Lecture 2: Intensity Surfaces and Gradients

Visualizing Images
Recall two ways of visualizing an image

Intensity pattern 2d array of numbers

We “see it” at this level Computer works at this level

Bridging the Gap
Motivation: we want to visualize images at a level high enough to retain human insight, but low enough to allow us to readily translate our insights into mathematical notation and, ultimately, computer algorithms that operate on arrays of numbers.

Images as Surfaces
Surface height proportional to pixel grey value (dark-low, light-high)

Examples
Note: see demoImSurf.m in matlab examples directory on course web site if you want to generate plots like these.

Examples
Mean = 164 Std = 1.8
Mean = 111  Std = 15.4

How does this visualization help us?
**Terrain Concepts**

Basic notions:
- Uphill / downhill
- Contour lines (curves of constant elevation)
- Steepness of slope
- Peaks/Valleys (local extrema)

More mathematical notions:
- Tangent Plane
- Normal vectors
- Curvature

Gradient vectors (vectors of partial derivatives) will help us define/compute all of these.

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**Math Example : 1D Gradient**

Consider function \( f(x) = 100 - 0.5 * x^2 \)

Gradient is \( \frac{df(x)}{dx} = - x \)

Geometric interpretation:
gradient at \( x_0 \) is slope of tangent line to curve at point \( x_0 \)

\[ \Delta y / \Delta x = \frac{df(x)}{dx} \bigg|_{x_0} \]

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Math Example: 1D Gradient

\[ f(x) = 100 - 0.5 * x^2 \]

\[ \frac{df(x)}{dx} = -x \]

Gradients on this side of peak are positive
Gradients on this side of peak are negative

Note: Sign of gradient at point tells you what direction to go to travel "uphill"

Math Example: 2D Gradient

\[ f(x,y) = 100 - 0.5 * x^2 - 0.5 * y^2 \]

\[ \frac{df(x,y)}{dx} = -x \quad \frac{df(x,y)}{dy} = -y \]

Gradient = \[ \frac{df(x,y)}{dx}, \frac{df(x,y)}{dy} \] = \[ -x, -y \]

Gradients are vector of partial derivs wrt x and y axes

Math Example: 2D Gradient

\[ f(x,y) = 100 - 0.5 * x^2 - 0.5 * y^2 \]

Gradient = \[ \frac{df(x,y)}{dx}, \frac{df(x,y)}{dy} \] = \[ -x, -y \]

Let \( g = [g_x, g_y] \) be the gradient vector at point/pixel \( (x_0, y_0) \)

Vector \( g \) points uphill (direction of steepest ascent)

Vector \(-g\) points downhill (direction of steepest descent)

Vector \( [g_y, -g_x] \) is perpendicular, and denotes direction of constant elevation. i.e. normal to contour line passing through point \( (x_0, y_0) \)

Math Example: 2D Gradient

\[ f(x,y) = 100 - 0.5 * x^2 - 0.5 * y^2 \]

Gradient = \[ \frac{df(x,y)}{dx}, \frac{df(x,y)}{dy} \] = \[ -x, -y \]

And so on for all points

Image Gradient

The same is true of 2D image gradients. The underlying function is numerical (tabulated) rather than algebraic. So need numerical derivatives.
**Numerical Derivatives**

See also T&V, Appendix A.2

Taylor Series expansion

\[ f(x + h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{3!}h^3f'''(x) + \mathcal{O}(h^4) \]

Manipulate:

\[ f(x + h) - f(x) = hf'(x) + \frac{1}{2}h^2f''(x) + \mathcal{O}(h^3) \]

Finite forward difference

\[ \frac{f(x + h) - f(x)}{h} = f'(x) + \mathcal{O}(h) \]

Finite backward difference

Taylor Series expansion

\[ f(x - h) = f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{3!}h^3f'''(x) + \mathcal{O}(h^4) \]

Manipulate:

\[ f(x) - f(x - h) = hf'(x) - \frac{1}{2}h^2f''(x) + \mathcal{O}(h^3) \]

\[ \frac{f(x) - f(x - h)}{h} = f'(x) + \mathcal{O}(h) \]

Finite backward difference

Taylor Series expansion

\[ f(x + h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{3!}h^3f'''(x) + \mathcal{O}(h^4) \]

Manipulate:

\[ f(x + h) = f(x - h) + 2hf'(x) + \frac{2}{3!}h^3f'''(x) + \mathcal{O}(h^4) \]

Finite central difference

\[ \frac{f(x + h) - f(x - h)}{2h} = f'(x) + \mathcal{O}(h^2) \]

Finite central difference

**Example: Temporal Gradient**

A video is a sequence of image frames I(x,y,t).

Consider the sequence of intensity values observed at a single pixel over time.

Each frame has two spatial indices x, y and one temporal (time) index t.
Temporal Gradient (cont)

What does the temporal intensity gradient at each pixel look like over time?

Example: Spatial Image Gradients

\[ \frac{I(x+1,y) - I(x-1,y)}{2} \]

\[ \frac{I(x,y+1) - I(x,y-1)}{2} \]

\[ I_x = \frac{dI(x,y)}{dx} \]

\[ I_y = \frac{dI(x,y)}{dy} \]

Functions of Gradients

Magnitude of gradient \( \text{sqrt}(I_x^2 + I_y^2) \)

Measures steepness of slope at each pixel

Angle of gradient \( \text{atan2}(I_y, I_x) \)

Denotes similarity of orientation of slope

What else do we observe in this image?

Enhanced detail in low contrast areas (e.g. folds in coat; imaging artifacts in sky)

Next Time: Linear Operators

Gradients are an example of linear operators, i.e. value at a pixel is computed as a linear combination of values of neighboring pixels.