Signals and Linear and Time-Invariant Systems in Discrete Time

- Properties of signals and systems (difference equations)
- Time-domain analysis
  - ZIR, system characteristic values and modes
  - ZSR, unit-pulse response and convolution
  - stability, eigenresponse and transfer function
- Frequency-domain analysis

Time-domain analysis of discrete-time LTI systems

- Discrete-time signals
- Difference equation single-input, single-output systems in discrete time
- The zero-input response (ZIR): characteristic values and modes
- The zero (initial) state response (ZSR): the unit-pulse response, convolution
- System stability
- The eigenresponse and (zero state) system transfer function
Discrete-time signal by sampling a continuous-time signal

- Consider a continuous-time signal $x : \mathbb{R} \rightarrow \mathbb{R}$ sampled every $T > 0$ seconds

$$x(kT + t_0) =: x[k] \quad \text{for } k \in \mathbb{Z},$$

where

- $t_0$ is the sampling time of the 0th sample, and
- $T$ is assumed less than the Nyquist sampling period of $x$, and
- $x[k]$ (with square brackets) is the $k$th sample itself.

- Here $x[\cdot]$ is a discrete-time signal defined on $\mathbb{Z}$.

Example of sampling with $t_0 = 0$ and positive signal $x$
Introduction to signals and systems in discrete time

- A discrete-time function (or signal) \( x : A \rightarrow B \) is one with countable (time) domain \( A \).
- We will take the range \( B = \mathbb{R} \) or \( B = \mathbb{C} \).
- Typically, we will herein take domain \( A = \mathbb{Z} \) or \( \mathbb{Z}^{\leq n} \) for some (finite) integer \( n \geq 0 \).
- Some properties of signals are as in continuous time: e.g., periodic, causal, bounded, even or odd.
- Similarly, some signal operations are as in continuous time: e.g., spatial shift/scale, superposition, time reflection, and (integer valued) time shift.

Time scaling: decimation and interpolation

- Time scaling can be implemented in continuous time prior to sampling at a fixed rate, or the sampling rate itself could be varied (again recall the Nyquist sampling rate).
- In discrete time, a signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) can be decimated (subsampled) by an integer factor \( L \neq 0 \) to create the signal \( x_L \) defined by
  \[
  x_L[k] = x[kL], \quad \forall k \in \mathbb{Z},
  \]
  i.e., \( x_L \) is defined only by every \( L \)th sample of \( x \).
- A discrete-time signal \( x \) can also be interpolated by an integer factor \( L > 0 \) to create \( x_L \) satisfying
  \[
  x_L[kL] = x[k], \quad \forall k \in \mathbb{Z}.
  \]
- For an interpolated signal \( x_L \), the values of \( x_L[r] \) for \( r \) not a multiple of \( L \) (i.e., \( \forall k \in \mathbb{Z} \) s.t. \( r \neq kL \)) can be set in different ways, e.g., between consecutive samples:
  - (piecewise constant) hold: \( x_L[r] = x_L[L \lfloor r/L \rfloor] = x[r/L] \)
  - linear interpolation:
    \[
    x_L[r] = x[L \lfloor r/L \rfloor] + \frac{r - L \lfloor r/L \rfloor}{L}(x[L \lfloor r/L \rfloor] + 1 - x[L \lfloor r/L \rfloor])
    \]
Time scaling: decimation and interpolation - Questions

- Is the functional mapping $x \rightarrow x_L$ causal for linear interpolation?
- Is the hold causal?
- **Exercise:** Show that if a periodic, continuous-time signal $x(t)$, with period $T_0$, is periodically sampled every $T$ seconds, then the resulting discrete-time signal $x[k]$ is periodic if and only if $T/T_0$ is rational.

Unit pulse $\delta$, unit step $u$, unit delay $\Delta$, and convolution *

- Some important signals in discrete time are as those in continuous time, e.g., polynomials, exponentials, unit step.
- In discrete time, rather than the (unit) impulse, there is unit pulse (Kronecker delta):
  \[
  \delta[k] = \begin{cases} 
  1 & \text{if } k = 0 \\
  0 & \text{else}
  \end{cases}
  \]
- Any discrete-time signal $x$ can thus be written as
  \[
  x[k] = \sum_{r=-\infty}^{\infty} x[r] \delta[k-r] = \sum_{r=-\infty}^{\infty} x[k-r] \delta[r]
  = (x \ast \delta)[k]
  \]
- or just $x = x \ast \delta$, i.e., the unit pulse $\delta$ is the identity of discrete-time convolution.
- Define the operator $\Delta$ as unit delay (time-shift), i.e., $\forall$ signals $y$ and $\forall k, r \in \mathbb{Z},$
  \[
  (\Delta y)[k] := y[k-r].
  \]
- The discrete-time unit step $u$ satisfies $\delta = u - \Delta u$, equivalently: $\forall k \in \mathbb{Z},$
  \[
  \delta[k] = u[k] - u[k-1] \quad \text{and} \quad u[k] = \sum_{r=0}^{\infty} (\Delta^r \delta)[k] = \sum_{r=0}^{\infty} \delta[k-r].
  \]
Exercise: For any signal causal $f\{f[k], k \geq 0\}$, show that
\[
\forall k \geq 0, (f * u)[k] = \sum_{r=0}^{k} f[r].
\]

Exponential signals in discrete time

- Real-valued exponential (geometric) signals have the form $x[k] = A\gamma^k$, $k \in \mathbb{Z}$, where $A, \gamma \in \mathbb{R}$.
- Consider the scalar $z = \gamma e^{j\Omega} \in \mathbb{C}$ with $\gamma > 0$, $\Omega \in \mathbb{R}$, where again $j := \sqrt{-1}$.
- Generally, complex-valued exponential signals have the (polar) form
\[
x[k] = Ae^{j\phi}z^k = A\gamma^k e^{j(\Omega k + \phi)}, \quad k \in \mathbb{Z},
\]
where w.l.o.g. we can take $-\pi < \Omega, \phi \leq \pi$ and real $A > 0$.

Exercise: Show this complex-valued exponential is periodic if and only if $\Omega/\pi$ is rational.

- By the Euler-De Moivre identity,
\[
x[k] = A\gamma^k e^{j(\Omega k + \phi)} = A\gamma^k \cos(\Omega k + \phi) + jA\gamma^k \sin(\Omega k + \phi), \quad k \in \mathbb{Z}.
\]
• In the figure, \( f \) is an input signal that is being transformed into an output signal, \( y \), by the depicted system (box).

• To emphasize this functional transformation, and clarify system properties, we will write the output signal \( (\text{i.e., system “response” to the input } f) \) as

\[
y = Sf,
\]

where, again, we are making a statement about functional equivalence:

\[
\forall k \in \mathbb{Z}, \quad y[k] = (Sf)[k].
\]

• Again, \( Sf \) is not \( S \) “multiplied by” \( f \), rather a functional transformation of \( f \).

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**SISO systems (cont)**

• The \( n \) signals \( \{x_1, x_2, \ldots, x_n\} \) are the internal states of the system.

• The states can be taken as outputs of unit-delay operators, \( \Delta \), i.e.,

\[
\forall k \in \mathbb{Z}, \quad (\Delta y)[k] = y[k-1].
\]

• Some properties of systems are as in continuous time: e.g., linear, time invariant, causal, memoryless, stable (with different conditions for stability as we shall see).
For linear and time-invariant systems in discrete time, relate output $y$ to input $f$ via difference equation in standard (time-advance operator) form:

$$\forall k \geq -n, \quad y[k + n] + a_{n-1}y[k + n - 1] + \ldots + a_1y[k + 1] + a_0y[k] = b_mf[k + m] + b_{m-1}f[k + m - 1] + \ldots + b_1f[k + 1] + b_0f[k],$$

given

- scalars $a_k$ for $0 \leq k \leq n$, with $a_n := 1$, and scalars $b_k$ for $0 \leq k \leq m$,
- $a_0 \neq 0$ or $b_0 \neq 0$ (so that $P, Q$ are of minimal degree), and
- initial conditions $y[-n], y[-n + 1], \ldots, y[-2], y[-1]$.

Compact representation of the above difference equation:

$$Q(\Delta^{-1}) y = P(\Delta^{-1}) f,$$

where polynomials

$$Q(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad P(z) = \sum_{k=0}^{m} b_k z^k,$$

$\Delta^{-1}$ is the unit time-advance operator: $(\Delta^{-1} y)[k] \equiv y[k + 1], (\Delta^{-r} y)[k] \equiv y[k + r]$.

Discussion: conditions for causality and difference equation in $\Delta$

**Exercise**: Show that the difference equation $Q(\Delta^{-1}) y = P(\Delta^{-1}) f$ is not causal if $\deg(P) = m > n = \deg(Q)$, i.e., the system is not proper.

An anti-causal difference equation can be implemented simply using memory to store a sliding window of prior values of the input $f$ and delaying the output.

**Example**: Decoding B (bidirectional) frames of MPEG video.
Numerical solution to difference equation by recursive substitution

- Given the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \), the input \( f[k] \) for \( k \geq 0 \), and initial conditions \( y[-n], \ldots, y[-1] \),
- one can recursively solve for \( y(y[k] \) for \( k \geq 0 \) by rewriting the system equation as
  \[
  y[k + n] = -\sum_{r=0}^{n-1} a_r y[k + r] + \sum_{r=0}^{m} b_r f[k + r] \quad \text{for} \quad k \geq -n
  \]
  \[
  \Rightarrow y[k] = -\sum_{r=0}^{n-1} a_r y[k + r - n] + \sum_{r=0}^{m} b_r f[k + r - n] \quad \text{for} \quad k \geq 0.
  \]
- For example, the difference equation in standard form,
  \[
  y[k + 1] + 3y[k] = 7f[k + 1] \quad \text{for} \quad k \geq -1,
  \]
  can be rewritten as
  \[
  y[k] = -3y[k - 1] + 7f[k] \quad \text{for} \quad k \geq 0.
  \]
- So, given \( f \) and \( y[-1] \) we can recursively compute
  \[
  y[0] = -3y[-1] + 7f[0], \quad y[1] = -3y[0] + 7f[1], \quad y[2] = -3y[1] + 7f[2], \quad \text{etc.}
  \]
- **Exercise:** If \( f = u \) and \( y[-1] = 7 \) then find \( y[3] \) for this example.

Approach to closed-form solution: ZIR and ZSR

- The total response \( y \) of \( P(\Delta^{-1})f = Q(\Delta^{-1})y \) to the given initial conditions and input \( f \) is a sum of two parts:
  - the ZSR, \( y_{ZS} \), which solves
    \[
    P(\Delta^{-1})f = Q(\Delta^{-1})y_{ZS} \quad \text{with zero i.c.'s, i.e., with} \quad 0 = y[-n] = \ldots = y[-1];
    \]
  - the ZIR, \( y_{ZI} \), which solves
    \[
    0 = Q(\Delta^{-1})y_{ZI} \quad \text{with the given initial conditions}.
    \]
- The total response \( y \) of the system to \( f \) and the given initial conditions is, by linearity,
  \[
  y = y_{ZI} + y_{ZS}.
  \]
- We will determine the ZIR by finding the characteristic modes of the system.
- We will determine the ZSR by convolution of the input with the (zero state) unit-pulse response, the latter also in terms of characteristic modes.
• Consider again the difference equation:
\[ \forall k \geq -1, \quad y[k + 1] + 3y[k] = 7f[k + 1], \]
• *i.e.* \( Q(z) = z + 3 \) with degree \( n = 1 \), and \( P(z) = 7z \) with degree \( m = 1 \),
• **Exercise:** Show that the following system corresponds to this difference equation.

\[ f \to \; 7 \; \to \; \Delta \; \to \; -3\Delta y \to \; y \]

• By recursive substitution, the total response is, \( \forall k \geq -1 \):
\[
y[k] = -3y[k - 1] + 7f[k] \\
= -3(-3y[k - 2] + 7f[k - 1]) + 7f[k] \\
= (-3)^2y[k - 2] - 3 \cdot 7f[k - 1] + 7f[k] \\
= \ldots \\
= (-3)^{k+1}y[-1] + \sum_{r=0}^{k} 7(-3)^{k-r}f[r] \\
= (-3)^{k+1}y[-1] + \sum_{r=0}^{\infty} h[k - r]f[r] =: (-3)^{k+1}y[-1] + (h * f)[k],
\]
• where \( h[k] := 7(-3)^k u[k] \) is the (zero state) unit-pulse response,
• \( y[-1] \) is the given \( (n = 1) \) initial condition, and
• we have defined the discrete-time convolution operator with \( \sum_{r=0}^{\infty}(...) := 0. \)
Total response - example (cont)

- **Exercise:** Prove by induction this expression for \( y[k] \) for all \( k \geq -1 \).

- **Exercise:** Prove convolution is commutative: \( h * f = f * h \).

- So, we can write the total response \( y = y_{\text{ZI}} + y_{\text{ZS}} \) starting from the time of oldest initial condition:

\[
\forall k \geq -1, \quad y_{\text{ZI}}[k] = (-3)^{k+1}y[-1]
\]

\[
\forall k \geq -1, \quad y_{\text{ZS}}[k] = u[k] \sum_{r=0}^{k} 7(-3)^{k-r}f[r] = u[k](h * f)[k]
\]

where \( y_{\text{ZS}}[k] = 0 \) when \( k < 0 \).

- Obviously, this example involves a linear, time-invariant and causal system as described by the difference equation above.

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Total response - discussion

- Note that in CMPSC 360, we don’t restrict our attention to linear and time-invariant difference equations.

- We use recursive substitution to guess at the form of the solution and then verify our guess by an inductive proof.

- In this course, we will describe a systematic approach to solve any LTIC difference equation, \( i.e., \) to solve for the output of a DT-LTIC system given the input and initial conditions.

- And again as in continuous time, we will see important insights about discrete-time signals and LTIC systems through frequency-domain representations and analysis.
Note that \( \forall k, \Delta^{-r}z^k = z^{k+r} = z^r z^k \), i.e., the \( r \)-units time-advance operator, \( \Delta^{-r} \), is replaced by the scalar \( z^r \) for all \( r \in \mathbb{Z} \).

Our objective is to solve for the ZIR, i.e., solve
\[
Q(\Delta^{-1})y \equiv 0 \text{ given } y[-n], y[-n+1], \ldots, y[-2], y[-1].
\]

Note that exponential (or "geometric") functions, \( \{z^k \mid k \in \mathbb{Z}\} \) for \( z \in \mathbb{C} \), are eigenfunctions of time-shift operators of the form \( Q(\Delta^{-1}) \) for a polynomial \( Q \).

