Time-domain analysis of discrete-time LTI systems

- Discrete-time signals
- Difference equation single-input, single-output systems in discrete time
- The zero-input response (ZIR): characteristic values and modes
- The zero (initial) state response (ZSR): the unit-pulse response, convolution
- System stability
- The eigenresponse and (zero state) system transfer function
• Consider a continuous-time signal \( x : \mathbb{R} \rightarrow \mathbb{R} \) sampled every \( T > 0 \) seconds
  \[
  x(kT + t_0) =: x[k] \quad \text{for } k \in \mathbb{Z},
  \]
  where
  - \( t_0 \) is the sampling time of the \( 0^{th} \) sample, and
  - \( T \) is assumed less than the Nyquist sampling period of \( x \), and
  - \( x[k] \) (with square brackets) is the \( k^{th} \) sample itself.

• Here \( x[\cdot] \) is a discrete-time signal defined on \( \mathbb{Z} \).

Example of sampling with \( t_0 = 0 \) and positive signal \( x \)
Introduction to signals and systems in discrete time

- A discrete-time function (or signal) \( x : A \rightarrow B \) is one with countable (time) domain \( A \).
- We will take the range \( B = \mathbb{R} \) or \( B = \mathbb{C} \).
- Typically, we will herein take domain \( A = \mathbb{Z} \) or \( A = \mathbb{Z}^{\leq n} \) for some (finite) integer \( n \geq 0 \).
- Some properties of signals are as in continuous time: e.g., periodic, causal, bounded, even or odd.
- Similarly, some signal operations are as in continuous time: e.g., spatial shift/scale, superposition, time reflection, and (integer valued) time shift.

Time scaling: decimation and interpolation

- Time scaling can be implemented in continuous time prior to sampling at a fixed rate, or the sampling rate itself could be varied (again recall the Nyquist sampling rate).
- In discrete time, a signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) can be decimated (subsampling) by an integer factor \( L \neq 0 \) to create the signal \( x_L \) defined by
  \[
x_L[k] = x[kL], \quad \forall k \in \mathbb{Z},
\]
  i.e., \( x_L \) is defined only by every \( L \)th sample of \( x \).
- A discrete-time signal \( x \) can also be interpolated by an integer factor \( L > 0 \) to create \( x_L \) satisfying
  \[
x_L[kL] = x[k], \quad \forall k \in \mathbb{Z}.
\]
- For an interpolated signal \( x_L \), the values of \( x_L[r] \) for \( r \) not a multiple of \( L \) (i.e., \( \forall k \in \mathbb{Z} \) s.t. \( r \neq kL \)) can be set in different ways, e.g., between consecutive samples:
  - (piecewise constant) hold: \( x_L[r] = x_L[L[r/L]] = x[r/L] \)
  - linear interpolation:
    \[
x_L[r] = x[r/L] + \frac{r - L[r/L]}{L}(x[r/L] + 1) - x[r/L])
    \]
Time scaling: decimation and interpolation - Questions

- Is the functional mapping \( x \rightarrow x_L \) causal for linear interpolation?
- Is the hold causal?

**Exercise**: Show that if a periodic, continuous-time signal \( x(t) \), with period \( T_0 \), is periodically sampled every \( T \) seconds, then the resulting discrete-time signal \( x[k] \) is periodic if and only if \( T/T_0 \) is rational.

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Unit pulse \( \delta \), unit step \( u \), unit delay \( \Delta \), and convolution *

- Some important signals in discrete time are as those in continuous time, e.g., polynomials, exponentials, unit step.
- In discrete time, rather than the (unit) impulse, there is unit pulse (Kronecker delta):

\[
\delta[k] = \begin{cases} 
1 & \text{if } k = 0 \\
0 & \text{else} 
\end{cases}
\]

- Any discrete-time signal \( x \) can thus be written as

\[
x[k] = \sum_{r=-\infty}^{\infty} x[r] \delta[k-r] = \sum_{r=-\infty}^{\infty} x[k-r] \delta[r] = (x \ast \delta)[k]
\]

