The Rules of Single-Player Craps

- First roll a pair of (assumed fair) six-sided dice.
  - If the outcome (dice sum) of the first roll $X$ is 7 or 11 then the player wins,
  - else if the $X \in \{2, 3, 12\}$ then the player loses ("craps out"),
  - else define the "point" $X \in \{4, 5, 6, 8, 9, 10\}$ and the player continues to roll the dice.

- For each subsequent (independent) roll of the dice:
  - If the player rolls the point $X$ then they win,
  - else if the player rolls 7 then they lose,
  - else they roll again.
Define $T$ as the number of rolls in the craps game.

Note that $\{T > 1\} = \{X \in A\}$, where $A = \{4, 5, 6, 8, 9, 10\}$.

Let $Y$ be the outcome of a dice toss independent of $X$.

The outcome of (last) toss $T > 1$ is $\sim (Y|Y \in \{X, 7\})$, i.e., is distributed as $Y$ conditioned on $Y \in \{X, Y\}$.

Let $W$ be the winning event.

Generally, all games of chance in casinos have the property that the probability of winning is less than half, $P(W) < 0.5$, i.e., one is more likely to lose.
The probability of winning craps

\[ P(W) = P(W, T = 1) + P(W, T > 1) \]
\[ = P(X \in \{7, 11\}) + \sum_{i \in A} P(X = i, Y = i | Y \in \{i, 7\}) \]
\[ = P(X \in \{7, 11\}) + \sum_{i \in A} P(X = i)P(Y = i | Y \in \{i, 7\}) \]
\[ = P(X \in \{7, 11\}) + \sum_{i \in A} P(X = i)^2 / P(Y \in \{i, 7\}) \]
\[ = \frac{6 + 2}{36} + \frac{(\frac{3}{36})^2}{\frac{3+6}{36}} + \frac{(\frac{4}{36})^2}{\frac{4+6}{36}} + \frac{(\frac{5}{36})^2}{\frac{5+6}{36}} + \frac{(\frac{5}{36})^2}{\frac{5+6}{36}} + \frac{(\frac{4}{36})^2}{\frac{4+6}{36}} + \frac{(\frac{3}{36})^2}{\frac{3+6}{36}} \]
\[ = \frac{244}{495} = 0.493 \]
Exercises

- Find the distribution of $T$. Hint: It is geometric.
- Argue that $W$ and $T$ are conditionally independent given \{ $T > 1$ \}.
- Simulate 1000 independent, complete craps games and plot the running sample mean and sample standard deviation of the empirical probability of winning.