That is, for any non-zero scalar \( z \in \mathbb{C} \), if we substitute \( y[k] = z^k \forall k \in \mathbb{Z} \) we get:
\[
\forall k \in \mathbb{Z}, \ (Q(\Delta^{-1})y)[k] = Q(\Delta^{-1})z^k = Q(z)z^k.
\]

So, to solve \( Q(z)z^k \equiv 0 \text{ for all time } k \geq 0 \), when \( z \neq 0 \) we require
\[
Q(z) = 0, \text{ the characteristic equation of the system.}
\]

If \( z \) is a root of the characteristic polynomial \( Q \) of the system, then
- \( z \) would be a characteristic value of the system, and
- the signal \( \{z^k\}_{k \geq 0} \) is a characteristic mode of the system when \( z \neq 0 \), i.e.,
\[
Q(\Delta^{-1})z^k = 0 \quad \forall k \geq 0.
\]

Since \( Q \) has degree \( n \), there are \( n \) roots of \( Q \) in \( \mathbb{C} \), each a system characteristic value.
• Let $n' \leq n$ be the number of non-zero roots of $Q$, i.e., $\tilde{Q}(z) = Q(z)/z^{n-n'}$ is a polynomial satisfying $\tilde{Q}(0) \neq 0$.

• Though there may be some repeated roots of the characteristic polynomial $Q$, there will always be $n'$ different, linearly independent characteristic modes, $\mu_k$, i.e.,

$$\forall k \geq -n, \sum_{r=1}^{n'} c_r \mu_r[k] = 0 \iff \forall r, \text{ scalars } c_r = 0.$$

• When $n = n'$, by system linearity, we will be able to write

$$\forall k \geq -n, \ y_{ZI}[k] = \sum_{r=1}^{n} c_r \mu_r[k],$$

for scalars $c_r \in \mathbb{C}$ that are found by considering the given initial conditions

$$y[k] = \sum_{r=1}^{n} c_r \mu_r[k] \text{ for } k \in \{-n, ..., -2, -1\},$$

i.e., $n$ equations in $n$ unknowns ($c_r$).

• The linear independence of the modes implies linear independence of these $n$ equations in $c_r$, and so they have a unique solution.

ZIR - the case of different, non-zero, real characteristic values

• If there are $n$ different non-zero roots of $Q$ in $\mathbb{R}$, $z_1, z_2, ..., z_n$, then there are $n$ characteristic modes: for $r \in \{1, 2, ..., n\}$,

$$\forall \text{ time } k, \mu_r[k] = z_r^k.$$

• Therefore,

$$\forall k \geq -n, \ y_{ZI}[k] = \sum_{r=1}^{n} c_r z_r^k.$$

• The $n$ unknown scalars $c_r \in \mathbb{R}$ can be solved using the $n$ equations:

$$y[k] = \sum_{r=1}^{n} c_r z_r^k, \text{ for } k \in \{-n, -n+1, ..., -2, -1\}.$$
Example: Consider the difference equation:
\[ \forall k \geq -3, \quad 2y[k+3] - 10y[k+2] + 12y[k+1] = 3f[k+2], \]
with \( y[-2] = 1 \) and \( y[-1] = 3 \).

That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, \ P(z) = (3/2)z \) and \( m = 1 \).

So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and the ZIR is
\[ \forall k \geq -n = -2, \quad y_{Z|I}[k] = c_1 3^k + c_2 2^k \]

Using the initial conditions to find the scalars \( c_1, c_2 \):
\[
1 = y[-2] = c_1 3^{-2} + c_2 2^{-2} \quad \text{and} \quad 3 = y[-1] = c_1 3^{-1} + c_2 2^{-1}.
\]

Exercise: Now solve for \( c_1 \) and \( c_2 \).

Note: When a coefficient \( c \) is worked out to be zero, it may not be exactly zero in practice, and the corresponding characteristic mode \( z^k \) will increasingly contribute to ZIR \( y_{Z|I} \) over time if \( |z| > 1 \) (i.e., an “unstable” mode in discrete time).

ZIR - the case of not-real characteristic values

The characteristic polynomial \( Q \) may have non-real roots, but such roots come in complex-conjugate pairs because \( Q \)'s coefficients \( a_k \) are all real.

For example, if the characteristic polynomial is
\[ Q(z) = (z - 1)(z^2 - 2z - 2) \]
then the characteristic values (\( Q \)'s roots) are
\[ -1, \ 1 \pm j\sqrt{3} \quad \text{again recalling} \ j = \sqrt{-1}. \]

Because we have three different characteristic values \( \in \mathbb{C} \), we can specify three corresponding characteristic modes,
\[ (-1)^k, (1 + j\sqrt{3})^k, (1 - j\sqrt{3})^k, \quad \forall k \geq 0, \]
and construct the ZIR as
\[ \forall k \geq -n = -3, \quad y_{Z|I}[k] = c_1 (-1)^k + c_2 (1 + j\sqrt{3})^k + c_3 (1 - j\sqrt{3})^k \]
\[ = c_1 (-1)^k + c_2 2^k e^{jk\pi/3} + c_3 2^k e^{-jk\pi/3} \]
where
- \( c_1 \in \mathbb{R} \) and \( c_2 = c_3 \in \mathbb{C} \) so that \( y_{Z|I} \) is real-valued, and again,
- these scalars are determined by the \( n = 3 \) given (real) initial conditions: \( y[-3], y[-2], y[-1] \).
By the Euler-De Moivre identity for the previous example,
\[ y_{ZI}[k] = c_1(-1)^k + (c_2 + c_3)2^k \cos(k\pi/3) + j(c_2 - c_3)2^k \sin(k\pi/3) \]
\[ = c_1(-1)^k + 2\Re\{c_2\}2^k \cos(k\pi/3) - 2\Im\{c_2\}2^k \sin(k\pi/3) \]

Again, because all initial conditions are real and \( Q \) has real coefficients, \( y_{ZI} \) is real valued and so \( c_3 = \overline{c_2} \Rightarrow c_2 + c_3, j(c_2 - c_3) \in \mathbb{R} \).

In general, consider two complex conjugate characteristic values \( v \pm jq \) corresponding to two complex-valued characteristic modes \( |z|^k e^{\pm jk\angle z} \), where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).

One can use Euler’s identity to show that the corresponding real-valued characteristic modes are
\[ |z|^k \cos(k\angle z), \; |z|^k \sin(k\angle z) \]

Consider the case where at least one characteristic value is of order \( > 1 \), i.e., there are repeated roots of the characteristic polynomial, \( Q \).

For example, \( Q(z) = (z + 0.75)^3(z - 0.5) \) has a triple (twice repeated) root at \(-0.75\) and a single root at \(0.5\).

Again, \( \{(-0.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})(-0.75)^k \equiv 0 \) follows from
\[ (\Delta^{-1} + .75)(-.75)^k = \Delta^{-1}(-.75)^k + .75(-.75)^k \]
\[ = (-.75)^{k+1} + .75(-.75)^k \]
\[ = 0. \]

Similarly, \( (0.5)^k \) is a characteristic mode since \( (\Delta^{-1} - 0.5)(0.5)^k \equiv 0 \).

Also, \( \{k(-.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})k(-.75)^k \equiv 0 \) follows from
\[ (\Delta^{-1} + .75)^2k(-.75)^k \]
\[ = (\Delta^{-2} + 1.5\Delta^{-1} + (.75)^2)k(-.75)^k \]
\[ = \Delta^{-2}k(-.75)^k + 1.5\Delta^{-1}k(-.75)^k + (.75)^2k(-.75)^k \]
\[ = (k + 2)(-.75)^{k+2} + 1.5(k + 1)(-.75)^{k+1} + (.75)^2k(-.75)^k \]
\[ = (-.75)^{k+2}((k + 2) - 2(k + 1) + k) \]
\[ = 0. \]
ZIR - the case of repeated characteristic values (cont)

- Similarly, \(\{k^2(-.75)^k\}\) is also a characteristic mode because 
  \((\Delta^{-1} + .75)^3k^2(-.75)^k = 0\).

- Note that without three such linearly independent characteristic modes
  \(\{(-.75)^k, k(-.75)^k, k^2(-.75)^k ; k \geq 0\}\)
for the twice-repeated (triple) characteristic value -.75, the initial conditions will create an
"overspecified" set of \(n\) equations involving fewer than \(n\) “unknown” coefficients \((c_k)\) of
the linear combination of modes forming the ZIR.

- For this example,
  
  \[
  y_{ZI}[k] = c_0(-0.75)^k + c_1k(-0.75)^k + c_2k^2(-0.75)^k + c_3(0.5)^k, \quad k \geq -4.
  \]

- If the given initial conditions are, say,
  \(y[-4] = 12, \ y[-3] = 6, \ y[-2] = -5, \ y[-1] = 10,\)
the four equations to solve for the four unknown coefficients \(c_k\) are:

  \[
  \begin{align*}
  y_{ZI}[-4] &= (-.75)^{-4}c_0 + (-4)(-.75)^{-4}c_1 + (-4)^2(-.75)^{-4}c_2 + (.5)^{-4}c_3 = 12 \\
  y_{ZI}[-3] &= (-.75)^{-3}c_0 + (-3)(-.75)^{-3}c_1 + (-3)^2(-.75)^{-3}c_2 + (.5)^{-3}c_3 = 6 \\
  y_{ZI}[-2] &= (-.75)^{-2}c_0 + (-2)(-.75)^{-2}c_1 + (-2)^2(-.75)^{-2}c_2 + (.5)^{-2}c_3 = -5 \\
  y_{ZI}[-1] &= (-.75)^{-1}c_0 + (-1)(-.75)^{-1}c_1 + (-1)^2(-.75)^{-1}c_2 + (.5)^{-1}c_3 = 10
  \end{align*}
  \]

ZIR - general case of repeated, non-zero characteristic values

- In general, a set of \(r\) linearly independent modes corresponding to a non-zero characteristic
value \(z \in \mathbb{C}\) repeated \(r - 1\) times are
  \[k^{r-1}z^k, k^{r-2}z^k, ..., kz^k, z^k, \quad \text{for } k \geq 0.\]

- Also, if \(v \pm jq\) are characteristic values repeated \(r - 1\) times, with \(v, q \in \mathbb{R}\) and \(q \neq 0,\)
we can use the \(2k\) real-valued modes
  
  \[
  k^a|z|^k \cos(k\angle z), k^a|z|^k \sin(k\angle z), \quad \text{for } a \in \{0, 1, 2, ..., r - 1\},
  \]
where \(|z| = \sqrt{v^2 + q^2}\) and \(\angle z = \arctan(q/v)\).
ZIR - when some characteristic values are zero

• Again let \( n' \leq n \) be the number of non-zero roots of \( Q \) (characteristic values),

• \( i.e., r := n - n' \geq 0 \) is the order (1+repetition) of the characteristic value \( 0 \), and

• \( r \geq 0 \) is the smallest index such that (the coefficient of \( Q \)) \( a_r \neq 0 \).

• So, there is a polynomial \( \tilde{Q} \) such that \( Q(z) = z^r \tilde{Q}(z) \) and \( \tilde{Q}(0) \neq 0 \).

• Because the constant signal zero cannot be a characteristic mode, we add \( r = n - n' \)
time-advanced unit-pulses:

\[
\forall k \geq -n, \ y_{ZI}[k] = \sum_{i=1}^{r} C_i \delta[k + i] + y_N[k] = C_r \delta[k + r] + C_{r-1} \delta[k + r - 1] + ... + C_1 \delta[k + 1] + y_N[k]
\]

where \( y_N \) is a “natural response” (linear combination of \( n' \) characteristic modes).

• The \( n \) initial conditions are then met by the \( r \) coefficients \( C_i \) of the advanced unit pulses
together with the \( n' = n - r \) coefficients of the characteristic modes in \( y_N \).

ZIR - when some characteristic values are zero - example

• Consider a fourth-order system with characteristic polynomial

\( Q(z) = z^2(z + 1)^2 \).

• Thus the poles are \( 0, -1 \) each repeated and the (non-zero) characteristic modes are \(( -1 )^k, k(-1)^k \).

• So, the ZIR is, for \( k \geq -4 \):

\[
y_{ZI}[k] = C_2 \delta[k + 2] + C_1 \delta[k + 1] + c_1 (-1)^k + c_2 k(-1)^k
\]

• That is, the ZIR has four unknown coefficients \( C_2, C_1, c_1, c_2 \) to account for the four (given)
initial conditions \( y[-4], y[-3], y[-2], y[-1] \).
Zero State Response - the unit-pulse response

- Recall the LTIC system
  \[ \sum_{r=0}^{n} a_r \Delta^{-r} y = Q(\Delta^{-1}) y = P(\Delta^{-1}) f := \sum_{r=0}^{m} b_r \Delta^{-r} f \]
  with \( a_n = 1, a_0 \neq 0 \) or \( b_0 \neq 0, m \leq n \).

- We can express any input signal
  \[ f[k] = \sum_{r=0}^{\infty} f[r] \delta[k - r] \quad \forall k \geq 0, \quad i.e., \forall f, f = f * \delta. \]

- So the unit pulse \( \delta \) is the identity of the convolution operator in discrete time.

- Thus, by LTI, the ZSR \( y_{ZS} \) is the convolution of input \( f \) and ZSR \( h \) to unit pulse \( \delta \),
  \[ y_{ZS}[k] = \sum_{r=0}^{\infty} f[r] h[k - r] = (f * h)[k], \quad \forall k \geq 0, \]

- \( h \) is called the unit-pulse response of the LTIC system, \( i.e. \),
  \[ Q(\Delta^{-1}) h = P(\Delta^{-1}) \delta \land h[k] = 0 \quad \forall k < 0. \]

Computing an LTIC system’s unit-pulse response, \( h \)

- For the LTIC system in standard form, if \( a_0 \neq 0 \) then
  \[ h = (b_0/a_0) \delta + y_N u \]
  where \( y_N \) is a natural response of the system (linear combination of characteristic modes).

- Note that \( h[k] = 0 \) for all \( k < 0 \) owing to the unit step \( u \).

- The \( n \) scalars of the natural response \( y_N \) component of \( h \) are solved using
  \[ (Q(\Delta^{-1}) h)[k] = (P(\Delta^{-1}) \delta)[k] \quad \text{for} \quad k \in \{-n, -n + 1, ..., -2, -1\} \]

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Unit-pulse response when zero is a characteristic value

- If \( r \geq 0 \) is the smallest index such that \( a_r \neq 0 \) (\( 0 \) is a char. mode of order \( r \)), then may need to add \( r \) delayed unit-pulse terms to \( h \):

\[
h(r) = \sum_{i=0}^{r-1} A_i \delta^i + (b_0/a_r)\Delta^r \delta + y_0 u,
\]

where

- by definition of the standard form of the difference equation, if \( r > 0 \), \( a_0 = 0 \) so \( b_0 \neq 0 \), and
- \( r \leq n \) since \( 0 \neq a_n := 1 \).

- So if \( r = 0 \) \((\text{i.e.}, a_0 \neq 0)\), then \( A_0 = b_0/a_0 \) as above, where \( \sum_{i=0}^{-1}(...) := 0 \).

- Exercise: Prove \( A_r = b_0/a_r \) for \( 0 \leq r \leq n \).

- Thus, zero is a characteristic value of degree \( r \geq 0 \), and

- there are \( r \) characteristic modes that will all be zero.

- The additional unit-pulse terms introduce \( r \) degrees of freedom in the form of the coefficients \( A_0, A_1, \ldots, A_{r-1} \) to accommodate the \( n = r + n' \) initial conditions of the unit-pulse response: \( h[-n] = h[-n+1] = \ldots = h[-2] = h[-1] = 0 \).