- or just \( x = x \ast \delta \), i.e., the unit pulse \( \delta \) is the identity of discrete-time convolution.
- Define the operator \( \Delta \) as unit delay (time-shift), i.e., \( \forall \) signals \( y \) and \( \forall k, r \in \mathbb{Z} \),

\[
(\Delta^* y)[k] := y[k-r].
\]

- The discrete-time unit step \( u \) satisfies \( \delta = u - \Delta u \), equivalently: \( \forall k \in \mathbb{Z} \),

\[
\delta[k] = u[k] - u[k-1] \quad \text{and} \quad u[k] = \sum_{r=0}^{\infty} (\Delta^* \delta)[k] = \sum_{r=0}^{\infty} \delta[k-r].
\]
Unit pulse and unit step functions

- **Exercise**: For any signal causal $f \{f[k], k \geq 0\}$, show that
  \[ \forall k \geq 0, \ (f * u)[k] = \sum_{r=0}^{k} f[r]. \]

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Exponential signals in discrete time

- Real-valued exponential (geometric) signals have the form $x[k] = A \gamma^k, k \in \mathbb{Z}$, where $A, \gamma \in \mathbb{R}$.

- Consider the scalar $z = \gamma e^{j\Omega} \in \mathbb{C}$ with $\gamma > 0, \Omega \in \mathbb{R}$, where again $j := \sqrt{-1}$.

- Generally, complex-valued exponential signals have the (polar) form
  \[ x[k] = A e^{j \phi} z^k = A \gamma^k e^{j(\Omega k + \phi)}, \ k \in \mathbb{Z}, \]
  where w.l.o.g. we can take
  \[ -\pi < \Omega, \phi \leq \pi \text{ and real } A > 0. \]

- **Exercise**: Show this complex-valued exponential is periodic if and only if $\Omega/\pi$ is rational.

- By the Euler-De Moivre identity,
  \[ x[k] = A \gamma^k e^{j(\Omega k + \phi)} = A \gamma^k \cos(\Omega k + \phi) + j A \gamma^k \sin(\Omega k + \phi), \ k \in \mathbb{Z}. \]
Systems - single input, single output (SISO)

In the figure, \( f \) is an input signal that is being transformed into an output signal, \( y \), by the depicted system (box).

To emphasize this functional transformation, and clarify system properties, we will write the output signal (i.e., system “response” to the input \( f \)) as

\[
y = Sf,
\]

where, again, we are making a statement about functional equivalence:

\[
\forall k \in \mathbb{Z}, \quad y[k] = (Sf)[k].
\]

Again, \( Sf \) is not \( S \) “multiplied by” \( f \), rather a functional transformation of \( f \).

SISO systems (cont)

The \( n \) signals \( \{x_1, x_2, \ldots, x_n\} \) are the internal states of the system.

The states can be taken as outputs of unit-delay operators, \( \Delta \), i.e.,

\[
\forall k \in \mathbb{Z}, \quad (\Delta y)[k] = y[k - 1].
\]

Some properties of systems are as in continuous time: e.g., linear, time invariant, causal, memoryless, stable (with different conditions for stability as we shall see).
Difference equation for an discrete time, LTI, SISO system

• For linear and time-invariant systems in discrete time, relate output $y$ to input $f$ via difference equation in standard (time-advance operator) form:

$$\forall k \geq -n, \quad \forall k \geq -n, \quad y[k + n] + a_{n-1}y[k + n - 1] + ... + a_1y[k + 1] + a_0y[k]$$

$$= b_m f[k + m] + b_{m-1} f[k + m - 1] + ... + b_1 f[k + 1] + b_0 f[k],$$

given

– scalars $a_k$ for $0 \leq k \leq n$, with $a_n := 1$, and scalars $b_k$ for $0 \leq k \leq m$,

– $a_0 \neq 0$ or $b_0 \neq 0$ (so that $P, Q$ are of minimal degree), and

– initial conditions $y[-n], y[-n + 1], ..., y[-2], y[-1]$.

