

Algorithm Design and Analysis



LECTURE 2 An Example Problem

- Stable matching problem

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S. Raskhodnikova; based on slides by K. Wayne.



Stable Matching Problem

- **Goal:** Given n men and n women, find a "suitable" matching.
 - Participants rate members of opposite sex.
 - Each man lists women in order of preference from best to worst.
 - Each woman lists men in order of preference from best to worst.

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓	favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Xavier		Amy	Bertha	Clare		Amy	Yancey	Xavier	Zeus	
Yancey		Bertha	Amy	Clare		Bertha	Xavier	Yancey	Zeus	
Zeus		Amy	Bertha	Clare		Clare	Xavier	Yancey	Zeus	

Men's Preference Profile

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Amy		Yancey	Xavier	Zeus	
Bertha		Xavier	Yancey	Zeus	
Clare		Xavier	Yancey	Zeus	

Women's Preference Profile

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Review Questions

- In terms of n , what is the length of the input to the Stable Matching problem, i.e., the number of entries in the tables?
(Answer: $2n^2$)
- What is a simple lower bound on the running time of any algorithm for Stable Matching?
(Answer: $\geq n$. The algorithm needs that much time to print the output.)

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Stable Matching Problem

- **Perfect matching:**
 - Each man gets exactly one woman.
 - Each woman gets exactly one man.
- **Stability:** no incentive for some pair of participants to undermine assignment by joint action.
 - In matching M , an unmatched pair $m-w$ is **unstable** if man m and woman w prefer each other to their current partners.
 - Unstable pair $m-w$ could each improve by eloping.
- **Stable matching:** perfect matching with no unstable pairs.
- **Problem:** Given the preference lists of n men and n women, find a stable matching if one exists.

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Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓	favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Xavier		Amy	Bertha	Clare		Amy	Yancey	Xavier	Zeus	
Yancey		Bertha	Amy	Clare		Bertha	Xavier	Yancey	Zeus	
Zeus		Amy	Bertha	Clare		Clare	Xavier	Yancey	Zeus	

Men's Preference Profile

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Amy		Yancey	Xavier	Zeus	
Bertha		Xavier	Yancey	Zeus	
Clare		Xavier	Yancey	Zeus	

Women's Preference Profile

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Stable Matching Problem

- Q. Is assignment X-C, Y-B, Z-A stable?
- A. No. Bertha and Xavier will hook up.

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓	favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Xavier		Amy	Bertha	Clare		Amy	Yancey	Xavier	Zeus	
Yancey		Bertha	Amy	Clare		Bertha	Xavier	Yancey	Zeus	
Zeus		Amy	Bertha	Clare		Clare	Xavier	Yancey	Zeus	

Men's Preference Profile

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Amy		Yancey	Xavier	Zeus	
Bertha		Xavier	Yancey	Zeus	
Clare		Xavier	Yancey	Zeus	

Women's Preference Profile

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CSE 585 Stable Matching Problem

- Q. Is assignment X-A, Y-B, Z-C stable?

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓		favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Xavier		Amy	Bertha	Clare		Amy		Yancey	Xavier	Zeus	
Yancey		Bertha	Amy	Clare		Bertha		Xavier	Yancey	Zeus	
Zeus		Amy	Bertha	Clare		Clare		Xavier	Yancey	Zeus	

Men's Preference Profile Women's Preference Profile

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CSE 585 Stable Matching Problem

- Q. Is assignment X-A, Y-B, Z-C stable?
- A. Yes. X and Y got their first choice; Z is the last choice for every woman. No man can participate in an unstable pair.

	favorite ↓	1 st	2 nd	3 rd	least favorite ↓		favorite ↓	1 st	2 nd	3 rd	least favorite ↓
Xavier		Amy	Bertha	Clare		Amy		Yancey	Xavier	Zeus	
Yancey		Bertha	Amy	Clare		Bertha		Xavier	Yancey	Zeus	
Zeus		Amy	Bertha	Clare		Clare		Xavier	Yancey	Zeus	

Men's Preference Profile Women's Preference Profile

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CSE 585 Existence of Stable Matching

- Q. Do stable matchings always exist?
- A. Not obvious a priori.

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CSE 585 Stable Roommate Problem

- Stable roommate problem**
 - 2n people; each person ranks others from 1 to 2n-1.
 - Assign roommate pairs so that no unstable pairs.

	1 st	2 nd	3 rd	
Adam	B	C	D	A-B, C-D ⇒ B-C unstable A-C, B-D ⇒ A-B unstable A-D, B-C ⇒ A-C unstable
Bob	C	A	D	
Chris	A	B	D	
Doofus	A	B	C	

- Observation.** Stable matchings do not always exist for stable roommate problem.

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CSE 585 Propose-and-Reject Algorithm

- Propose-and-reject algorithm. [Gale-Shapley 1962]

```

Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
  Choose such a man m
  w = 1st woman on m's list to whom m has not yet proposed
  if (w is free)
    assign m and w to be engaged
  else if (w prefers m to her fiancé m')
    assign m and w to be engaged, and m' to be free
  else
    w rejects m
}
    
```

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CSE 585 Proof of Correctness: Termination

- Claim.** Algorithm terminates after at most n^2 iterations of while loop.
- Pf.** Each time through the loop a man proposes to a new woman. There are only n^2 possible proposals. ▀

	1 st	2 nd	3 rd	4 th	5 th		1 st	2 nd	3 rd	4 th	5 th
Victor	A	B	C	D	E	Amy	W	X	Y	Z	V
Wyatt	B	C	D	A	E	Bertha	X	Y	Z	V	W
Xavier	C	D	A	B	E	Clare	Y	Z	V	W	X
Yancey	D	A	B	C	E	Diane	Z	V	W	X	Y
Zeus	A	B	C	D	E	Erika	V	W	X	Y	Z

An instance where $n(n-1) + 1$ proposals required

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