From Last Time

Boolean Logic

Constants: 0 and 1

Operators: “&” and “|” and “!”

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!0 = 1
!1 = 0

The Lady or the Tiger

- A prisoner must choose between two rooms, one of which contains a lady, and the other a tiger.
- If he chooses the former, he marries the lady; if he chooses the latter, he gets eaten by the tiger.
- The king of a certain land decides to try his prisoners this way
- But he will put signs on the doors of the rooms, and tell the prisoners facts about the signs.
- If the prisoner is clever and can reason logically he’ll save his life

The First Day

- The king explained to the prisoners that each of the two rooms contained either a lady or a tiger,
- but it could be that there were tigers in both rooms, or ladies in both rooms.
- **Prisoner 1**: The doors had the following signs:
  
  Door 1: In this room there is a lady and in the other room there is a tiger.
  
  Door 2: In one of these rooms there is a lady and in one of these rooms there is a tiger.

- The king pointed to the signs on the doors and said
  
  "One of the signs is true and one of the signs is false."

Which door should the prisoner choose?

Representing this Problem

Consider the properties we need to represent:

- Each room contains either a lady or a tiger
- We can make statements about a particular room
- We can make statements about some room (without being specific)
- We can make statements about the signs
Representing this Problem
How can we represent these using symbols (whose values can be True or False)?

There may be many possible representations. We would like a representation that can only encode possible situations.

We shouldn’t be able to represent:

- There is a tiger and a lady in Room 1; OR
- There are three tigers in Room 2

One Solution: Boolean Logic!

Represent each room with a symbol: \( R_1, R_2 \)

- Assigning the value True to a symbol means the corresponding room contains a lady.
- Assigning the value False to a symbol means the corresponding room contains a tiger.

Does this satisfy our requirements?

We can now represent what is written on each sign.

One Solution

- Door 1: In this room there is a lady and in the other room there is a tiger.
- Using our representation: \( R_1 \) is TRUE and \( R_2 \) is FALSE
- In Logic: \( R_1 \& !R_2 \)
- Door 2: In one of these rooms there is a lady and in one of these rooms there is a tiger.
- (\( R_1 \) is TRUE and \( R_2 \) is FALSE) or (\( R_1 \) is FALSE and \( R_2 \) is TRUE)
- \((R_1\&!R_2) + (\neg R_1\&R_2)\)
- or \( R_1 \oplus R_2 \)

Observe that the value of these expressions is TRUE (1) when the signs are correct and FALSE (0) otherwise.

One Solution

How can we represent the King’s rules?

"One of the signs is true and one of the signs is false."

We can represent this rule by observing that this is the same as saying:

- Sign 1 is true and Sign 2 is false OR
- Sign 1 is false and Sign 2 is true

\( \text{Sign1} \oplus \text{Sign2} \)
The Representation of Prisoner 1

Door 1
In this room there is a lady and in the other room there is a tiger.

Door 2
In one of these rooms there is a lady and in one of these rooms there is a tiger.

Representing each sign:

- Sign 1: \( R_1 \& \neg R_2 \)
- Sign 2: \( R_1 \oplus R_2 \)

Now add that one sign is true and the other false:

\[ (R_1 \& \neg R_2) \oplus (R_1 \oplus R_2) \]

What can we do with this?

Exhaustive Enumeration

Given the formula

\[ (R_1 \& \neg R_2) \oplus (R_1 \oplus R_2) \]

How can we go about finding a solution? (What is a solution?)

How can a computer go about finding a solution?

- The simplest way is to exhaustively check all possibilities behind the doors
- How many are there?
- What are all the possibilities?

Enumerating Possible Solutions

We have two variables \( R_1 \) and \( R_2 \). Each of these can be either True (1) or False (0). This means we have Four possible situations:

1. \( R_1 = \text{True}; R_2 = \text{True} \)
2. \( R_1 = \text{True}; R_2 = \text{False} \)
3. \( R_1 = \text{False}; R_2 = \text{True} \)
4. \( R_1 = \text{False}; R_2 = \text{False} \)

- If a truth assignment makes the formula True, then the assignment satisfies all the conditions.
- If a truth assignment makes the formula False, then the assignment violates some condition.
Enumerating Possible Solutions

- Applying each possible assignment to our problem and simplifying, we see that only one reduces to 1

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<tr>
<th>(R1)</th>
<th>(R2)</th>
<th>((R1&amp;!R2) \oplus (R1 \oplus R2))</th>
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\(\Leftarrow\text{Solution}\)

- Generalizing, given \(N\) variables there are \(2^N\) truth assignments
- This number gets large very quickly, making it impractical for humans to enumerate \((2^{10} = 1024)\)

The Second Trial

Door 1
At least one of these rooms contains a lady.

Door 2
A tiger is in the other room

The king pointed to the signs on the doors and said

"They are either both true or both false."

Which door should the prisoner choose?

The king’s statement can be reworded as:

Sign 1 is true if and only if Sign 2 is true.

or

if Sign 1 is true then Sign 2 is true; AND if Sign 2 is true then Sign 1 is true.