• Compact representation of the above difference equation:

$$Q(\Delta^{-1}) y = P(\Delta^{-1}) f,$$ where polynomials

$$Q(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad P(z) = \sum_{k=0}^{m} b_k z^k,$$ and

$\Delta^{-1}$ is the unit time-advance operator: $(\Delta^{-1} y)[k] \equiv y[k + 1], (\Delta^{-r} y)[k] \equiv y[k + r]$

Discussion: conditions for causality and difference equation in $\Delta$

• **Exercise:** Show that the difference equation $Q(\Delta^{-1}) y = P(\Delta^{-1}) f$ is not causal if $\text{deg}(P) = m > n = \text{deg}(Q)$, i.e., the system is not proper.

• A not anti-causal difference equation can be implemented simply using memory to store a sliding window of prior values of the input $f$ and delaying the output.

• **Example:** Decoding B (bidirectional) frames of MPEG video.
Numerical solution to difference equation by recursive substitution

- Given the system $Q(\Delta^{-1})y = P(\Delta^{-1})f$, the input $f[k]$ for $k \geq 0$, and initial conditions $y[-n], \ldots, y[-1]$, one can recursively solve for $y(k)$ for $k \geq 0$ by rewriting the system equation as
  
  $y[k + u] = -\sum_{r=0}^{m} a_r y[k + r] + \sum_{r=0}^{m} b_r f[k + r]$ for $k \geq -n$
  
  $\Rightarrow y[k] = -\sum_{r=0}^{m} a_r y[k + r - n] + \sum_{r=0}^{m} b_r f[k + r - n]$ for $k \geq 0$.

- For example, the difference equation in standard form,
  
  $y[k + 1] + 3y[k] = 7f[k + 1]$ for $k \geq -1$,
  
  can be rewritten as
  
  $y[k] = -3y[k - 1] + 7f[k]$ for $k \geq 0$.

- So, given $f$ and $y[-1]$ we can recursively compute
  

- Exercise: If $f = u$ and $y[-1] = 7$ then find $y[3]$ for this example.

Approach to closed-form solution: ZIR and ZSR

- The total response $y$ of $P(\Delta^{-1})f = Q(\Delta^{-1})y$ to the given initial conditions and input $f$ is a sum of two parts:
  
  - the ZSR, $y_{ZS}$, which solves
    
    $P(\Delta^{-1})f = Q(\Delta^{-1})y_{ZS}$ with zero i.c.'s, i.e., with $0 = y[-n] = \ldots = y[-1]$;
  
  - the ZIR, $y_{ZI}$, which solves
    
    $0 = Q(\Delta^{-1})y_{ZI}$ with the given initial conditions.

- The total response $y$ of the system to $f$ and the given initial conditions is, by linearity,
  
  $y = y_{ZI} + y_{ZS}$.

- We will determine the ZIR by finding the characteristic modes of the system.

- We will determine the ZSR by convolution of the input with the (zero state) unit-pulse response, the latter also in terms of characteristic modes.
Total response - example (cont)

- Consider again the difference equation:
  \[ y[k+1] + 3y[k] = 7f[k+1], \]
  \( \forall k \geq -1, \)

- i.e., \( Q(z) = z + 3 \) with degree \( n = 1 \), and \( P(z) = 7z \) with degree \( m = 1 \),

- Exercise: Show that the following system corresponds to this difference equation.

\[
\begin{array}{c}
-3 \\
\Delta \\
y \leftarrow \Delta \leftarrow \Delta 
\end{array}
\]

- By recursive substitution, the total response is, \( \forall k \geq -1: \)
  \[
  y[k] = -3y[k-1] + 7f[k] \\
  = -3(-3y[k-2] + 7f[k-1]) + 7f[k] \\
  = (-3)^2y[k-2] - 3 \cdot 7f[k-1] + 7f[k] \\
  = \ldots \\
  = (-3)^{k+1}y[-1] + \sum_{r=0}^{k} (-3)^{k-r}f[r] \\
  =: (-3)^{k+1}y[-1] + \sum_{r=0}^{\infty} h[k-r]f[r] =: (-3)^{k+1}y[-1] + (h \ast f)[k],
  \]

- where \( h[k] := 7(-3)^k u[k] \) is the (zero state) unit-pulse response,

- \( y[-1] \) is the given \( (n = 1) \) initial condition, and

- we have defined the discrete-time convolution operator with \( \sum_{r=0}^{\infty} (...) := 0. \)
Exercise: Prove by induction this expression for $y[k]$ for all $k \geq -1$.