Computing the ZSR - example 1

- Recall that the difference equation \( y = 7f - 3\Delta y \) corresponds to the above system; in standard form:

\[
\forall k \geq -1, \quad y[k+1] + 3y[k] = 7f[k+1].
\]

with \( Q(z) = z + 3, P(z) = 7z \) and \( n = 1 = m \).

- Since the system characteristic value is \(-3 \) and \( b_0 = 0 \), the (zero state) unit-pulse response has the form \( h[k] = c(-3)^k u[k] \).

- The scalar \( c \) is solved by evaluating the above difference equation at time \( k = -1 \):

\[
(Q(\Delta^{-1})h)[-1] = (P(\Delta^{-1})\delta)[-1]
\]

\[
i.e., \quad h[0] + 3h[-1] = 7\delta[0]
\]

\[
\Rightarrow c + 3 \cdot 0 = 7 \cdot 1, \quad c = 7
\]
Computing the ZSR - example 1 (cont)

- So, \( h[k] = 7(-3)^k u[k] \).
- If the input is \( f[k] = 4(0.5)^k u[k] \), the system ZSR is, for all \( k \geq 0 \),
  \[
  y_{zs}[k] = \sum_{r=0}^{k} h[r] f[k - r] = \sum_{r=0}^{k} 7(-3)^r 4(0.5)^{k-r} \\
  = 28(0.5)^k \sum_{r=0}^{k} (-6)^r = 28(0.5)^k \frac{(-6)^{k+1} - 1}{-6 - 1} u[k] \\
  = (24(-3)^k + 4(0.5)^k) u[k].
  \]
- Note how the ZIR \( y_{zi} \) has a term that is a characteristic mode (excited by the input \( f \))
  and a term that is proportional to the input \( f \) (this forced response is an eigenresponse).
- **Exercise:** For the difference equation, \( y[k+1] + 3y[k] = 7f[k] \ \forall k \geq -1 \): draw the
  block diagram, show that \( h[k] = 21(-3)^{k-1} u[k] + (7/3)\delta[k] \), and find the ZSR to
  the above input \( f \).
- **Exercise:** Read “sliding tape” method to compute convolution in Lathi, p. 595.

---

Computing the unit pulse response - example 2

- Find the ZSR of the following system to input \( f[k] = 2(-5)^k u[k] \):

\[
\begin{align*}
  & 1.5 \rightarrow y \\
  f & \rightarrow + \\
  & \Delta \rightarrow 3 \\
  & \Delta \rightarrow -6 \\
  & 3 \rightarrow -6 \\
  & -6 \rightarrow f
\end{align*}
\]

- **Exercise:** show the difference equation for this system (in direct canonical form) is:
  \( \forall k \geq 0 \), \( y[k+2] - 5y[k+1] + 6y[k] = 1.5f[k+1] \)
- That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, P(z) = 1.5z \) and \( m = 1 \).
- So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and \( b_0 = 0 \) so the unit-pulse response
  \( h[k] = (c_1 3^k + c_2 2^k) u[k] \).

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Computing the unit pulse response - example 2 (cont)

• To find the constants, evaluate the difference equation at \( k = -1 \):

\[
2h[1] - 10h[0] + 12h[-1] = 3\delta[0] \\
\Rightarrow 2h[1] - 10h[0] = 3 \\
(2 \cdot 3 - 10 \cdot 1)c_1 + (2 \cdot 2 - 10 \cdot 1)c_2 = 3 \\
\Rightarrow -4c_1 + -6c_2 = 3
\]

and at \( k = -2 \):

\[
2h[0] - 10h[-1] + 12h[-2] = 3\delta[-1] \Rightarrow 12h[0] = 0 \Rightarrow h[0] = 0
\]

\[
\Rightarrow c_1 + c_2 = 0.
\]

• Thus, \( c_2 = -1.5 = -c_1 \) so that \( h[k] = (-1.5(3)^k + 1.5(2)^k)u[k] \) and for \( k \geq 0 \)

\[
y_{ZS}[k] = (h \ast f)[k] = \sum_{r=0}^{k} h[r]f[k-r].
\]

• **Exercise:** Write the ZSR as a sum of system modes \( 2^k \) and \( 3^k \) and a (force) term like the input, here taken as \( f[k] = 4(-5)^ku[k] \).

Convolution - other important properties

• Again, for a LTI system with impulse response \( h \) and input \( f \), the ZSR is \( y_{ZS} = f \ast h \), where

\[
(f \ast h)[k] = \sum_{r=\infty}^{\infty} f[r]h[k-r]
\]

• By simply changing the dummy variable of summation to \( r' = h - r \), can show convolution is commutative: \( f \ast h = h \ast f \).

• One can directly show that convolution \( f \ast h \) is a bi-linear mapping from pairs of signals \( (f, h) \) to signals \( (y_{ZS}) \), consistent with convolution’s commutative property and the (zero state) system with impulse response \( h \) being LTI;

• that is, \( \forall \) signals \( f, g, h \) and scalars \( \alpha, \beta \in \mathbb{C} \),

\[
(\alpha f + \beta g) \ast h = \alpha(f \ast h) + \beta(g \ast h)
\]

• By changing order of summation (Fubini’s theorem), one can easily show that convolution is associative, i.e., \( \forall \) signals \( f, g, h \),

\[
(f \ast g) \ast h = f \ast (g \ast h).
\]
Convolution - other important properties (cont)

• We’ll use these properties when composing more complex systems from simpler ones.

• By just changing variables of integration, we can show how to exchange time-shift with convolution, i.e., \( \forall \) signals \( f, h : \mathbb{Z} \rightarrow \mathbb{C} \) and times \( k \in \mathbb{Z} \),

\[
(\Delta^k f) * h = \Delta^k (f * h);
\]

recall how convolution represents the ZSR of linear and time-invariant systems.

• By the ideal sampling property, recall that the identity signal for convolution is the unit pulse \( \delta \), i.e., \( \forall \) signals \( f \),

\[
f * \delta = \delta * f = f
\]

• Exercise: Adapt the proofs of these properties in continuous time to this discrete-time case.

• Exercise: In particular, show that if \( f \) and \( h \) are causal signals, then \( y = f * h \) is causal; i.e., if the unit-pulse response \( h \) of a system is a causal signal, then the system is causal.

System stability - ZIR - asymptotically stable

• Consider a SISO system with input \( f \) and output \( y \).

• Recall that the ZIR \( y_{ZI} \) is a linear combination of the system’s characteristic modes, where the coefficients depend on the initial conditions, possibly including some initial unit-pulse terms if zero is a characteristic value (system pole).

• A system is said to be asymptotically stable if for all initial conditions,

\[
\lim_{k \rightarrow \infty} y_{ZI}[k] = 0.
\]

• So, a system is asymptotically stable if and only if all of its characteristic values have magnitude less than 1.
System stability - ZIR - asymptotically stable: Example

- If the characteristic polynomial $Q(z) = (z - 0.5)(z^2 + 0.0625)$, then
- the system’s characteristic values (roots of $Q$) are $0.5, \pm 0.25j$ each with magnitude less than one,
- and the ZIR is of the form,
  \[ y_{ZIR}[k] = (c_1(0.5)^k + c_2(0.25j)^k + c_2(-0.25j)^k)u[k] \]
  \[ = (c_1(0.5)^k + 2\text{Re}\{c_2\}(0.25)^k \cos(k\pi/2) - 2\text{Im}\{c_2\}(0.25)^k \sin(k\pi/2))u[k], \]
  
- recalling that $j^k = e^{ik\pi/2}$.
- So, $y_{ZIR}[k] \rightarrow 0$ as $k \rightarrow \infty$ for all $c_1, c_2$ (i.e., for all initial conditions), and
- hence is asymptotically stable.

System stability - bounded signals

- A signal $y$ is said to be bounded if
  \[ \exists M < \infty \text{ s.t. } \forall k \in \mathbb{Z}, \quad |y[k]| \leq M; \]
  otherwise $y$ is said to be unbounded.
- For example, $y[k] = 0.25\left(\frac{1+j\sqrt{3}}{2}\right)^k u[k]$ is bounded (can use $M = 0.25$).
- Also, $3 \cos(5k)$ is bounded (can use $M = 3$).
- But both $2^k \cos(5k)$ and $3 \cdot (-2)^k$ are unbounded.
System stability - ZIR - marginally stable

- A system is said to be marginally stable if it is not asymptotically stable but \( y_{ZIR} \) is always (for all initial conditions) bounded.

- A system is marginally stable if and only if
  - it has no characteristic values with magnitude strictly greater than 1,
  - it has at least one characteristic value with magnitude exactly 1, and
  - all magnitude-1 characteristic values are not repeated.

- That is, a marginally stable system has
  - some characteristic modes of the form \( \cos(\Omega k) \) or \( \sin(\Omega k) \),
  - while the rest of the modes are all of the form \( k^r|z|^k \cos(\Omega k) \) or \( k^r|z|^k \sin(\Omega k) \),
    with \( |z| < 1 \) and integer degree \( r \geq 0 \).

  - Exercise: Explain why we can take \( \Omega \in (-\pi, \pi] \) without loss of generality.

  - Note: the dimension of frequency \( \Omega \) is \([\Omega] = \text{radians per unit time}\).

System stability - ZIR - marginally stable: Example

- The characteristic polynomial is \( Q(z) = z(z^2 + 1)(z - 0.25) \) gives characteristic values \( 0, 0.25, \pm j \).

- then the system is marginally stable with modes \((0.25)^k \cos(k\pi/2), \sin(k\pi/2)\),

- the last two of which are bounded but do not tend to zero as time \( k \to \infty \).
A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.

For example, the system with $Q(z) = (z^2 - 0.5)(z + 3)$ is unstable owing to the characteristic value $-3$ with unbounded mode $(-3)^k$.

For another example, if the characteristic polynomial is $Q(z) = (z^2 + 1)^2(z - 0.5)$ then the purely imaginary characteristic values $\pm j$ are repeated, and hence the two additional modes $k \sin(k\pi/2), k \cos(k\pi/2)$ are unbounded, so this system is unstable.

Similarly, if $Q(z) = (z^2 - 1)^2(z - 0.5)$ then the characteristic values $\pm 1$ are repeated and the modes $k$ and $k(-1)^k$ are unbounded, so this system is unstable too.
• A SISO system is said to be *Bounded Input, Bounded Output* (BIBO) stable if ∀ bounded input signals \( f \), the ZSR \( y_{ZS} \) is bounded.

• A sufficient condition for BIBO stability is absolute summability of the unit-pulse response,
  \[
  \sum_{k=0}^{\infty} |h[k]| < \infty.
  \]

• To see why: If the input \( f \) is bounded (by \( M_f \) with \( 0 \leq M_f < \infty \)) then \( \forall k \geq 0 \):
  \[
  |y_{ZS}[k]| = |(f * h)[k]| \\
  = \left| \sum_{r=0}^{k} f[k-r]h[r] \right| \\
  \leq \sum_{r=0}^{k} |f[k-r]| |h[r]| \quad \text{(by the triangle inequality)} \\
  \leq \sum_{r=0}^{k} M_f |h[r]| \\
  \leq M_f \sum_{r=0}^{\infty} |h[r]| =: M_y < \infty,
  \]

• The condition of absolute summability of the unit-pulse response,
  \[
  \sum_{r=0}^{\infty} |h[r]| < \infty,
  \]
  is also necessary for, and hence equivalent to, BIBO stability.

• If any component characteristic mode of \( h \) is unbounded, then \( h \) will not to be absolutely summable.

• Thus, if the system (ZIR) is asymptotically stable it will be BIBO stable; the converse is also true.
• Recall that for any polynomial \( Q \) and \( z \in \mathbb{C} \) (including \( s = jw, \ w \in \mathbb{R} \)),
\[
Q(\Delta^{-1})z^k = Q(z)z^k, \ \forall k \geq 0.
\]

• So, if we guess that a “particular” solution of the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \) with input \( f[k] = Az^ku[k] \) is of the form \( y_0[k] = AH(z)z^k = H(z)f[k], k \geq 0 \), then we get by substitution that \( \forall k \geq 0, z \in \mathbb{C} \),
\[
(Q(\Delta^{-1})y_0)[k] = (P(\Delta^{-1})f)[k] \Rightarrow AH(z)Q(z)z^k = AP(z)z^k \\
\Rightarrow H(z) = P(z)/Q(z).
\]

• The “rational polynomial” \( H = P/Q \) is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

• So, the ZSR (forced response + characteristic modes) would be of the form:
\[
y_{ZS}[k] = (AH(z)z^k + \text{linear combination of char. modes})u[k].
\]

• Recall that for the example with \( Q(z) = z + 3 \) and \( P(z) = 7z \), we computed the unit-pulse response \( h[k] = 7(-3)^ku[k] \) and the ZSR to input \( f[k] = 4(0.5)^ku[k] \) as \( y_{ZS}[k] = (24(-3)^k + 4(0.5)^k)u[k] \).

• Here, note that \( H(0.5) = P(0.5)/Q(0.5) = 1 \), i.e., the forced response component of \( y_{ZS} \) is \( H(0.5)f[k] = 1 \cdot 4(0.5)^ku[k] \).

ZSR - unit-pulse response \( h \), transfer function \( H \), and eigenresponse

• \( y_{ZS}[k] = (H(z)Az^k + \text{linear combination of char. modes})u[k] \) is the ZSR to input \( f[k] = Az^ku[k] \), where \( H(z) = P(z)/Q(z) \).

• The eigenresponse is a special case of the forced response for exponential inputs.

• If \( |z| = 1 \), i.e., \( z = e^{j\Omega} \) for some \( \Omega \in (-\pi, \pi) \) (w.l.o.g.), and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:
\[
y[k] \rightarrow AH(e^{j\Omega})e^{j\Omega k} \quad \text{as} \ k \rightarrow \infty.
\]

• Since \( y = h * f \), we get that as \( k \rightarrow \infty \) for a LTIC and asymptotically stable system,
\[
y_{ZS}[k] = \sum_{r=0}^{k} h[r]Ae^{j\Omega(k-r)}
\]
\[
= Ae^{j\Omega k} \sum_{r=0}^{k} h[r]e^{-j\Omega r} \rightarrow Ae^{j\Omega k}H(e^{j\Omega}),
\]
\[
\Rightarrow \sum_{r=0}^{\infty} h[r]e^{-j\Omega r} = H(e^{j\Omega}), \ \forall \Omega \in (-\pi, \pi).
The LTIC system transfer function $H$ is the $z$-transform of the system unit-pulse response $h$:

$$H(z) = \sum_{k=0}^{\infty} h[k]z^{-k},$$

where $z \in C$ is in $H$’s “region of convergence”.

- Note that $H(e^{j\Omega})$ is periodic since $H(e^{j\Omega}) = H(e^{j\Omega+2\pi k})$ for any integer $k$.

- For the $z$-transform (and DTFS) we will use this notation for $H$, but for the DTFT we will write $H(\Omega)$ instead of $H(e^{j\Omega})$.

