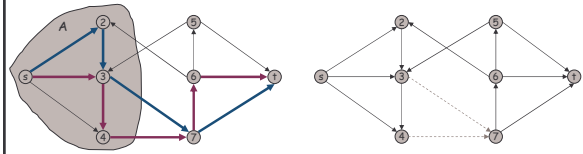


### Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf.  $\geq$

- Suppose max number of edge-disjoint paths is k.
- Then max flow value is k.
- Max-flow min-cut  $\Rightarrow$  cut (A, B) of capacity k.
- Let F be set of edges going from A to B.
- $|F| = k$  and disconnects t from s. •



## 7.7 Extensions to Max Flow

### Circulation with Demands

Circulation with demands.

- Directed graph  $G = (V, E)$ .
  - Edge capacities  $c(e)$ ,  $e \in E$ .
  - Node supply and demands  $d(v)$ ,  $v \in V$ .
- $\downarrow$   
 demand if  $d(v) > 0$ ; supply if  $d(v) < 0$ ; transshipment if  $d(v) = 0$

Def. A **circulation** is a function that satisfies:

- For each  $e \in E$ :  $0 \leq f(e) \leq c(e)$  (capacity)
- For each  $v \in V$ :  $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$  (conservation)

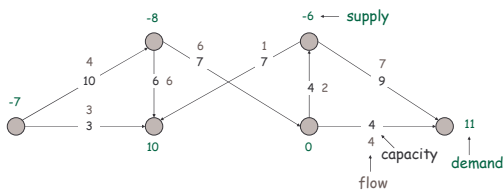
Circulation problem: given  $(V, E, c, d)$ , does there exist a circulation?

### Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

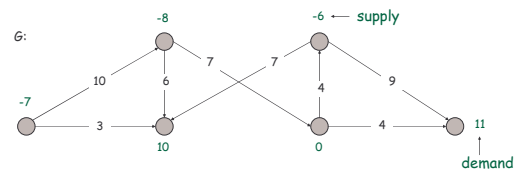
$$\sum_{v: d(v) > 0} d(v) = \sum_{v: d(v) < 0} -d(v) =: D$$

Proof. Sum conservation constraints for every demand node v.



### Circulation with Demands

Max flow formulation.

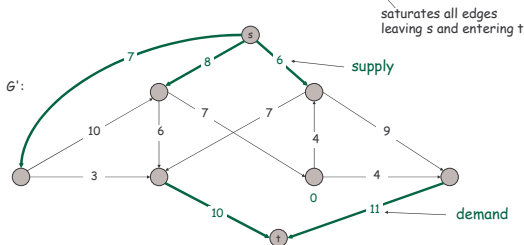


### Circulation with Demands

#### Max flow formulation.

- Add new source  $s$  and sink  $t$ .
- For each  $v$  with  $d(v) < 0$ , add edge  $(s, v)$  with capacity  $-d(v)$ .
- For each  $v$  with  $d(v) > 0$ , add edge  $(v, t)$  with capacity  $d(v)$ .

· Claim:  $G$  has circulation iff  $G'$  has max flow of value  $D$ .



### Circulation with Demands

**Integrality theorem.** If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

**Proof.** Follows from max flow formulation and integrality theorem for max flow.

**Characterization.** Given  $(V, E, c, d)$ , there does **not** exist a circulation iff there exists a node partition  $(A, B)$  such that  $\sum_{v \in B} d_v > \text{cap}(A, B)$

**Proof idea.** Look at min cut in  $G'$ .

demand by nodes in  $B$  exceeds supply of nodes in  $B$  plus max capacity of edges going from  $A$  to  $B$

### Circulation with Demands and Lower Bounds

#### Feasible circulation.

- Directed graph  $G = (V, E)$ .
- Edge capacities  $c(e)$  and lower bounds  $\ell(e)$ ,  $e \in E$ .
- Node supply and demands  $d(v)$ ,  $v \in V$ .

**Definition.** A **circulation** is a function that satisfies:

- For each  $e \in E$ :  $\ell(e) \leq f(e) \leq c(e)$  (capacity)
- For each  $v \in V$ :  $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$  (conservation)

**Circulation problem with lower bounds.** Given  $(V, E, \ell, c, d)$ , does there exist a circulation?

### Circulation with Demands and Lower Bounds

#### Idea. Model lower bounds with demands.

- Send  $\ell(e)$  units of flow along edge  $e$ .
- Update demands of both endpoints.



**Theorem.** There exists a circulation in  $G$  iff there exists a circulation in  $G'$ . If all demands, capacities, and lower bounds in  $G$  are integers, then there is a circulation in  $G$  that is integer-valued.

**Proof sketch.**  $f(e)$  is a circulation in  $G$  iff  $f'(e) = f(e) - \ell(e)$  is a circulation in  $G'$ .