Exercise: Prove convolution is commutative: $h * f = f * h$.

So, we can write the total response $y = y_{ZI} + y_{ZS}$ starting from the time of oldest initial condition:

\[
\forall k \geq -1, \quad y_{ZI}[k] = (-3)^{k+1}y[-1]
\]
\[
\forall k \geq -1, \quad y_{ZS}[k] = u[k] \sum_{r=0}^{k} 7(-3)^{k-r}f[r] = u[k](h * f)[k]
\]

where $y_{ZS}[k] = 0$ when $k < 0$.

Obviously, this example involves a linear, time-invariant and causal system as described by the difference equation above.

Total response - discussion

Note that in CMPSC 360, we don’t restrict our attention to linear and time-invariant difference equations.

We use recursive substitution to guess at the form of the solution and then verify our guess by an inductive proof.

In this course, we will describe a systematic approach to solve any LTIC difference equation, i.e., to solve for the output of a DT-LTIC system given the input and initial conditions.

And again as in continuous time, we will see important insights about discrete-time signals and LTIC systems through frequency-domain representations and analysis.
ZIR - the characteristic values

• Note that \( \forall k \), \( \Delta^{-r} = z^{k+r} = z^r z^k \), i.e., the \( r \)-units time-advance operator, \( \Delta^{-r} \), is replaced by the scalar \( z^r \) for all \( r \in \mathbb{Z} \).

• Our objective is to solve for the ZIR, i.e., solve

\[
Q(\Delta^{-1})y \equiv 0 \quad \text{given } y[-n], y[-n + 1], ..., y[-2], y[-1].
\]

• Note that exponential (or "geometric") functions, \( \{z^k \mid k \in \mathbb{Z}\} \) for \( z \in \mathbb{C} \), are eigenfunctions of time-shift operators of the form \( Q(\Delta^{-1}) \) for a polynomial \( Q \).

• That is, for any non-zero scalar \( z \in \mathbb{C} \), if we substitute \( y[k] = z^k \) \( \forall k \in \mathbb{Z} \) we get:

\[
\forall k \in \mathbb{Z}, \quad (Q(\Delta^{-1})y)[k] = Q(\Delta^{-1})z^k = Q(z)z^k.
\]

• So, to solve \( Q(z)z^k \equiv 0 \) for all time \( k \geq 0 \), when \( z \neq 0 \) we require

\[
Q(z) = 0, \quad \text{the characteristic equation of the system.}
\]

ZIR - the characteristic values (cont)

• If \( z \) is a root of the characteristic polynomial \( Q \) of the system, then
  
  – \( z \) would be a characteristic value of the system, and
  
  – the signal \( \{z^k\}_{k \geq 0} \) is a characteristic mode of the system when \( z \neq 0 \), i.e.,

\[
Q(\Delta^{-1})z^k = 0 \quad \forall k \geq 0.
\]

• Since \( Q \) has degree \( n \), there are \( n \) roots of \( Q \) in \( \mathbb{C} \), each a system characteristic value.
ZIR - the characteristic values (cont)

- Let \( n' \leq n \) be the number of non-zero roots of \( Q \), i.e., \( \tilde{Q}(z) = Q(z)/z^{n-n'} \) is a polynomial satisfying \( \tilde{Q}(0) \neq 0 \).

- Though there may be some repeated roots of the characteristic polynomial \( Q \), there will always be \( n' \) different, linearly independent characteristic modes, \( \mu_k \), i.e.,
  \[
  \forall k \geq -n, \sum_{r=1}^{n'} c_r \mu_r[k] = 0 \iff \forall r, \text{ scalars } c_r = 0.
  \]

- When \( n = n' \), by system linearity, we will be able to write
  \[
  \forall k \geq -n, \ y_{ZI}[k] = \sum_{r=1}^{n} c_r \mu_r[k],
  \]
  for scalars \( c_r \in \mathbb{C} \) that are found by considering the given initial conditions
  \[
  y[k] = \sum_{r=1}^{n} c_r \mu_r[k] \text{ for } k \in \{-n, \ldots, -2, -1\},
  \]
  i.e., \( n \) equations in \( n \) unknowns \( c_r \).