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**Frequency-domain methods for discrete-time signals**

- Discrete-Time Fourier Series (DTFS) of periodic signals
- Discrete-Time Fourier Transform (DTFT)
- sampled data systems
  * DFT & FFT
- $z$-transform for (complete) transient response
- eigenresponse
- canonical system realization of a difference equation
Discrete-time Fourier series of periodic signals

- For all \( r, N \in \mathbb{Z} \), note that the signal \( \{ \exp(jr\frac{2\pi}{N}k) \mid k \in \mathbb{Z} \} \) "repeats itself" every \( N > 0 \) units of (discrete) time \( k \), in particular

\[
\forall r \in \mathbb{Z}, \quad e^{jr\frac{2\pi}{N}k} \big|_{k=0} = 1 = e^{jr\frac{2\pi}{N}k} \big|_{k=N}
\]

- Also the signals \( \{ \exp(jr\frac{2\pi}{N}k) \mid k \in \mathbb{Z} \} \equiv \{ \exp(jr'\frac{2\pi}{N}k) \mid k \in \mathbb{Z} \} \) whenever \( r' = r \mod N \).

- Suppose \( N \) is the period of periodic signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) and \( \Omega_o = 2\pi/N \) be the fundamental frequency of \( x \) (recall \( [\Omega_o] = \) radians/unit-time).

- We can write \( x \) as a Discrete-Time Fourier Series (DTFS):

\[
\forall k \in \mathbb{Z}, \quad x[k] = \sum_{r=0}^{N-1} D_r e^{j\Omega_o k}.
\]

where \( r \) indexes \( x \)'s \( N \) harmonics.

- Note that the DTFS can also be written for any discrete-time signal \( x : A \to \mathbb{R} \) defined over any finite interval of time, e.g., \( A = \{0, 1, 2, ..., N-1\} \) or \( A = \{-N, -N+1, ..., -1\} \) for integer \( N < \infty \).

Discrete-time Fourier series of periodic signals (cont)

- Consider the \( N \) signals \( \xi_r[k] := e^{j\Omega_o k} \) over any time-interval \( A \) of length \( N \).

- Equivalently consider these \( N \) signals \( \xi_r \) as \( N \)-vectors in \( \mathbb{R}^N \), i.e., the \( k^{th} \) entry of vector \( \xi_r \) is \( \xi_r[k] \).

- If these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are linearly independent, then they will form a basis spanning all other signals \( x : A \to \mathbb{R} \), equivalently all other vectors \( x \in \mathbb{R}^N \).

- i.e., any such \( x \) can be written as a linear combination of the \( \{ \xi_r \}_{r=0}^{N-1} \) giving the DTFS of \( x \):

\[
x_r = \sum_{r=0}^{N-1} D_r \xi_r.
\]

- If we show that these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are orthogonal then
  - linear independence follows
  - the \( r^{th} \) coordinate \( D_r \) (DTFS coefficients) is found by simply projecting \( x \) onto the vector \( \xi_r \):

\[
D_r = \langle x, \xi_r \rangle / ||\xi_r||^2.
\]
DTFS - coefficients (cont)

• Consider any period of \( x : \mathbb{Z} \to \mathbb{R} \), say \( \{0, 1, 2, ..., N - 1\} \).

• First note that for any \( v \in \mathbb{Z} \) that is not a multiple of \( N \) (so \( \exp(jv\Omega) = \exp(j(2\pi/N)) \neq 1 \)), the geometric series
  \[
  \sum_{k=0}^{N-1} \exp(jv\Omega) = \sum_{k=0}^{N-1} \left( \exp(j2\pi/N) \right)^k = \frac{\exp(jv(2\pi/N))N - \exp(jv(2\pi/N))0}{\exp(j2\pi/N) - 1} = 0.
  \]

• Thus, for any \( r \neq v \in \mathbb{Z} \) such that \( N \nmid (v - r) \), the inner product \( \langle \xi_r, \xi_v \rangle = \sum_{k=0}^{N-1} \exp(jr(2\pi/N)k)\exp(jv(2\pi/N)k) = \sum_{k=0}^{N-1} \exp((r-v)(2\pi/N)k) = 0 \), recalling that the inner product is conjugate-linear in the second argument so that \( \langle x, x \rangle = ||x||^2 \) when \( x \) is \( \mathbb{C} \)-valued.

• So, these signals are orthogonal and the DTFS coefficients of \( N \)-periodic \( x \) are
  \[
  D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \exp(-jr\Omega) = \frac{\langle x, \{\exp(jr\Omega)k\} \rangle}{||\{\exp(jr\Omega)k\}||^2}, \quad \Omega_o = \frac{2\pi}{N}.
  \]

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DTFS - checking coefficients

• Let’s now compute the inner product of \( \xi_v \), for any \( v \in \{0, 1, ..., N - 1\} \), with the DTFS of \( N \)-periodic \( x \):
  \[
  \langle x, \{\exp(jv\Omega)k\} \rangle = \sum_{k=0}^{N-1} x[k] \exp(-jv\Omega) = \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} D_r \exp(jr\Omega) \exp(-jv\Omega) = \sum_{r=0}^{N-1} D_r \exp((r-v)\Omega) \Omega = \sum_{r=0}^{N-1} D_r \Omega \delta(r - v) = D_v \Omega
  \]

• Again, we have verified the DTFS coefficients is
  \[
  D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \exp(-jr\Omega) = \frac{\langle x, \{\exp(jr\Omega)k\} \rangle}{||\{\exp(jr\Omega)k\}||^2}, \quad \Omega_o = \frac{2\pi}{N}
  \]

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**DTFS - example**

**Problem:** Identify the DTFS coefficients (if they exist) for

\[ x[k] = 7 \sin(5.7\pi k) + 2 \cos(3.2\pi k), \quad k \in \mathbb{Z}. \]

**Solution:**

- First note that the two components of \( x \) are periodic, so their sum is periodic. *(Why is this so in discrete time?)*

- Since \( \sin \) and \( \cos \) have period \( 2\pi \), we can subtract integer multiples of \( 2\pi \) to get

\[ x[k] = 7 \sin(1.7\pi k) + 2 \cos(1.2\pi k). \]

- \( 1.7\pi k \) is an integer multiple of \( 2\pi \) when (integer) \( k = 20 \), and when \( k = 5 \) for \( 1.2\pi k \), so least common multiple of these periods is \( k = 20 \).

- (Show that one can alternatively find the greatest common divisor of the component frequencies.)
DTFS - example (cont)

- Thus, the period of $x$ is $N = 20$ and the fund. frequ. is $\Omega_o = 2\pi/N = 0.1\pi$.

- By Euler’s identity and adding $2\pi k$ to the negative exponents,
  \[
  x[k] = \frac{7}{2j}e^{j1.7\pi k} - \frac{7}{2j}e^{-j1.7\pi k} + e^{j1.2\pi k} + e^{-j1.2\pi k}
  = -3.5je^{j1.7\pi k} + 3.5je^{j0.3\pi k} + e^{j1.2\pi k} + e^{j0.8\pi k}.
  \]

- So, the DTFS of $x[k] = \sum_{r=0}^{19} D_r e^{jr0.1\pi k}$ with
  \[
  D_{17} = -3.5j = 3.5e^{-j\pi/2}, \quad D_3 = 3.5j = 3.5e^{j\pi/2}, \quad D_{12} = 1, \quad \text{and} \quad D_8 = 1;
  \]
  else $D_r = 0$ (incl. the fundamental $r \in \{1,19\}$ & DC $r = 0$ components).

DTFS - example and exercise

- **Example:** The DTFS of an even rectangle wave with period $N = 6$ and duty cycle 3:
  \[
  x[k] = \sum_{\ell=-\infty}^{\infty} \Delta^6(\Delta^{-1}u - \Delta^2u)[k] = \sum_{\ell=-\infty}^{\infty} (u[k+1-6\ell] - u[k-2-6\ell])
  \]
  is
  \[
  = \sum_{r=0}^{5} D_r e^{jr\Omega_o k},
  \]
  where the fund. freq. $\Omega_o = 2\pi/6$ and, $\forall r \in \mathbb{Z},$
  \[
  D_r = \frac{1}{6} \sum_{k=-3}^{2} x[k]e^{-jr\Omega_o k} = \frac{1}{6} \sum_{k=-1}^{1} 1 \cdot e^{-jr(2\pi/6)k} = \frac{1}{6}(1 + 2 \cos(r(2\pi/6)k)).
  \]

- **Exercise:** Plot $x[k]$ as a function of time $k$ and plot its (periodic) spectrum:
  $\forall r \in \{0,1,2,...,5\}, \ell \in \mathbb{Z},$
  \[
  \hat{X}(r 2\pi/6 + 2\pi \ell) = D_r.
  \]
DTFS - Parseval’s theorem

- The average power of the $N$-periodic discrete-time signal $x$ is
  \[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} x[k]x^*[k];
  \]
equivalently, the sum could be taken over any interval of length $N \in \mathbb{Z}^\geq 0$.

- Substituting the Fourier series of $x$ separately for $x[k]$ and $x^*[k]$ (using a different summation-index variable for each substitution), leads to Parseval’s theorem
  \[
P_x = \sum_{r=0}^{N-1} |D_r|^2.
  \]

- Parseval’s theorem can be used to determine the amount of periodic signal $x$’s power resides in a given frequency band $[\Omega_1, \Omega_2] \subset [0, 2\pi]$ radians/unit-time:
  1. determine the harmonics $r\Omega_o$ of $x$ that reside in this band, i.e., integers $r \in [\Omega_1/\Omega_o, \Omega_2/\Omega_o]$ where $x$’s fundamental frequency $\Omega_o = 2\pi/N$.
  2. sum just over these harmonics to get the answer, $\sum_{\Omega_o \leq r \leq \Omega_o} |D_r|^2$.

---

DTFS - Parseval’s theorem example

- Find the fraction of $x$’s average power in the frequency band $[0.4\pi, 1.1\pi]$ where
  \[
  \forall k \in \mathbb{Z}, \quad x[k] = \sum_{v=-\infty}^{\infty} (3\delta[k-4v] - 4\delta[k-1-4v])
  \]
- **Solution:** $x$ has period $N = 4$ and average power
  \[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{4} \sum_{k=0}^{3} |x[k]|^2 = \frac{1}{4} (3^2 + (-4)^2 + 0^2 + 0^2) = \frac{25}{4}
  \]
- $x$ has fundamental frequency $\Omega_o = 2\pi/N = \pi/2$ radians/unit-time and discrete-time Fourier coefficients
  \[
  D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k]e^{-j(r\Omega_o)k} = \frac{1}{4} \left( 3 - 4e^{-j\pi/2} \right), \quad 0 \leq r \leq N-1 = 3.
  \]
- The harmonics $r$ of $x$ that reside in $[0.4\pi, 1.1\pi]$ satisfy $0.4\pi \leq r\Omega_o = r\pi/2 \leq 1.1\pi$, i.e. $r \in \{1, 2\}$.
- So, by Parseval’s theorem, the answer is $(|D_1|^2 + |D_2|^2)/P_x$.  

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Periodic extensions

- Consider signal $x : \mathbb{Z} \to \mathbb{R}$ having finite support $\{-M, -M + 1, \ldots, 0, \ldots, M - 1, M\}$ for $0 < M < \infty$; i.e., $\forall |k| > M, x[k] = 0$.

- For $N \geq M$, define $2N$-periodic $x^{(N)}$ such that

\[
x^{(N)}[k] = \begin{cases} 
  x[k] & \text{if } |k| \leq M \\
  0 & \text{if } M < |k| \leq N
\end{cases}
\]

- $x^{(N)}$ is a periodic extension of the finite-support signal $x$, where again $x^{(N)}$'s period is $2N$ and

\[
\lim_{N \to \infty} x^{(N)} = x.
\]

DTFS of periodic extension leading to DTFT

- For $r \in \{-N + 1, -N + 2, \ldots, N - 1, N\}$, the DTFS of $x^{(N)}$ has coefficients

\[
D_r^{(N)} = \frac{1}{2N} \sum_{k=-N+1}^{N} x^{(N)}[k] e^{-jr\frac{2\pi}{2N}k}
\]

\[
= \frac{1}{2N} \sum_{k=-M}^{M} x[k] e^{-jr\frac{2\pi}{2N}k}
\]

\[
= \frac{1}{2N} \sum_{k=-\infty}^{\infty} x[k] e^{-jr\frac{2\pi}{2N}k}
\]

\[
= \frac{1}{2N} X \left( r \frac{2\pi}{2N} \right)
\]

where the Discrete-Time Fourier Transform (DTFT) of (aperiodic) $x : \mathbb{Z} \to \mathbb{R}$ is $X : \mathbb{R} \to \mathbb{C}$:

\[
X(\Omega) := \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} =: (F x)(\Omega), \quad \Omega \in \mathbb{R}
\]

- Note that Fourier integrals (spectra of discrete-time signals) are periodic, repeating themselves every $2\pi$ radians: $\forall \Omega \in \mathbb{R}, \ell \in \mathbb{Z},$

\[
X(\Omega) = X(\Omega + \ell 2\pi).
\]
Inverse DTFT by Fourier Integral

- Thus, $\forall k \in \mathbb{Z}$,
  
  $x[k] = \lim_{N \to \infty} x^{(N)}[k]$

  $= \lim_{N \to \infty} \sum_{r=-N+1}^{N} D_r^{(N)} e^{j \frac{2\pi}{2N}k}$

  $= \lim_{N \to \infty} \sum_{r=-N+1}^{N} X \left( r \frac{2\pi}{2N} \right) e^{j \frac{2\pi}{2N}k} \cdot \frac{1}{2N} \cdot \frac{2\pi}{2\pi}$

  $= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega$

  where the last equality is by Riemann integration with $2\pi/(2N) \to d\Omega$.

- Thus, we have derived the inverse DTFT by Fourier integral of $X$ giving (aperiodic) $x$,

  $\forall k \in \mathbb{Z}, \ x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega =: (F^{-1}X)[k]$.  

DTFT Examples - exponential signal

- If $x = \delta$ then obviously $X \equiv 1$.

- The geometric signal $x[k] = \gamma^k u[k]$ for scalar $\gamma$ s.t. $|\gamma| < 1$ has DTFT

  $X(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k$

  $= \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{1}{(1 - \gamma) \cos(\Omega) + j\gamma \sin(\Omega)}$

- Note that

  $|X(\Omega)| = \frac{1}{(1 - \gamma \cos(\Omega))^2 + \gamma^2 \sin^2(\Omega)} = \frac{1}{1 + \gamma^2 - 2\gamma \cos(\Omega)}$

  $\angle X(\Omega) = -\arctan \left( \frac{\gamma \sin(\Omega)}{1 - \gamma \cos(\Omega)} \right)$
DTFT Examples - exponential signal (cont)

• The plots above are for $\gamma = 0.5$.

• Note how $X$ has period $2\pi$.

• Exercise: What are the maximum and minimum values of $|X|$, i.e., how would this plot depend on $\gamma > 0$? Plot $x$ and $\Delta X$. How do these plots differ when $-1 < \gamma < 0$?

• Exercise: Find the DTFT of anticausal signal $x[k] = \gamma^k u[-k]$ for scalar $\gamma$ s.t. $|\gamma| > 1$.

• Exercise: Find the DTFT of $x[k] = |k|$, $k \in \mathbb{Z}$, for scalar $\gamma$ s.t. $|\gamma| < 1$.

