Lecture 19:
Essential and Fundamental Matrices

Epipolar Geometry

image1  image 2

Epipole: location of cam2 as seen by cam1.
Epipole: location of cam1 as seen by cam2.

Corresponding points lie on conjugate epipolar lines

This Lecture...

image1  image 2

Given a point in one image, how do we determine the corresponding epipolar line to search along in the second image?

Essential Matrix

The essential and fundamental matrices are 3x3 matrices that "encode" the epipolar geometry of two views.

Motivation: Given a point in one image, multiplying by the essential/fundamental matrix will tell us which epipolar line to search along in the second view.

Essential Matrix

\[ P_r = R(P_l - T) \]
\[ \Rightarrow P_r^T RSP_l = 0 \]
\[ \Rightarrow P_r^T EP_l = 0 \]

\[ E = RS \] is "essential matrix"
**Essential Matrix Properties**

\[ E = RS \]

- has rank 2
  \( \Rightarrow \) has both a left and right nullspace (important!!!)
- depends only on the **EXTRINSIC** Parameters (R & T)

**Longuet-Higgins equation**

\[ P_r^T EP_l = 0 \]

\[ p_l = \frac{f_l}{Z_l} I_l \quad p_r = \frac{f_r}{Z_r} I_r \]

\[ \begin{pmatrix} \frac{Z_r}{f_r} p_r \end{pmatrix}^T E \begin{pmatrix} \frac{Z_l}{f_l} p_l \end{pmatrix} = 0 \]

\[ p_r^T E p_l = 0 \]

**Longuet-Higgins Makes Sense**

- Note, there is nothing magic about Longuet-Higgins equation.
- A film point can also be thought of as a viewing ray. They are equivalent.
  - \((u,v)\) 2D film point
  - \((u,v,f)\) 3D point on film plane
  - \(k(u,v,f)\) viewing ray into the scene
  - \(k(X, Y, Z)\) ray through point P in the scene
    [hint: \(k=f/Z\), and we have \(u=fX/Z, v=fY/Z\)].

**Epipolar Lines**

- Let \(l\) be a line in the image:
  \[ au + bv + c = 0 \]
- Using homogeneous coordinates:
  \[ \vec{p} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \quad \vec{t} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \]
  \[ \vec{p}^T \vec{t} = \vec{t}^T \vec{p} = 0 \]

**Epipolar Lines**

- Remember:
  \[ p_r^T E p_l = 0 \]
  \[ \vec{t}_r = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \]
  \[ p_r \text{ belongs to epipolar line in the right image defined by} \]
  \[ \vec{t}_r = E p_l \]
Epipolar Lines

- Remember:
  \[ p_l^T E p_l = 0 \]
  \[ l_l = E^T p_r \]

Epipoles

- Remember: epipoles belong to the epipolar lines
  \[ e_r^T E p_l = 0 \quad p_r^T E e_l = 0 \]
- And they belong to all the epipolar lines
  \[ e_r^T E = 0 \quad E e_l = 0 \]

We can use this to compute the location of the epipoles. There will be an example, shortly...

Essential Matrix Summary

Longuet-Higgins equation
  \[ p_r^T E p_l = 0 \]

Epipolar lines:
  \[ \tilde{p}_r^T \tilde{l}_r = 0 \quad \tilde{p}_l^T \tilde{l}_l = 0 \]
  \[ \tilde{l}_r = E p_l \quad \tilde{l}_l = E^T p_r \]

Epipoles:
  \[ e_r^T E = 0 \quad E e_l = 0 \]

Fundamental Matrix

The essential matrix uses \textit{camera} coordinates

To use image coordinates we must consider the \textit{intrinsic} camera parameters:

\[ p_l = \hat{M}_l p_l \]
\[ p_r = \hat{M}_r p_r \]

Fundamental Matrix Properties

\[ F = M_r^{-T} R S M_l^{-1} \]
- has rank 2
- depends on the \textit{intrinsic} and \textit{extrinsic} Parameters (f, etc.; R & T)

Analogous to essential matrix. The fundamental matrix also tells how pixels (points) in each image are related to epipolar lines in the other image.

short version: The same equation works in pixel coordinates too!
Example

\[ F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \]

\[ x = 343.5300 \quad y = 221.7005 \]

normalize so sum of squares of first two terms is 1 (optional)

Example

\[ F = \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \]

\[ x = 343.5300 \quad y = 221.7005 \]

Example

\[ (205.5526 \quad 80.5 \quad 1.0) \begin{pmatrix} -0.00310695 & -0.0025646 & 2.96584 \\ -0.028094 & -0.00771621 & 56.3813 \\ 13.1905 & -29.2007 & -9999.79 \end{pmatrix} \]

\[ L = (0.0010 & -0.0030 & -0.4851) \]

\[ \Rightarrow (0.3211 & -0.9470 & -151.39) \]

Example
Example

where is the epipole?

\[ F \cdot e_L = 0 \]

vector in the right nullspace of matrix \( F \)

However, due to noise, \( F \) may not be singular. So instead, next best thing is eigenvector associated with smallest eigenvalue of \( F \)

>> \[[u,d] = \text{eigs}(F' \cdot F)\]

\[ u = \begin{bmatrix} -0.0003 & -0.0618 & -0.9981 \\ -0.0056 & -0.9981 & 0.0618 \\ 1.0000 & 0.0001 & 0.0001 \end{bmatrix} \]

\( d = 1.0e8 \cdot \begin{bmatrix} -1.0000 & 0 & 0 \\ 0 & -0.0000 & 0 \\ 0 & 0 & -0.0000 \end{bmatrix} \)

eigenvector associated with smallest eigenvalue

>> \( uu = u(:,3) \)

\[ uu = \begin{bmatrix} -0.9981 & 0.0618 & 0.0001 \end{bmatrix} \]

>> \( uu / uu(3) \) : to get pixel coords

\(-19021.8 \quad 1177.97 \quad 1.0\)