- The linear independence of the modes implies linear independence of these \( n \) equations in \( c_r \), and so they have a unique solution.

ZIR - the case of different, non-zero, real characteristic values

- If there are \( n \) different non-zero roots of \( Q \) in \( \mathbb{R} \), \( z_1, z_2, \ldots, z_n \), then there are \( n \) characteristic modes: for \( r \in \{1, 2, \ldots, n\} \),
  \[
  \forall \text{ time } k, \mu_r[k] = z_r^k.
  \]
- Therefore,
  \[
  \forall k \geq -n, \ y_{ZI}[k] = \sum_{r=1}^{n} c_r z_r^k.
  \]
- The \( n \) unknown scalars \( c_r \in \mathbb{R} \) can be solved using the \( n \) equations:
  \[
  y[k] = \sum_{r=1}^{n} c_r z_r^k, \text{ for } k \in \{-n, -n + 1, \ldots, -2, -1\}.
  \]
• **Example:** Consider the difference equation:

\[
\forall k \geq -3, \quad 2y[k + 3] - 10y[k + 2] + 12y[k + 1] = 3f[k + 2],
\]

with \(y[-2] = 1\) and \(y[-1] = 3\).

• That is, \(Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2)\) and \(n = 2\), \(P(z) = (3/2)z\) and \(m = 1\).

• So, the \(n = 2\) characteristic values are \(z = 3, 2\) and the ZIR is

\[
\forall k \geq -n = -2, \quad y_{ZI}[k] = c_1 3^k + c_2 2^k
\]

• Using the initial conditions to find the scalars \(c_1, c_2\):

\[
1 = y[-2] = c_1 3^{-2} + c_2 2^{-2} \quad \text{and} \quad 3 = y[-1] = c_1 3^{-1} + c_2 2^{-1}.
\]

• **Exercise:** Now solve for \(c_1\) and \(c_2\).

• Note: When a coefficient \(c\) is worked out to be zero, it may not be exactly zero in practice, and the corresponding characteristic mode \(z^k\) will increasingly contribute to ZIR \(y_{ZI}\) over time if \(|z| > 1\) (i.e., an "unstable" mode in discrete time).

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### ZIR - the case of not-real characteristic values

• The characteristic polynomial \(Q\) may have non-real roots, but such roots come in complex-conjugate pairs because \(Q\)'s coefficients \(a_k\) are all real.

• For example, if the characteristic polynomial is

\[
Q(z) = (z - 1)(z^2 - 2z - 2)
\]

then the characteristic values (\(Q\)'s roots) are

\(-1, \quad 1 \pm j\sqrt{3} \quad \text{again recalling} \quad j = \sqrt{-1}.
\)

• Because we have three different characteristic values \(\in \mathbb{C}\), we can specify three corresponding characteristic modes,

\((-1)^k, (1 + j\sqrt{3})^k, (1 - j\sqrt{3})^k, \quad \forall k \geq 0,\)

and construct the ZIR as

\[
\forall k \geq -n = -3, \quad y_{ZI}[k] = c_1 (-1)^k + c_2 (1 + j\sqrt{3})^k + c_3 (1 - j\sqrt{3})^k
\]

\[
= c_1 (-1)^k + c_2 2^k e^{k\pi j/3} + c_3 2^k e^{-k\pi j/3}
\]

where

- \(c_1 \in \mathbb{R}\) and \(c_2 = \overline{c_3} \in \mathbb{C}\) so that \(y_{ZI}\) is real-valued, and again,
- these scalars are determined by the \(n = 3\) given (real) initial conditions: \(y[-3], y[-2], y[-1]\).
ZIR - not-real characteristic values with real characteristic modes

- By the Euler-De Moivre identity for the previous example,
  \[ y_{ZI}[k] = c_1(-1)^k + (c_2 + c_3)2^k \cos(k\pi/3) + j(c_2 - c_3)2^k \sin(k\pi/3) \]

- Again, because all initial conditions are real and \( Q \) has real coefficients, \( y_{ZI} \) is real valued and so \( c_3 = \overline{c_2} \Rightarrow c_2 + c_3, j(c_2 - c_3) \in \mathbb{R} \).