---

DTFT Examples - Square and Triangle Pulse

• For $T \in \mathbb{Z}^{>0}$, the even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ (i.e., $x[k] = u[k + T] - u[k - (T + 1)]$), has DTFT

$$X(\Omega) = \sum_{k=0}^{T} 1 - e^{-j\Omega k} = 1 + 2 \sum_{k=1}^{T} \cos(k\Omega), \Omega \in \mathbb{R}.$$  

• Exercise (even rectangle pulse in frequency domain): Show that for fixed $\Omega'$ s.t. $0 < \Omega' < \pi$,

$$\mathcal{F}^{-1}\{\Delta_{-\Omega'}u - \Delta_{\Omega'}u\}[k] = \frac{\Omega'}{\pi} \text{sinc}(\Omega'k), \quad k \in \mathbb{Z}.$$  

• For $T \in \mathbb{Z}^{>0}$, the odd triangle pulse with support $2T + 1$, $x[k] \equiv k(\Delta^{-T}u[k] - \Delta^{T+1}u[k])$ has DTFT

$$X(\Omega) = \sum_{k=-T}^{T} ke^{-j\Omega k} = -2j \sum_{k=1}^{T} k \sin(k\Omega), \Omega \in \mathbb{R}.$$  

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For fixed time $K_0$, clearly
\[ \mathcal{F}\{\delta[k-K_0]\}(\Omega) = e^{jK_0\Omega}, \]
where here $\delta$ is the unit pulse.

Note that $e^{jK_0\Omega}$ is a sinusoidal function of $\Omega$ with period $2\pi/K_0$ radians.

**Exercise:** For fixed frequency $\Omega_0$ radians/unit-time, show that
\[ \mathcal{F}\{e^{-j\Omega_0 k}\}(\Omega) = 2\pi \sum_{v=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi v), \]
where here $\delta$ is the Dirac impulse (in the frequency domain $\Omega \in \mathbb{R}$). Hint: work with $\mathcal{F}^{-1}$.

So, the DTFT of a $N$-periodic signal with Fourier series
\[ \sum_{r=0}^{N-1} D_r e^{jr2\pi k/N} \xmapsto{\mathcal{F}} 2\pi \sum_{v=-\infty}^{\infty} \sum_{r=0}^{N-1} D_r \delta(\Omega - v2\pi N + 2\pi v), \]
\[ \text{i.e., shift in time by } K_0 \text{ corresponds to product with sinusoid of period } 2\pi/K_0 \text{ (linear phase shift) in frequency domain.} \]

**Exercise:** Prove the dual property that if fixed $\Omega_0 \in \mathbb{R}$ and $X = \mathcal{F}\{x\}$ then
\[ \mathcal{F}\{x[k]e^{j\Omega_0 k}\}(\Omega) = X(\Omega - \Omega_0), \]
\[ \text{i.e., modulation (multiplication by a sinusoid) in time domain results in frequency shift.} \]
DTFT - convolution properties

• Let \( X_r = \mathcal{F}\{x_r\} \) for \( r \in \{1, 2\} \).

\[
\mathcal{F}\{x_1 \ast x_2\}(\Omega) := \sum_{k=-\infty}^{\infty} (x_1 \ast x_2)[k]e^{-j\Omega k}
\]
\[
= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[l]x_2[k-l]e^{-j(k-l)\Omega}e^{-j\Omega l} \quad \text{i.e., } x e^{j\Omega} e^{-j\Omega} = 1
\]
\[
= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k']e^{-jk'\Omega}e^{-j\Omega l} \quad \text{where } k' = k - l
\]
\[
= \sum_{l=-\infty}^{\infty} x_1[l]e^{-j\Omega l} \sum_{k=-\infty}^{\infty} x_2[k']e^{-jk'\Omega} =: X_1(\Omega)X_2(\Omega)
\]

• **Exercise:** Prove the dual property that

\[
\mathcal{F}\{x_1 x_2\}(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(v)X_2(\Omega - v)dv.
\]

• **Exercise:** Use the convolution properties to prove the time and frequency shift properties. Hint: \((\Delta^K, \delta) * x = \Delta^K * x\).

• **Exercise:** Show that DTFT is a linear operator.

DTFT - Parseval's Theorem

• The energy of a signal DT \( x \) is

\[
E_x := \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} (\mathcal{F}^{-1}X)[k](\mathcal{F}^{-1}X)[k]
\]
\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')e^{j\Omega k}d\Omega' \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega k}d\Omega
\]
\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \overline{X(\Omega)} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega k}e^{-j\Omega k}d\Omega
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \overline{X(\Omega)} \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{j\Omega k}e^{-j\Omega k} \right) d\Omega d\Omega' \]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \overline{X(\Omega)} \frac{1}{2\pi} \left( 2\pi \delta(\Omega - \Omega') \right) d\Omega d\Omega' \]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')\overline{X(\Omega')}d\Omega' = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega')|^2 d\Omega' \quad \text{recalling that for fixed } \Omega': \mathcal{F}^{-1}\{2\pi \delta(\Omega - \Omega')\}[k] = e^{-j\Omega k} \text{ & } \int_{-\pi}^{\pi} \overline{X(\Omega)}\delta(\Omega - \Omega')d\Omega = \overline{X(\Omega')}.
\]
The even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ has energy

$$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-T}^{T} 1^2 = 2T + 1.$$ 

Recall its DTFT is $X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k}$, so

$$\frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{2\pi} X(\Omega)\overline{X(\Omega)} d\Omega = \frac{1}{2\pi} \int_{2\pi} \sum_{k=-T}^{T} e^{-j\Omega k} \sum_{k=-T}^{T} e^{+j\Omega k} d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi} \left( \sum_{k=-T}^{T} 1 + \sum_{k<k'} e^{j(k'-k)\Omega} \right) d\Omega$$

$$= \sum_{k=-T}^{T} \frac{1}{2\pi} \int_{2\pi} 1 d\Omega + \sum_{k<k'} \frac{1}{2\pi} \int_{2\pi} e^{j(k'-k)\Omega} d\Omega = \sum_{k=-T}^{T} 1 + 0$$

$$= 2T + 1.$$

**Exercise:** Repeat this calculation using $X(\Omega) = 1 + 2 \sum_{k=1}^{T} \cos(\Omega k)$.

**Exercise:** Compute amount of energy of $x$ in the frequency band $[-\pi/6, \pi/6]$ radians/unit-time, i.e.,

$$\frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} |X(\Omega)|^2 d\Omega$$
• Consider a SISO, DT-LTIC system described by the difference equation
\[ Q(\Delta^{-1})y = P(\Delta^{-1})f, \]
where \( f \) is the input and \( y \) is the ZSR (output).

• Recall that by the time-shift property,
\[ Q(e^{j\Omega})Y_{ZS}(\Omega) = P(e^{j\Omega})F(\Omega) \Rightarrow Y_{ZS}(\Omega) = H(\Omega)F(\Omega). \]

• We now re-derive from first principles the eigenresponse by first recalling that the ZSR
\[ y_{ZS} = f * h \] where \( h \) is the unit-pulse response.

• Taking DTFTs, \( Y_{ZS} = H \) \( F \) where \( H = Fh \) is the transfer function.

• Suppose the system is BIBO/asymptotically stable, i.e., the \( n \) roots of \( Q \) (system char.
modes/poles) \( z \) all have modulus \( |z| < 1 \).

• The ZSR will consist of a forced response plus characteristic modes, where the latter will
\( \to 0 \) over time (our stability assumption) so that the forced response becomes the steady-
state response.

The forced response to a persistent sinusoidal input
\[ f[k] = A_f e^{j(\Omega_k + \phi)} \]
will be of the form
\[ y_{ss}[k] = A_y e^{j(\Omega_k + \phi)} \]
where (for \( k \geq 0 \),
\[ Q(e^{j\Omega_k})y_{ss}[k] = (Q(\Delta^{-1})y_{ss})[k] = (P(\Delta^{-1})f)[k] = P(e^{j\Omega_k})f[k]. \]
\[ \Rightarrow y_{ss}[k] = \frac{P(e^{j\Omega_k})}{Q(e^{j\Omega_k})}f[k] \]

Also, the ZSR \( y_{ZS} = h * f \), i.e., for all time \( k \geq 0 \):
\[ y_{ZS}[k] = \sum_{v=0}^{k} h[v]A_f e^{j(\Omega(v-k) + \phi)} = f[k] \sum_{v=0}^{k} h[v]e^{-j\Omega_v} \]
\[ \to f[k]H(\Omega_v) =: y_{ss}[k] \text{ as } k \to \infty. \]
Equating the forced responses (steady-state response for a stable system), we again get that the system transfer function is

\[ H(\Omega) = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})} = (Fh)(\Omega). \]

Note that \( \forall k \in \mathbb{Z}, H(\Omega) = H(\Omega + 2\pi k). \)

Also, we write \( H(\Omega) \) not \( H(e^{j\Omega}) \) for the DTFT.

So, the eigenresponse of a BIBO/asymptotically stable SISO, DT-LTIC system is the steady-state response to a sinusoid:

\[ f[k] = A_f e^{j(\Omega_0 k + \phi_f)} \rightarrow H(\Omega_0) f[k] = A_y e^{j(\Omega_0 k + \phi_y)} = y_{ss}[k] \]

The system magnitude response (gain) is \( |H(\Omega)| = |P(e^{j\Omega})|/|Q(e^{j\Omega})| \), i.e., \( A_y = A_f |H(\Omega_0)| \).

The system phase response is \( \angle H(\Omega) = \angle P(e^{j\Omega}) - \angle Q(e^{j\Omega}) \), i.e., \( \phi_y = \phi_f + \angle H(\Omega_0) \).

---

### Eigenresponse - example

**Problem:** For the system \( 2y[k] = 0.6y[k - 1] - 7f[k] \) find the steady-state response (if it exists) to \( f[k] = 4 \cos(5k) u[k] \).

**Solution:** The difference equation in standard form is

\( (Q(\Delta^{-1})y)[k] = y[k + 1] - 0.3y[k] = -3.5f[k + 1] = (P(\Delta^{-1})f)[k] \), where \( Q(z) = z - 0.3 \) and \( P(z) = -3.5z \).

The sole system characteristic value (root of \( Q, \) system pole) is 0.3, hence the system is BIBO/asymptotically stable.

By Euler’s identity \( f[k] = (2e^{j5k} + 2e^{j(-5)k})u[k] \)

By linearity, the eigenresponse is therefore

\( 2H(5)e^{j5k} + 2H(-5)e^{j(-5)k}, \)

where \( H(\Omega) = P(e^{j\Omega})/Q(e^{j\Omega}) = -3.5e^{j\Omega}/(e^{j\Omega} - 0.3) = \frac{3.5}{e^{j\Omega} - 0.3}, \angle H(\Omega) = \pi + \Omega - \arctan\left(\frac{-\sin(\Omega)}{\cos(\Omega) - 0.3}\right) \).

**Exercise:** Show that the eigenresponse is also simply \( |H(5)|4 \cos(5k + \angle H(5)) \).
2D Image Processing Example

- Apply 1-dimensional filtering to a 2-dimensional (2D) image by separately performing row and column operations.
- For $256 \times 256$ pixel (2D) image,

$$f = \begin{bmatrix} f[1,1] & f[1,2] & \ldots & f[1,256] \\ f[2,1] & f[2,2] & \ldots & f[2,256] \\ \vdots & \vdots & \ddots & \vdots \\ f[256,1] & f[256,2] & \ldots & f[256,256] \end{bmatrix}$$

- If $f[k,i]$ represents the 8-bit (grey) intensity of the pixel in row $k$ and column $i$ (i.e., 8 bits per pixel or bpp), then the "raw" image size will be $256^3 \text{bits} = 16 \text{Mb} = 2 \text{MB}$.
- Each of $f$’s rows of pixels can be processed by a system with unit-pulse response $h$ to obtain a new row of pixels, and thus a new image $y$:

$$\forall k, \ f[k,\cdot] \rightarrow h \rightarrow y[k,\cdot]$$

- Alternatively, each of $f$’s columns of pixels can be processed by a system with unit-pulse response $h$ to obtain a new column of pixels, and thus a new image $y$:

$$\forall i, \ f[\cdot,i] \rightarrow h \rightarrow y[\cdot,i]$$

---

Image Processing: High-Pass and Low-Pass Filtering

- The system $h$ may have a specific signal processing objective.
- The output pixels $y[k,i]$ may be quantized to fewer bpp than those of the input, thus achieving image compression.
- The simple low-pass filter (L)

$$h[k] = \frac{1}{2}(\delta[k] + \delta[k-1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] + f[k-1])$$

can capture shading and texture in the image.
- The simple high-pass filter (H)

$$h[k] = \frac{1}{2}(\delta[k] - \delta[k-1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] - f[k-1])$$

can capture edges in the image.
- Typically more compression possible in higher-frequency bands (H).
Image Processing: Tandem Row and Column Filtering

\[ f \rightarrow \text{row filtering} \rightarrow \text{column filtering} \rightarrow y \]

- Define \( y_{\text{LH}} \) as the output of
  \[ f \rightarrow \text{L} \rightarrow \text{H} \rightarrow y_{\text{LH}} \]

- Similarly define \( y_{\text{LL}}, y_{\text{HH}} \) and \( y_{\text{HL}} \).

- The \( y \) images are downsampled by a factor of four (two in each direction).

- The \( y_{\text{LL}} \) image will have a lot of energy while \( y_{\text{HH}} \) will have the least energy.

- This motivates non-uniform quantization (bit allotment per pixel) of these images.

- Together with a coding strategy for the quantized images (particularly for the regions of zero pixel-values), this is the basic approach used in JPEG leading to very good compression, e.g., from 8 bpp to 0.2-0.5 bpp.

Sampling Continuous-Time Signals (A/D)

- Consider continuous-time signal \( x \) with \( X = \mathcal{F}x \).

- Recall that by sampling at period \( T \) with impulses in continuous time \( t \in \mathbb{R} \), we get
  \[
  x_T(t) := \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} x(kT)e^{-jkTw} =: X_T(w),
  \]
  equivalently,
  \[
  X_T(w) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( w - \frac{2\pi v}{T} \right).
  \]

- Now define the sampled process in discrete-time \( k \in \mathbb{Z} \) and its DTFT,
  \[
  \hat{x}[k] := x(kT) \xrightarrow{\mathcal{F}} \hat{X}(\Omega) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{-j\Omega k}.
  \]

- Substituting \( w = \Omega/T \) we get
  \[
  \hat{X}(\Omega) = X_T \left( \frac{\Omega}{T} \right) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( \frac{\Omega - v2\pi}{T} \right).
  \]

- Exercise: Read decimation (downsampling) and interpolation (upsampling) of Lathi Figs. 8.17 & 10.9.
Sampling Continuous-Time Signals - example

- We are particularly interested in the case where
  - the continuous-time signal $x$ is band-limited, i.e., $\exists w' > 0$ s.t. $X(w) = 0$ for $|w| > w'$, and
  - the sampling frequency is greater than Nyquist’s, i.e., $2\pi/T > 2w' \Rightarrow w'T < \pi$.

- **Example:** For fixed $w' > 0$, consider the cts-time signal $x(t) = Asinc(w't)$ with FT
  \[
  X(w) = \frac{A\pi}{w'}(u(w + w') - u(w - w')).
  \]

- Sampling $x$ at period $T < \pi/w'$ we get the discrete-time signal $x[k] = Asinc(w'kT)$.

