Algorithm Design and Analysis

CSE 565

LECTURES 29 and 30

Network Flow

- Application:
  Bipartite Matching

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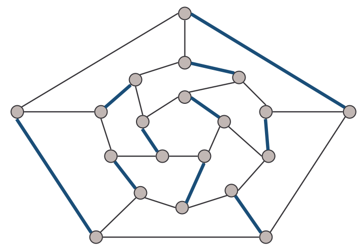
Matching

Matching.

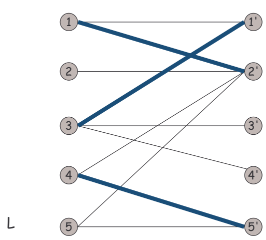
- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



Bipartite Matching

Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in M.
- Max matching: find a max cardinality matching.



matching
1-2', 3-1', 4-5'

Bipartite Matching

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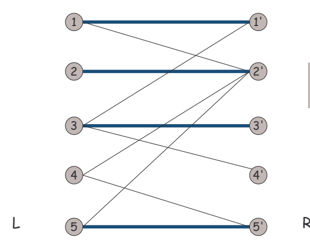
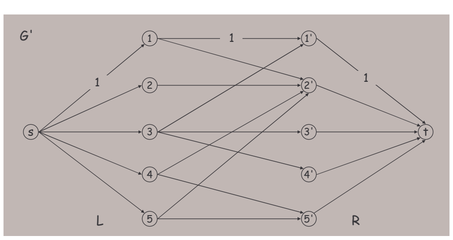


max matching
1-1', 2-2', 3-3' 4-4'

Bipartite Matching

Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from L to R, and assign capacity 1 (or infinity).
- Add source s, and capacity 1 edges from s to each node in L.
- Add sink t, and capacity 1 edges from each node in R to t.

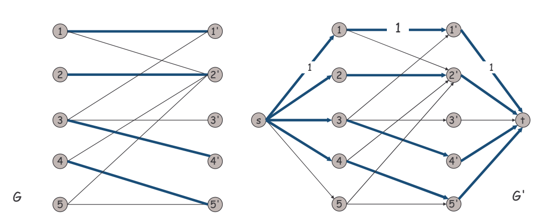


Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

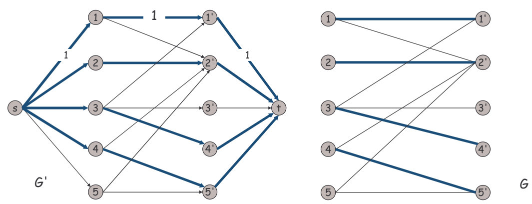
Pf. $\leq$

- Given max matching M of cardinality k.
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has cardinality k. ▪



Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.
Pf. $\geq$
- Let f be a max flow in G' of value k.
- Integrality theorem $\Rightarrow$ k is integral; all capacities are 1 $\Rightarrow$ f is 0-1.
- Consider M = set of edges from L to R with $f(e) = 1$.
  - each node in L and R participates in at most one edge in M
  - $|M| = k$: consider cut $(L \cup s, R \cup t)$ ▪



Perfect Matching

Def. A matching $M \subseteq E$ is perfect if each node appears in exactly one edge in M.

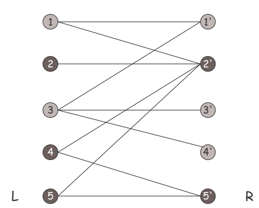
Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings.
- Clearly we must have $|L| = |R|$.
- What other conditions are necessary?
- What conditions are sufficient?

Perfect Matching

Notation. Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S.

Observation. If a bipartite graph $G = (L \cup R, E)$, has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.
Pf. Each node in S has to be matched to a different node in N(S).

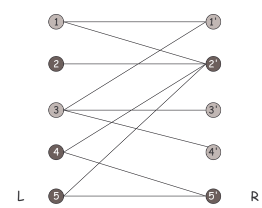


No perfect matching:
$S = \{ 2, 4, 5 \}$
$N(S) = \{ 2', 5' \}$.

Marriage Theorem

Marriage Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. $\Rightarrow$ This was the previous observation.



No perfect matching:
$S = \{ 2, 4, 5 \}$
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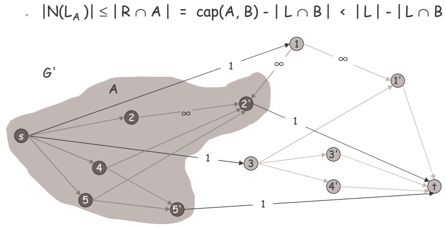
Proof of Marriage Theorem

Pf. $\Leftarrow$ Suppose G does not have a perfect matching.
- Formulate as a max flow problem with ∞ constraints on edges from L to R and let (A, B) be min cut in G'.
- By max-flow min-cut, $cap(A, B) < |L|$.
- Choose $S = L \cap A$.
- $cap(A, B) = |L \cap B| + |R \cap A|$.
- Since min cut can't use ∞ edges: $N(S) \subseteq R \cap A$.
- $|N(L_A)| \leq |R \cap A| = cap(A, B) - |L \cap B| < |L| - |L \cap B| = |S|$. ▪



$S = \{2, 4, 5\}$
$L \cap B = \{1, 3\}$
$R \cap A = \{2', 5'\}$
$N(S) = \{2', 5'\}$

Bipartite Matching: Running Time

Which max flow algorithm to use for bipartite matching?
- Generic augmenting path: $O(m \text{ val}(f^*)) = O(mn)$.
- Capacity scaling: $O(m^2 \log C) = O(m^2)$.
- Shortest augmenting path (not covered in class): $O(m\, n^{1/2})$.

Non-bipartite matching.
- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m\, n^{1/2})$. [Micali-Vazirani 1980]