Representing the Second
The statements

1. At least one of these rooms contains a lady
2. A tiger is in the other room

can be represented as

\[R1|R2\]
\[!R1\]

Now add that either both are true or both are false

\[((R1|R2)&!R1) \mid (!R1|R2)&!!R1)\]
Representing the Second
We can again apply all four truth assignments to see which ones satisfy the formula.

| $R_1$ | $R_2$ | $((R_1|R_2)\&!R_1)$ | $(!((R_1|R_2)\&!!R_1)$ |
|-------|-------|---------------------|---------------------|
| 0     | 0     | 0                   | 0                   |
| 0     | 1     | 1                   | ⇐ Solution          |
| 1     | 0     | 0                   | 0                   |
| 1     | 1     | 0                   | 0                   |

Simplifying Formulas
We can use the following rules to simplify a formula:

- The operators $\&$ and $|$ are associative and commutative: $A\&B = B\&A$ $A|B = B|A$ $(A\&B)\&C = A\&(B\&C)$ $(A|B)|C = A|(B|C)$
- Double Negation: $!!A = A$
- Distributivity: $A\&(B\&C) = (A\&B)\&(A\&C)$
- Distributivity: $A|(B\&C) = (A|B)\&(A|C)$
- DeMorgan’s $!(A|B) = !A\&!B$
- DeMorgan’s $!(A\&B) = !A\|!B$
- $A\&(A|B) = A|(A\&B) = A$
- $A\&!A = 0$, $A\|!A = 1$
- $A\&0 = 0$, $A\|0 = A$, $A\&1 = A$, $A\|1 = 1$

Simplifying Formulas
Applying these appropriately to $((R_1|R_2)\&!R_1)$ $(!((R_1|R_2)\&!!R_1)$

we get

$!R_1\&R_2$

which we can easily see has only one possible solution.

The Third Trial
Door 1
Either a tiger is in this room or a lady is in the other room.

Door 2
A lady is in the other room

The king pointed to the signs on the doors and said "Again, they are either both true or both false."

Which door should the prisoner choose?
Representing the Third Trial

The signs

1. Either a tiger is in this room or a lady is in the other room.
2. A lady is in the other room
   can be represented as
   \[ R_1 \lor R_2 \]
   \[ R_1 \]
   Now add that either both are true or both are false
   \[ (\neg R_1 \land \neg R_2) \lor (R_1 \land R_2) \]
   Can we simplify and solve this formula?

The Second Day

The King decided to make the second day’s puzzles more difficult.
He decided to use the following RULES for the signs:

1. If a lady is in Room 1, then the sign on that door is true; otherwise it is false.
2. If a lady is in Room 2, then the sign on that door is false; otherwise it is true.

Again, each room contained either a lady or a tiger, but not both.
But both rooms may contain ladies; or both tigers.

The Fourth Trial

For the fourth trial, the king put the same sign on both rooms:

Door 1
Both rooms contain ladies.

Door 2
Both rooms contain ladies.

Which room should the prisoner pick?

Representing the Fourth Problem

Rewording the King’s rules:

1. Sign 1 is true iff (if and only if) Room 1 contains a lady
2. Sign 2 is true iff (if and only if) Room 2 contains a tiger

Put another way:

1. if Room 1 contains a lady then Sign 1 is true
2. if Room 1 contains a tiger then Sign 1 is false
3. if Room 2 contains a lady then Sign 2 is false
4. if Room 2 contains a tiger then Sign 2 is true
### New Logical Operator

- The expression If A then B is called an *implication*.
- Logically, it means that whenever A is true, B must also be true.
- We write it as $A \supset B$ and read it as *A implies B*.

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### Representing the Fourth Problem

Both signs read

*Both rooms contain ladies.*

which we can encode as

$R_1 \& R_2$

Now let’s add the King’s rules (using the last wording above):

$R_1 \supset (R_1 \& R_2) \& \neg R_1 \supset (R_1 \& R_2) \&$

$R_2 \supset (R_1 \& R_2) \& \neg R_2 \supset (R_1 \& R_2)$

Can we simplify this formula?

A rule to help us:

\[(A \supset B) = \neg A \| B\]

### The Fifth Trial

The fifth trial began before the King had time to put the signs on the doors:

???
This room contains a tiger

???
Both rooms contain tigers

"Which sign goes on which door?" asked the prisoner.

"It doesn’t matter. You can solve the puzzle without that information," replied the King.

What does the King’s statement mean? How can we represent it?
The Fifth Trial (cont)

The King’s statement tells us that either:

1. The signs only make sense one way. (The other way produces a contradiction.) OR
2. The signs make sense either way and produce the same unique solution.

So we should consider both arrangements of signs:

1. This room contains a tiger
2. Both rooms contain tigers

and

1. Both rooms contain tigers
2. This room contains a tiger

Representing the Fifth Trial

Putting the signs this way

1. This room contains a tiger
2. Both rooms contain tigers

translates to

\[ R_1 \supset !R_1 \land \\neg R_1 \supset !!R_1 \land \\neg R_2 \supset (\neg R_1 \land \neg R_2) \land \\neg R_2 \supset (!\neg R_1 \land \neg R_2) \]

which has no solution. Why?

Representing the Fifth Trial

Hence, the signs must go this way:

1. Both rooms contain tigers
2. This room contains a tiger

and translate to

\[ R_1 \supset (!\neg R_1 \land \neg R_2) \land \\neg R_1 \supset (!\neg R_1 \land \neg R_2) \land \\neg R_2 \supset !R_2 \land \\neg R_2 \supset !!R_2 \]

Can we simplify this?

Complicating the Problem

Suppose the King decides to make things harder by either:

- Increasing the number of doors; or
- Allowing rooms to be empty

How can we accommodate these?