- Using inverse DTFT, recall that we can easily check that the DTFT of $x$ is,
  \[
  \hat{X}(\Omega) = \sum_{v=-\infty}^{\infty} \frac{A\pi}{w'T}(u(\Omega + w'T - 2\pi v) - u(\Omega - w'T - 2\pi v))
  = \sum_{v=-\infty}^{\infty} \frac{1}{T}X\left(\frac{\Omega - 2\pi v}{T}\right),
  \]
  noting $\forall T > 0$, $u(\frac{\Omega}{T} \pm w') = u(\frac{1}{T}(\Omega \pm w'T)) = u(\Omega \pm w'T)$, $\Omega := \Omega - 2\pi v$.

**Sampling Continuous-Time Signals - example (cont), $w'T < \pi$**
Suppose the signal $f$ is sampled every $T_s$ seconds, i.e., at sampling frequency $w_s := 2\pi/T_s$.

Recall Poisson’s identity (the Fourier series of the picket-fence function)

$$p_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}$$

Let’s rederive the relationship between the spectrum of a sampled continuous-time signal and its discrete-time counterpart by first defining the discrete-time signal

$$\hat{f}[k] = f(kT_s).$$

We want to relate the (continuous-time) Fourier transform of $f$ to the (discrete-time) Fourier transform of $\hat{f}$,

$$\hat{F}(\Omega) := \sum_{k=-\infty}^{\infty} \hat{f}[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f(kT_s)e^{-j\Omega k}.$$
• Now consider a discrete time signal $\hat{y}[k]$.

• We implement at D/A with a $T_s$-second hold, i.e., construct the continuous-time signal

$$y(t) := \sum_{k=-\infty}^{\infty} \hat{y}[k]r_{T_s}(t - kT_s),$$

where

$$r_{T_s}(t) := u(t) - u(t - T_s) \overset{\mathcal{F}}{\rightarrow} T_s \text{sinc}(wT_s/2)e^{-jwT_s/2} =: R_{T_s}(w).$$

• Note that $y$ is in the form of a convolution, so:

$$Y(w) = \sum_{k=-\infty}^{\infty} \hat{y}[k]R_{T_s}(w)e^{-jwkT_s},$$

$$= R_{T_s}(w)\hat{Y}(wT_s).$$

---

Sampled Data Systems: equivalent cts-time transfer function

• Consider a digital system $\hat{H}(\Omega)$ (or $\hat{H}(e^{j\Omega})$ depending on notation), whose (ZS) output is $\hat{y}$ when the input is $\hat{f}$, i.e., $\hat{Y} = \hat{H}\hat{F}$.

• The equivalent continuous-time transformation of the tandem system

$$f \rightarrow \begin{array}{c} \text{A/D (} T_s\text{-sample)} \\
\end{array} \overset{\mathcal{L}}{\rightarrow} \hat{H}(\Omega) \overset{\mathcal{F}}{\rightarrow} \begin{array}{c} \text{D/A (} T_s\text{-hold)} \\
\end{array} \rightarrow y$$

with input $f$ has (ZS) output

$$Y(w) = R_{T_s}(w)\hat{Y}(wT_s) = R_{T_s}(w)\hat{H}(wT_s)\hat{F}(wT_s)$$

$$= R_{T_s}(w)\hat{H}(wT_s)\frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s).$$

• Exercise: Show that if $f$ is band-limited by $w_s/2$ (i.e., $w_s$ is greater than $f$’s Nyquist frequency) and the previous sampled data system is followed by an ideal low-pass filter with bandwidth $w_s/2$, then the equivalent (continuous-time) transfer function is

$$H(w) = \hat{H}(wT_s)T_s^{-1}R_{T_s}(w)(u(w + w_s/2) - u(w - w_s/2))$$
Note that the term in the transfer function $H$,

$$T_s^{-1} R_T(w)(u(w + w_s/2) - u(w - w_s/2)) = \text{sinc}(\Omega/2)(u(\Omega + \pi) - u(\Omega - \pi))$$

is not a constant function of $\Omega = wT_s$.

This distortion due to the hold function $R$ can be reduced by putting in tandem with $\hat{H}$ an equalizer system with transfer function approximately

$$\hat{R}^{-1}(\Omega) := \sum_{k=-\infty}^{\infty} \frac{u(\Omega + \pi - k2\pi) - u(\Omega - \pi - k2\pi)}{\text{sinc}((\Omega - k2\pi)/2)}$$

i.e.,

$$\hat{H}(\Omega) \rightarrow \hat{R}^{-1}(\Omega)$$

---

Sampled Data Systems: equalization of hold sinc($\Omega/2$) by $\hat{R}^{-1}(\Omega)$

- the hold (at left, $R$) distorts the signal by attenuating its higher frequency components
- the equalizer (at right, $R^{-1}$) amplifies at the higher frequencies to cancel out this distortion

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DFT and FFT - Reading Exercise on Computational Issues

- Read Lathi Sec. 5.2 and 5.3 re. continuous-time FS, FT
- Read Lathi Sec. 10.6 re. DTFS, DTFT

Transient analysis in discrete time by unilateral $z$-transform

- $z$-transform definition and region of convergence.
- Basic $z$-transform pairs and properties.
- Inverse $z$-transform of rational polynomials by Partial Fraction Expansion (PFE).
- Total transient response of SISO DT LTIC systems $Q(\Delta^{-1})y = P(\Delta^{-1})f$.
- The steady-state eigenresponse revisited.
- System composition and canonical realizations.
The unilateral $z$-transform & region of convergence

- The $z$-transform of a signal $x = \{x[k]\}_{k \geq 0}$ is
  \[ X(z) = (Zx)(z) = \sum_{k=0}^{\infty} x[k]z^{-k} := \lim_{K \to \infty} \sum_{k=0}^{K} x[k]z^{-k}, \]
  where $z \in \mathbb{C}$.

- If the signal $x$ is bounded by an exponential (geometric), i.e.,
  \[ \exists M, \gamma \in \mathbb{R}^+ \text{ such that } \forall k \in \mathbb{Z}_{\geq 0}, \ |x[k]| \leq M \gamma^k \ (i.e., \ -M \gamma^k \leq x[k] \leq M \gamma^k) \]
  then the series $X(z)$ converges in the region outside of a disk centered $0 \in \mathbb{C}$,
  \[ \{z \in \mathbb{C} | |z| > \gamma\}. \]

- To see why bounded by an exponential suffices, recall absolute convergence $\Rightarrow$ convergence:
  \[ \forall k \geq 0, \ |x[k]z^{-k}| = |x[k]| \cdot |z|^{-k} \leq M \gamma^k |z|^{-k} = M \left(\gamma/|z|\right)^k \]
  $\Rightarrow \sum_{k=0}^{\infty} |x[k]z^{-k}| \leq M \sum_{k=0}^{\infty} \left(\gamma/|z|\right)^k$ which converges if $\gamma/|z| < 1$.

---

Basic $z$-transform pairs and RoCs

- $\delta[k] \xrightarrow{Z} 1, \ z \in \mathbb{C}$
  \[ u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \ |z| > 1 \]
  \[ \beta^k u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{1}{1-\beta z^{-1}} = \frac{z}{z-\beta}, \ |z| > |\beta| \]
  \[ \{\beta^{-1}u[k-1]\}(z) \xrightarrow{Z} \sum_{k=1}^{\infty} \beta^{-1} z^{-k} = z^{-1} \sum_{k=0}^{\infty} \beta^{-1} z^{-k} = z^{-1} \frac{1}{1-\beta z^{-1}} = \frac{1}{z-\beta}, \ |z| > |\beta| \]
  \[ e^{j\Omega k} u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} e^{j\Omega k} z^{-k} = \frac{1}{1-e^{j\Omega} z^{-1}}, \ |z| > 1 \ (\beta = e^{j\Omega}) \]
  \[ k\beta^k u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} k\beta^k z^{-k} = \beta \frac{d}{d\beta} \sum_{k=0}^{\infty} \beta^k z^{-k} = \beta \frac{d}{d\beta} \frac{1}{1-\beta z^{-1}} = \frac{\beta z^{-1}}{(1-\beta z^{-1})^2}, \ |z| > |\beta| \]

**Exercise:** Find $Z\{A \cos(\Omega k + \phi) u[k]\}$ and $Z\{A \sin(\Omega k + \phi) u[k]\}$. 

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Basic $z$-transform properties: linearity

- The $z$-transform is a linear operator: for all scalars $a_1, a_2 \in \mathbb{C}$ and all signals $x_1, x_2 : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ with respective ROCs $C_1, C_2 \subset \mathbb{C}$,
  \[
  (\mathcal{Z}\{a_1x_1 + a_2x_2\})(z) = a_1(\mathcal{Z}x_1)(z) + a_2(\mathcal{Z}x_2)(z), \quad z \in C_1 \cap C_2.
  \]

- Note that
  \[
  \{z \mid |z| > \gamma_1\} \cap \{z \mid |z| > \gamma_2\} = \{z \mid |z| > \max\{\gamma_1, \gamma_2\}\} \subset \mathbb{C}.
  \]

Basic $z$-transform properties: advance time shift

- Advance time shift (no change in RoC): Let $X = \mathcal{Z}x$.
  \[
  \Delta^{-1}x \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k+1]z^{-k} = -zx[0] + \sum_{k=-1}^{\infty} x[k+1]z^{-k} \\
  = -zx[0] + z \sum_{k=-1}^{\infty} x[k+1]z^{-(k+1)} \\
  = -zx[0] + z \sum_{k'=0}^{\infty} x[k']z^{-k'} \\
  = -zx[0] + zX(z)
  \]

- Exercise: For $v \in \mathbb{Z}_{\geq 0}$ show by induction that
  \[
  (\mathcal{Z}\{\Delta^{-v}x\})(z) = -\sum_{k=1}^{v} z^kx[v-k] + z^vX(z)
  \]
Basic \( z \)-transform properties: delay time shift

- Delay time shift (no change in RoC). For \( v \in \mathbb{Z}^+ \),
  \[
  \Delta^v(xu) \overset{z}{\rightarrow} \sum_{k=0}^{\infty} x[k-v]u[k-v]z^{-k} = \sum_{k=v}^{\infty} x[k]z^{-k-v} = z^{-v}X(z).
  \]

- So in the “zero-state” (input-output) context (i.e., \( x[k]u[k] = 0 \) for \( k < 0 \)), we identify multiplying by \( z^{-1} \) in complex-frequency domain with the unit delay \( \Delta \) in the time domain.

- Delay \( v \in \mathbb{Z}^+ \) of non-causal \( x \):
  \[
  \Delta^v x \overset{z}{\rightarrow} \sum_{k=0}^{\infty} x[k-v]z^{-k} = \sum_{k=-v}^{\infty} x[k']z^{-k'-v} = \sum_{k'=-v}^{\infty} x[k']z^{-k'-v} + z^{-v}X(z).
  \]

Basic \( z \)-transform properties: frequency shift & convolution

- Let \( X = Zx \) with RoC \( C(\gamma) := \{ z \in \mathbb{C} \mid |z| > \gamma \} \).
  \[
  \beta^k x[k] \overset{z}{\rightarrow} \sum_{k=0}^{\infty} \beta^k x[k]z^{-k} = \sum_{k=0}^{\infty} x[k](z/\beta)^{-k} = X(z/\beta), \quad z \in C(\gamma/|\beta|).
  \]
  i.e., \( \times \beta^k \) in the time-domain is dilation by \( \beta \) in the \( z \)-domain.

- For signals \( x_1, x_2 : \mathbb{Z}^+ \to \mathbb{C} \) \( x_1[k], x_2[k] = 0 \) for \( k < 0 \), with respective ROCs \( C_1, C_2 \subset \mathbb{C} \),
  \[
  x_1 \ast x_2 \overset{z}{\rightarrow} \sum_{k=0}^{\infty} (x_1 \ast x_2)[k]z^{-k} = \sum_{k=0}^{\infty} \sum_{v=0}^{k} x_1[v]x_2[k-v]z^{-(k-v)}z^{-v} = \sum_{v=0}^{\infty} x_1[v]z^{-v} \sum_{k=v}^{\infty} x_2[k-v]z^{-(k-v)} = \sum_{v=0}^{\infty} x_1[v]z^{-v} \sum_{k=0}^{\infty} x_2[k']z^{-k'} = X_1(z)X_2(z), \quad z \in C_1 \cap C_2.
  \]
Basic $z$-transform properties: convolution, IVT & FVT

- So convolution in the time-domain is multiplication in the frequency domain.
- The converse is also true.
- Directly by definition of $X = Zx$, we get the initial value theorem
  \[ \lim_{z \to \infty} X(z) = x[0]. \]
- There is also a “final value” theorem for $\lim_{k \to \infty} x[k]$.

Total response of SISO LTIC systems

- We now study transient analysis of LTI difference equations using $z$-transforms.
- Recall our system is defined given polynomials $P, Q$, input $f$ and initial conditions:
  - $Q(\Delta^{-1})y = P(\Delta^{-1})f$, where $y$ is the (total) output and
  - input $f[k] = 0$ for $k < 0$,
  - degree of polynomial $Q = n \geq m =$ degree of polynomial $P$ (causal system),
  - $Q(z) = z^n + \sum_{v=0}^{n-1} a_v z^v$ (i.e., $a_n = 1$) and $P(z) = \sum_{v=0}^{m} b_v z^v$,
  - $a_n \neq 0$ or $b_n \neq 0$ for poly'ls $Q, P$ of minimum degree,
  - $n$ initial conditions $y[-n], y[-n+1], ..., y[-2], y[-1]$.
- We can restate the difference equation in terms of delays by delaying both sides by $n$ time-units (i.e., applying with $\Delta^n$), to get
  \[
  \Delta^n Q(\Delta^{-1})y = \Delta^n P(\Delta^{-1})f
  \]
  \[\Rightarrow \quad \hat{Q}(\Delta)y := \sum_{v=0}^{n} a_v \Delta^{n-v} y = \sum_{v=0}^{m} b_v \Delta^{n-v} f =: \hat{P}(\Delta)f \]
So, taking the $z$-transform of the (delay) difference equation, we get by the (delay) time-shift and linearity properties that

$$
\sum_{v=0}^{n} a_v \sum_{k=-v}^{-1} y[k] z^{-k-v} + \tilde{Q}(z^{-1}) Y(z) = \tilde{P}(z^{-1}) F(z)
$$

So, solving for the total response $Y$ we get

$$
Y(z) = \frac{\tilde{P}(z^{-1}) F(z)}{Q(z^{-1})} - \frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{-1} y[k] z^{-k-v}}{Q(z^{-1})} = Y_{ZS}(z) + Y_{ZI}(z)
$$

where the ZIR and ZSR in the complex-frequency ($z$) domain respectively are

$$
Y_{ZI}(z) := -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{-1} y[k] z^{-k-v}}{Q(z^{-1})} = -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{-1} y[k] z^{n-k-v}}{Q(z)}
$$

$$
Y_{ZS}(z) := \frac{\tilde{P}(z^{-1}) F(z)}{Q(z^{-1})} = \frac{P(z) F(z)}{Q(z)} = H(z) F(z) \quad \text{(transfer function $H$)}.
$$

Regarding this total transient response, note how

- the $z$-transform’s unilateral aspect captures the impact of initial conditions (ZIR), and
- a greater range of inputs $f$ than under DTFT through RoC $\subset \mathbb{C}$ (not just $|z| = 1$).

### Total response of SISO LTIC systems - example

- Suppose i.c. $y[-1] = -1$, input $f[k] = 2(-3)^k u[k]$ and output $y$ s.t.
  \[ \forall k \geq -1, \quad 2y[k+1] + 2y[k] = 3f[k+1] + 2f[k]. \]

- To find the total response $y$, we take the $z$-transform of the equivalent system: $\forall k \geq 0$,
  \[ 2y[k] + 2y[k-1] = 3f[k] + 2f[k-1] \]
  \[ \Rightarrow 2Y(z) + 2(z^{-1}Y(z) + y[-1]) = 3F(z) + 2z^{-1} F(z). \]

- So by the delay property for non-causal signals ($y$), the total response
  \[ Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} F(z) + \frac{-2y[-1]}{2 + 2z^{-1}} \]
  \[ = H(z) F(z) + \frac{-y[-1]}{1 + z^{-1}} \]
  \[ = Y_{ZS}(z) + Y_{ZI}(z) \]

with RoC for $Y$ being the intersection of those of $F$ and $H$. 

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Total response of SISO LTIC systems - example

- Here \( F(z) = \mathcal{Z}\{2(-3)^k\} = 2/(1 + 3z^{-1}) \), \( y[-1] = -1 \), so

\[
Y(z) = \frac{3 + 2z^{-1}}{3 + 2z^{-1}} \frac{2}{1 + 3z^{-1} + 1} + \frac{1}{1 + z^{-1}}
\]

\[
= \frac{(1 + z^{-1})(1 + 3z^{-1})}{(1 + z^{-1})(1 + 3z^{-1})} + \frac{1}{1 + z^{-1}}
\]

where for the last equality see PFE below (here in \( z^{-1} \)).

- Understanding that the ZIR begins at \( k = -1 \) (initial condition) and the ZSR at time \( k = 0 \), we get:

\[
\forall k \geq -1, \quad y[k] = (3.5(-3)^k - 0.5(-1)^k)u[k] + (-1)^k = y_{ZS}[k] + y_{ZI}[k],
\]

where we minded the ambiguity \( \mathcal{Z}x = \mathcal{Z}xu \).

- Exercise: Verify this solution using time-domain methods, i.e.,

\[
y = y_{ZI} + y_{ZS} = y_{ZI} + h \ast f,
\]

where \( h \) and \( y_{ZI} \) consist of char. modes.

Inverse \( z \)-transform of proper rational polynomials

- We now describe how to find \( \mathcal{Z}^{-1}X \) of causal signal \( X \) that is rational polynomial in \( z \), i.e., \( X(z) = M(z)/N(z) \) where \( M(z) \) and \( N(z) \) are polynomials in \( z \).

- If \( \deg(M) = \deg(N) + 1 \), we perform long division to write \( X = c + \bar{M}/N \) where \( \deg(N) = \deg(\bar{M}) \) and \( \mathcal{Z}^{-1}X = c\delta + \mathcal{Z}^{-1}\{\bar{M}/N}\} \).

- If \( \deg(M) = \deg(N) \) and \( M(0) = 0 \) (so \( z^{-1}M(z) \) is a polynomial), we can factor \( z \) from \( M \) to get

\[
X(z) = z^{-1}M(z) / N(z).
\]

- We will find \( \mathcal{Z}^{-1}X \) using PFE of the strictly proper rational polynomial \( z^{-1}M(z)/N(z) \).

- Alternatively, we could apply PFE on strictly proper rational polynomials in \( z^{-1} \), \( z^{-K}M(z)/(z^{-K}N(z)) \) where \( K := \deg(N) \), as in the previous example.
Partial Fraction Expansion (PFE) example in $z$ (not $z^{-1}$)

- For example, suppose
  \[ X(z) := \frac{z(3z + 2)}{z^2 - 0.64} = \frac{(3z + 2)}{(z + 0.8)(z - 0.8)} \]
  \[ = z \left( \frac{0.25}{z + 0.8} + \frac{2.75}{z - 0.8} \right) = 0.25\frac{z}{z + 0.8} + 2.75\frac{z}{z - 0.8} \]
  where PFE (below) gave the numerators (residues) 0.25 and 2.75.

- So,
  \[ (z^{-1}X)[k] = 0.25(-0.8)^k u[k] + 2.75(0.8)^k u[k] \]

- Note that the associated RoC of $X$ is $\{ z \in \mathbb{C} \mid |z| > 0.8 \}$. 

Partial Fraction Expansion (PFE) - preliminaries

- Let $K = \deg(N) = \deg(M)$ so that we can factor
  \[ N(z) = \prod_{k=1}^{K} (z - p_k), \]
  where the $p_k$ are the roots of $N$ (poles of $M/N$).

- We assume $M$ and $N$ have no common roots, i.e., no “pole-zero cancellation” issue to consider, so that the $p_k$ are the poles of $M/N$.

- Again, we assume $M(0) = 0$ (0 is a zero of $M/N$) and so $z^{-1}M(z)$ is a polynomial of degree $K - 1$.

- Note that the RoC for $M(z)/N(z)$ is $\{ z \in \mathbb{C} \mid |z| > \max_k |p_k| \}$. 
PFE - the case of no repeated poles

• Suppose there are no repeated poles for $M/N$, i.e., $\forall k \neq l, \ p_k \neq p_l$.
• In this case, we can write the PFE of $z^{-1}M(z)/N(z)$ as

$$\frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l}$$

$$\Rightarrow \frac{M(z)}{N(z)} = \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{z}{z - p_l} = \sum_{l=1}^{K} \frac{1}{z - p_l}$$

where the scalars (Heaviside coefficients) $c_l \in \mathbb{C}$ are

$$c_l = \frac{z^{-1}M(z)}{\prod_{k \neq l} (z - p_k)} \bigg|_{z = p_l} = \lim_{z \to p_l} \frac{z^{-1}M(z)}{N(z)}(z - p_l) = \frac{z^{-1}M(z)}{N(z)}(z - p_l) \bigg|_{z = p_l}.$$ 

• That is, to find the Heaviside coefficient $c_k$ over the term $z - p_k$ in the PFE, we have removed (covered up) the term $z - p_k$ from the denominator $N(z)$ and evaluated the remaining rational polynomial at $z = p_k$.

• This approach, called the Heaviside cover-up method, works even when $p$ is $\mathbb{C}$-valued.

• Given the PFE of $z^{-1}M/N$, $(Z^{-1}M/N)[k] = \sum_{i=1}^{K} c_i p_i^k u[k]$.

PFE - proof of Heaviside cover-up method

• To prove that the above formula for the Heaviside coefficient $c_l$ is correct, note that the claimed PFE of $z^{-1}M(z)/N(z)$ is

$$\sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k)$$

• Thus, the PFE equals $z^{-1}M(z)/N(z)$ if and only if the numerator polynomials are equal, i.e., iff

$$z^{-1}M(z) = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) =: \hat{M}(z).$$

• Again, two polynomials are equal if their degrees, $L$, are equal and either:
  - their coefficients are the same, or
  - they agree at $L + 1$ (or more) different points, e.g., two lines ($L = 1$) are equal if they agree at 2 ($= L + 1$) points.

• Since $z^{-1}M(z)$ is a polynomial of degree $< K$, it suffices to check that whether $z^{-1}M(z) = \hat{M}(z)$ for all $z = p_k, k \in \{1, 2, ..., K\}$, i.e., this would create $K$ equations in $< K$ unknowns (the coefficients of $\hat{M}$).
• But note that any pole $p_r$ of $z^{-1} M(z)/N(z)$ is a root of all but the $r^{th}$ term in $\hat{M}$, so that

$$\hat{M}(p_r) = c_r \prod_{k \neq r} (p_r - p_k)$$

$$= \left( \frac{z^{-1} M(z)}{\prod_{k \neq r} (z - p_k)} \right) \bigg|_{z = p_r} \prod_{k \neq r} (p_r - p_k)$$

$$= \frac{p_r^{-1} M(p_r)}{\prod_{k \neq r} (p_r - p_k)} \prod_{k \neq r} (p_r - p_k)$$

$$= p_r^{-1} M(p_r).$$

• Q.E.D.

---

PFE - the case of no repeated poles - example

• To find the inverse $z$-transform of a proper rational polynomial $X = M/N$ with $M(0) = 0$, first factor its denominator $N$ and factor $z$ from $M$, e.g.,

$$X(z) = \frac{z^3 + 5z^2}{z^3 + 9z^2 + 26z + 24} = \frac{z^2 + 5z}{(z + 4)(z + 3)(z + 2)}, \text{ for } |z| > 4.$$

• So, by PFE

$$X(z) = z \left( \frac{c_4}{z + 4} + \frac{c_3}{z + 3} + \frac{c_2}{z + 2} \right) = \frac{z^{-1} M(z)}{N(z)} \Rightarrow$$

$$z^{-1} M(z) = 1z^2 + 5z + 0 = c_4(z + 3)(z + 2) + c_3(z + 4)(z + 2) + c_2(z + 4)(z + 3) =: \hat{M}(z).$$

• We can solve for the 3 constants $c_k$ by comparing the 3 coefficients of quadratic $M$ and $\hat{M}$.

• The Heaviside cover-up method suggests we try $z = -2, -3, -4$ to solve for $c_2, c_3, c_4$:

$$c_4 = \frac{z^2 + 5z}{(z + 3)(z + 2)} \bigg|_{z = -4} = -2, \quad c_3 = \frac{z^2 + 5z}{(z + 4)(z + 2)} \bigg|_{z = -3} = 6, \quad c_2 = \frac{z^2 + 5z}{(z + 4)(z + 3)} \bigg|_{z = -2} = -3$$

• Thus, $x[k] = (z^{-1} X)[k] = (-2(-4)^k + 6(-3)^k - 3(-2)^k)u[k]$. 

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PFE - the case of a non-repeated, complex-conjugate pair of poles

- Again, recall that for polynomials with all coefficients \( \in \mathbb{R} \), all complex poles will come in complex-conjugate pairs, \( p_1 = \bar{p}_2 \).
- The case of non-repeated poles \( p_1, p_2 = \alpha \pm j\beta \ (\alpha, \beta \in \mathbb{R}, j := \sqrt{-1}) \) that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients \( c_1, c_2 \), i.e., \( c_1 = \bar{c}_2 \).
- In the PFE, we can alternatively combine the terms
  \[
  \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} = \frac{r_1z + r_0}{(z - \alpha)^2 + \beta^2}
  \]
  where by equating the two numerator polynomials’ coefficients,
  \[
  r_0 = -c_1p_2 - c_2p_1 = -2\text{Re}\{c_1p_2\} \in \mathbb{R} \quad \text{and} \quad r_1 = c_1 + c_2 = 2\text{Re}\{c_1\} \in \mathbb{R}.
  \]
- Exercise: Show that
  \[
  2|c| \cdot |p|^k \cos(k \angle p + \angle c) \xrightarrow{z} \frac{cz}{z - p} + \frac{\bar{c}z}{z - \bar{p}}
  \]

PFE - non-repeated complex-conjugate pair of poles - example

- To find the inverse \( z \)-transform of
  \[
  X(z) = \frac{3z^2 + 2z}{z^3 + 5z^2 + 10z + 12},
  \]
  first factor the denominator and divide the numerator by \( z \) to get
  \[
  X(z) = \frac{3z + 2}{(z^2 + 2z + 4)(z + 3)}.
  \]
- Note that the poles of \( X \) are \(-3\) and \(-1 \pm j\sqrt{3}\) (so \( X \)'s RoC is \(|z| > 3\)).
- So, we can expand \( X \) to
  \[
  X(z) = z\frac{r_1z + r_0}{z^2 + 2z + 4} + z\frac{c_3}{z + 3},
  \]
  where by the Heaviside cover-up method,
  \[
  c_3 = \frac{3z + 2}{z^2 + 2z + 4}\bigg|_{z=-3} = -1.
  \]
• To find $r_1, r_0$, we will compare coefficients of the numerator polynomials of $X$ (actually $z^{-1}X$) and its PFE, i.e.,

$$0z^2 + 3z + 2 = (r_1z + r_0)(z + 3) + c_3(z^2 + 2z + 4) \quad (\ast)$$

$$= (r_1 - 1)z^2 + (3r_1 + r_0 - 2)z + 3r_0 - 4.$$  

• Thus, by comparing coefficients

$$0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4$$

we get 

$$r_0 = 2 \quad \text{and} \quad r_1 = 1.$$  

• Note how $z = -3$ in $(\ast)$ gives $c_3 = -1$ as Heaviside cover-up did.

• Thus by substituting, we get

$$X(z) = z \frac{z + 2}{z^2 + 2z + 4} + z \frac{-1}{z + 3}$$

• Exercise: Show that

$$x[k] = (z^{-1}X)[k]$$

$$= \left(\sqrt{\frac{4}{3}} 2^k \cos(k\frac{2\pi}{3} - \frac{\pi}{6}) - (-3)^k\right) u[k].$$
PFE - the general case of repeated poles

• If a particular pole \( p \) of \( z^{-1}M(z)/N(z) \) is of order \( r \geq 1 \), i.e., \( N(z) \) has a factor \( (z-p)^{r} \), then the PFE of \( z^{-1}M(z)/N(z) \) has the terms

\[
\frac{c_1}{z-p} + \frac{c_2}{(z-p)^2} + \ldots + \frac{c_r}{(z-p)^r} = \sum_{k=1}^{r} \frac{c_k}{(z-p)^k} = \frac{z^{-1}M(z)}{N(z)} - \Phi(z)
\]

with \( c_k \in \mathbb{C} \forall k \in \{1, 2, \ldots, r\} \), where \( \Phi(z) \) represents the other PFE terms of \( z^{-1}M(z)/N(z) \) (i.e., corresponding to poles \( \neq p \)).

• Note that equating \( z^{-1}M(z)/N(z) \) to its PFE and multiplying by \( (z-p)^{r} \) gives

\[
\frac{z^{-1}M(z)}{N(z)}(z-p)^{r} = c_r + \sum_{k=1}^{r-1} c_k(z-p)^{r-k} + \Phi(z)(z-p)^{r}
\]

\[
\Rightarrow \left. \frac{z^{-1}M(z)}{N(z)}(z-p)^{r} \right|_{z=p} = c_r,
\]

i.e., Heaviside cover-up (of the entire term \( (z-p)^{r} \)) works for \( c_r \).

PFE - the general case of repeated poles (cont)

• To find \( c_{r-1} \), we differentiate the previous display to get

\[
\frac{d}{dz} \left( \frac{z^{-1}M(z)}{N(z)}(z-p)^{r} \right) = \sum_{k=1}^{r-1} c_k(r-k)(z-p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z-p)^{r}
\]

\[
= c_{r-1} + \sum_{k=1}^{r-2} c_k(r-k)(z-p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z-p)^{r}
\]

\[
\Rightarrow c_{r-1} = \left( \frac{d}{dz} \left( \frac{z^{-1}M(z)}{N(z)}(z-p)^{r} \right) \right) \bigg|_{z=p}
\]

• If we differentiate the original display \( k \in \{0, 1, 2, \ldots, r-1\} \) times and then substitute \( z = p \), we get (with \( 0! := 1 \))

\[
\left( \frac{d^k}{dz^k} \left( \frac{z^{-1}M(z)}{N(z)}(z-p)^{r} \right) \right) \bigg|_{z=p} = k!c_{r-k}
\]

\[
\Rightarrow c_{r-k} = \frac{1}{k!} \left( \frac{d^k}{dz^k} \left( \frac{z^{-1}M(z)}{N(z)}(z-p)^{r} \right) \right) \bigg|_{z=p}.
\]
To find the inverse z-transform of
\[ X(z) = \frac{z(3z+2)}{(z+1)(z+2)^3}, \]
write the PFE of \( X \) as
\[ X(z) = z \left( \frac{c_1}{z+1} + \frac{c_{2,1}}{z+2} + \frac{c_{2,2}}{(z+2)^2} + \frac{c_{2,3}}{(z+2)^3} \right), \]
so clearly the RoC of causal \( X \) is \(|z| > 2\).

By Heaviside cover-up
\[ c_1 = \left. \frac{3z+2}{(z+2)^3} \right|_{z=-1} = -1 \quad \text{and} \quad c_{2,3} = \left. \frac{3z+2}{(z+1)^3} \right|_{z=-2} = 4. \]

Also,
\[ c_{2,2} = \left. \frac{1}{1!} \left( \frac{d}{dz} \frac{3z+2}{z+1} \right) \right|_{z=-2} = \left. \frac{1}{1!(z+1)^3} \right|_{z=-2} = 1 \]
\[ c_{2,1} = \left. \frac{1}{2!} \left( \frac{d^2}{d^2z} \frac{3z+2}{z+1} \right) \right|_{z=-2} = \left. \frac{1}{2!(z+1)^3} \right|_{z=-2} = 1 \]
Thus,
\[ X(z) = \frac{z}{z+1} + \frac{1}{z+2} + \frac{z}{(z+2)^2} + \frac{4}{(z+2)^3} \quad \forall |z| > 2 \]
\[ \Rightarrow x[k] = (Z^{-1}X)[k] = \left( (-1)^k + \frac{k(-2)^{k-1}}{2} \right) u[k] \]

Exercise: Show by induction and integration by parts that: \( \forall m \in \mathbb{Z}^{>0}, \)
\[ \binom{k}{m} z^{k-m} u[k] \xrightarrow{Z} \frac{z}{(z-\gamma)^m} \]

Exercise: Find the ZSR \( y \) to input \( f[k] = 2^k u[k] = 2^{|k|/2} u[k] \) of the marginally stable system \( H(z) = 4/(z^2 + 1) \).
PFE of \( M/N \) when \( M(0) \neq 0 \)

- If \( M(0) \neq 0 \) (so cannot factor \( z \) from \( M(z) \)), then just perform long division if \( \deg(M) \geq \deg(N) \) to get a strictly proper rational polynomial, factor \( N \) to find the poles, and find the PFE as before.

- When taking inverse \( z \)-transform, recall the \( z \)-transform pair:

\[
\beta^{k-1}u[k-1] \xrightarrow{Z} \frac{1}{z-\beta}, \quad |z| > |\beta|
\]

PFE without factoring \( z \) from the numerator first

- For example, to find the ZSR to \( f[k] = 2(-1)^ku[k] \) of the system

\[
y[k + 1] - 4y[k] = 5f[k],
\]

take the \( z \)-transform to get

\[
Y_ZS(z) = H(z)F(z) = \frac{5}{z-4}F(z) = \frac{10z}{(z-4)(z+1)}
\]

\[
= \frac{8}{z-4} + \frac{2}{z+1} \quad \text{(by PFE)}
\]

\[
\Rightarrow y_{ZS}[k] = 8(4)^{k-1}u[k-1] + 2(-1)^{k-1}u[k-1]
\]

- Note that the unit-pulse response is

\[
h[k] = Z^{-1}(H)[k] = 5(4)^{k-1}u[k-1],
\]

and that, by delaying the difference equation to get

\[
y[k] = -4y[k-1] + 5f[k-1],
\]

we see that (the ZSR) \( y_{ZS}[0] = 0 \).

- Exercise: First factor \( z \) from the numerator of \( Y_{ZS} \) before PFE to show that

\[
y_{ZS}[k] = 2(4)^{k}u[k] - 2(-1)^{k}u[k].
\]

Is this result different? Check for \( k = 0 \) and \( k > 0 \).
The total response of a SISO LTI system to input $f$ is of the form

$$Y(z) = H(z)F(z) + \frac{P_1(z)}{Q(z)} F(z) + \frac{P_1(z)}{Q(z)} = Y_{QS}(z) + Y_{ZI}(z).$$

where $P_1$ depends on the initial conditions and the RoC is the intersection of that of input $F = Zf$ and the system characteristic modes.

Unlike for DTFT notation, here write $H(z) = \frac{P(z)}{Q(z)} = (Zh)(z)$.

Suppose the system is BIBO/asymptotically stable and the input is a sinusoid at frequency $\Omega_o$ radians per unit time, $f[k] = Ae^{j(\Omega_o k + \phi)}u[k] = Ae^{j\phi}(e^{j\Omega_o})^k u[k]$ with $A > 0$

$\Rightarrow F(z) = Ae^{j\phi}z/(z - e^{j\Omega_o})$ with RoC $|z| > 1$.

Since $e^{j\Omega_o}$ cannot be a system pole (owing to asymptotic stability all poles have modulus strictly less than one), we can use Heaviside cover-up on

$$Y_{QS}(z) = H(z)F(z) = z \frac{P(z)}{Q(z)}(z - e^{j\Omega_o}) Ae^{j\phi} \quad \text{to get}$$

$$Y_{QS}(z) = z \frac{H(e^{j\Omega_o})} {z - e^{j\Omega_o}} Ae^{j\phi} + \text{char. modes} = H(e^{j\Omega_o})F(z) + \text{char. modes}.$$

Thus, the total response of an asymptotically stable system to a sinusoidal input $f$ at frequency $\Omega_o$ is

$$y[k] = H(e^{j\Omega_o})f[k] + \text{linear combination of characteristic modes.}$$

So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as $k \to \infty$,

$$y[k] \to H(e^{j\Omega_o})f[k] = H(e^{j\Omega_o})Ae^{j(\Omega_o k + \phi)} = |H(e^{j\Omega_o})| Ae^{j(\Omega_o k + \phi + \angle H(e^{j\Omega_o}))},$$

where again,

$H = P/Q$ is the system’s transfer function,

$|H(e^{j\Omega_o})|$ is the system’s magnitude response at frequency $\Omega_o$ radians/unit-time, and

$\angle H(e^{j\Omega_o})$ is the system’s phase response at $\Omega_o$. 

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• Laplace’s approximation: the rate at which the total response converges to the eigenresponse response is according to the characteristic value of largest modulus,
  – which will be $< 1$ owing to the stability assumption,
  – i.e., giving the modes(s) that $\to 0$ slowest.

• In continuous-time systems, it’s the characteristic value of largest real part, which will be negative owing to stability assumption.

Canonical (ZS) system-realizations - direct form

• Consider the proper ($m \leq n$) transfer function

$$H(z) = \frac{P(z)}{Q(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{Y(z)}{F(z)}$$

• The direct-form realization employs the interior system state $X := F/Q$, i.e., $F = QX$ and $Y = PX$ where the former implies (with $a_n = 1$),

$$F(z) = \sum_{r=0}^{n} a_r z^r X(z) \Rightarrow z^n X(z) = F(z) - \sum_{r=0}^{n-1} a_r z^r X(z).$$

• For $n = 2$, there are two "system states" (outputs of unit delays), $X$ and $zX$ (respectively, $x[k]$ and $(\Delta^{-1}x)[k] = x[k + 1])$:

![Diagram](attachment:image.png)
• Now adding $Y = PX$, we finally get the direct-form canonical system-realization of $H$:

![Diagram]

• Again, state variables taken as outputs of unit delays, here: $x, \Delta^{-1}x, \ldots, \Delta^{-(n-1)}x$.

• If $b_n = b_2 \neq 0$, there is direct coupling of input and output, $H$ is proper but not strictly so, $h = Z^{-1}H$ has a unit-pulse component $b_2\delta$.

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Canonical system-realizations - direct form (cont)

• Note that this $n = 2$ example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for $n = 2$, $H(z) = P(z)/Q(z)$ where

$$Q(z) = z^2 + a_1z + a_0 = (z - \alpha)^2 + \beta^2$$

for $\alpha, \beta \in \mathbb{R}$, so the poles are $\alpha \pm j\beta$.  

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In the general case of a proper transfer function, we can use partial-fraction expansion

- grouping the terms corresponding to a complex-conjugate pair of poles, i.e., a second-order denominator, and
- using a direct-form realization for these terms.

Besides the PFE-based and direct-form realizations, there are other (zero-state) system realizations, e.g., "observer" canonical.

For proper rational-polynomial transfer functions $H = P/Q$, all of these realizations involve $n$ (degree of $Q$) unit delays, the output of each being a different interior state variable of the system.

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**Canonical system realizations by PFE - example**

$$H(z) = \frac{.3z^2 - .1}{z^2 - .1z - .3} = .3 + \frac{.3z - .01}{(z - .6)(z + .5)} = .3 + \frac{1.7/1.1}{z - .6} + \frac{.16/1.1}{z + .5}$$

Note that one cannot factor $z$ from the numerator of $H$.

**Exercise:** Find a realization for this transfer function $H$ by

1. splitting/forking the input signal $F$,
2. using the direct canonical form for each of these 3 terms of $H$ found by long division and PFE, and
3. summing three resulting output signals to get the (ZS) output $Y = HF$.  

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Digital Proportional-Integral (PI) system

- Consider a continuous-time signal $x$ sampled every $T$ seconds,
  \[ \forall k \in \mathbb{Z}^+, \quad x[k] = x(kT), \]
  and its integral $y(t) = \int_0^t x(\tau) d\tau$.
- The sampled integral can be approximated, $y(kT) \approx y[k]$, by the trapezoidal rule,
  \[ y[k] = y[k-1] + \frac{x[k-1] + x[k]}{2}. \]
- In the complex-frequency domain,
  \[
  Y(z) = Y(z)z^{-1} + \frac{X(z)z^{-1} + X(z)T}{2} \Rightarrow \frac{Y(z)}{X(z)} = \frac{T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}.
  \]

Digital PI system (cont)

- So, a digital PI transfer function would be of the form,
  \[ G(z) = K_p + \frac{K_i}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}. \]
  for constants $K_p$, $K_i$.
- In practice, PID or PI systems $G$ are commonly used to control a plant $H$, where $G$ may
  be in series with $H$ or in the feedback branch.
- Exercises:
  - Draw the direct-form canonical realization for $G$.
  - Draw the block diagram for the closed-loop system with negative feedback:
    \[ Y = HX \text{ and } X = F - GY \] where $H$ is the (open-loop) system.
  - Find the closed-loop transfer function $Y/F$ and recall the pole placement problem to
    stabilize $H$. 
Recursive Least Squares (RLS) Filter - Introduction

- Consider a LTI system with input \( f \) and output \( y \),

\[
y[k] = \sum_{r=0}^{K} h[k - r]f[r] + v[k], \quad k \in \mathbb{Z},
\]

where \( v \) is an additive noise process and \( K \) is the maximum system order.

- The system (unit-pulse response) \( h \) is not known.

- Past values of the output \( y \) are observed (known).

- At time \( k \), the objective is to forecast the next output \( \hat{y}[k + 1] \), based on the assumed known/observed quantities:
  - the next input \( f[k + 1] \),
  - the past \( R \) input-output pairs \{ \( f[r], y[r] \) \}_{k-R+1 \leq r \leq k}.

-- RLS objective and \( R^{th} \)-order linear tap filter

- The output of an \( R^{th} \)-order RLS tap-filter at time \( k \) is,

\[
\hat{y}_k[i] = \sum_{r=i-R+1}^{i} \eta_k[i - r]f[r], \quad i \leq k + 1.
\]

- The objective of this filter at time \( k \) is to accurately estimate the system output \( y[k + 1] \) with \( \hat{y}_k[k + 1] \) by choosing the \( R \) filter coefficients

\[
\eta_k[k - R + 1], \ldots, \eta_k[k - 1], \eta_k[k]
\]

that minimize the following sum-of-square-error objective:

\[
\mathcal{E}_k = \sum_{r=k-R+1}^{k} \lambda^{k-r}|y[r] - \hat{y}_k[r]|^2 = \sum_{r=k-R+1}^{k} \lambda^{k-r}|e_k[r]|^2
\]

where

- \( \lambda > 0 \) is a forgetting factor and
- error \( e_k[r] := y[r] - \hat{y}_k[r] \).
RLS filter

- So, to minimize $\mathcal{E}_k$, substitute $\hat{y}_k[r]$ into $\mathcal{E}_k$ and solve

$$0 = \frac{\partial \mathcal{E}_k}{\partial \eta_k[i]} \text{ for } i \in \{k - R + 1, \ldots, k - 1, k\}.$$ 

- That is, $R$ equations in $R$ unknowns: for $i \in \{k - R + 1, \ldots, k - 1, k\}$,

$$0 = \sum_{r=k-R+1}^{k} 2\lambda^{k-r} e_k[r] \frac{\partial e_k[r]}{\partial \eta_k[i]}$$

$$= \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(y[r] - \hat{y}_k[r]) \left( -\frac{\partial \hat{y}_k[r]}{\partial \eta_k[i]} \right)$$

$$= \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(\hat{y}_k[r] - y[r]) f[r - i]$$

- **Exercise:** Prove the last equality.

RLS filter (cont)

- Substituting $\hat{y}_k[r]$, rewrite these equations to get the following $R$ equations in $R$ unknowns $\eta_k[i]$ that are $\mathcal{E}_k$-minimizing: for $i \in \{k - R + 1, \ldots, k - 1, k\}$,

$$\sum_{r=k-R+1}^{k} \lambda^{k-r} f[r - i] \sum_{\ell=r-R+1}^{r} f[\ell] \eta_k[r - \ell] = \sum_{r=k-R+1}^{k} \lambda^{k-r} y[r] f[r - i]$$

- **Exercise:** Prove the last equality and write it in matrix form.

- **Exercise:** Research how the $\mathcal{E}_k$-minimizing filter parameters $\eta_k$ can be computed recursively, *i.e.*, using $\eta_{k-1}$.

- The filter order $R$ can also be “trial adapted” to discover the system order $K$ so that the error-minimizing filter parameters $\eta_k$ “track” the system unit-pulse response $h$ over time $k$.

- Note the required initial “warm-up” period of $R$ time-units where the outputs of system $h$ are simply observed and recorded and no estimates are made.

- **Exercise:** If there was no additive noise process $v$ and the system unit-pulse response $h$ had finite support (*i.e.*, a FIR system with $K < \infty$), show how $h$ can be deduced from input-output $(f, y)$ observations.