- In general, consider two complex conjugate characteristic values \( v \pm jq \) corresponding to two complex-valued characteristic modes \( |z|^k e^{\pm jk\angle z} \), where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).

- One can use Euler’s identity to show that the corresponding real-valued characteristic modes are
  \[ |z|^k \cos(k\angle z), \ |z|^k \sin(k\angle z) \]

ZIR - the case of repeated characteristic values

- Consider the case where at least one characteristic value is of order > 1, i.e., there are repeated roots of the characteristic polynomial, \( Q \).

- For example, \( Q(z) = (z + 0.75)^3(z - 0.5) \) has a triple (twice repeated) root at \(-0.75\) and a single root at \(0.5\).

- Again, \( \{(0.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})(-0.75)^k \equiv 0 \) follows from
  \[ \Delta^{-1} + .75)(-0.75)^k = \Delta^{-1}(-0.75)^k + .75(-0.75)^k \]
  \[ = (-.75)^{k+1} + .75(-.75)^k \]
  \[ = 0. \]

- Similarly, \( (0.5)^k \) is a characteristic mode since \( (\Delta^{-1} - 0.5)(0.5)^k \equiv 0 \).

- Also, \( \{k(-.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})k(-.75)^k \equiv 0 \) follows from
  \[ (\Delta^{-1} + .75)^2k(-.75)^k \]
  \[ = (\Delta^{-2} + 1.5\Delta^{-1} + (.75)^2)k(-.75)^k \]
  \[ = \Delta^{-2}k(-.75)^k + 1.5\Delta^{-1}k(-.75)^k + (.75)^2k(-0.75)^k \]
  \[ = (k + 2)(-.75)^{k+2} + 1.5(k + 1)(-.75)^{k+1} + (.75)^2k(-0.75)^k \]
  \[ = (-.75)^{k+2}((k + 2) - 2(k + 1) + k) \]
  \[ = 0. \]
ZIR - the case of repeated characteristic values (cont)

• Similarly, \( \{k^2(-.75)^k\} \) is also a characteristic mode because 
\[ (\Delta^{-1} + .75)^3 k^2(-.75)^k = 0. \]

• Note that without three such linearly independent characteristic modes 
\[ \{(-.75)^k, k(-.75)^k, k^2(-.75)^k ; k \geq 0\} \]
for the twice-repeated (triple) characteristic value -.75, the initial conditions will create an
"overspecified" set of \( n \) equations involving fewer than \( n \) "unknown" coefficients (\( c_k \)) of
the linear combination of modes forming the ZIR.

• For this example, 
\[ y_{ZI}[k] = c_0(-.75)^k + c_1 k(-.75)^k + c_2 k^2(-.75)^k + c_3 (0.5)^k, \quad k \geq -4. \]

If the given initial conditions are, say, 
\[ y[-4] = 12, \quad y[-3] = 6, \quad y[-2] = -5, \quad y[-1] = 10, \]
the four equations to solve for the four unknown coefficients \( c_k \) are:
\[
\begin{align*}
y_{ZI}[-4] &= (-.75)^{-4}c_0 + (-4)(-.75)^{-1}c_1 + (-4)^2(-.75)^{-4}c_2 + (.5)^{-4}c_3 = 12 \\
y_{ZI}[-3] &= (-.75)^{-3}c_0 + (-3)(-.75)^{-1}c_1 + (-3)^2(-.75)^{-3}c_2 + (.5)^{-3}c_3 = 6 \\
y_{ZI}[-2] &= (-.75)^{-2}c_0 + (-2)(-.75)^{-1}c_1 + (-2)^2(-.75)^{-2}c_2 + (.5)^{-2}c_3 = -5 \\
y_{ZI}[-1] &= (-.75)^{-1}c_0 + (-1)(-.75)^{-1}c_1 + (-1)^2(-.75)^{-1}c_2 + (.5)^{-1}c_3 = 10
\end{align*}
\]

ZIR - general case of repeated, non-zero characteristic values

• In general, a set of \( r \) linearly independent modes corresponding to a non-zero characteristic
value \( z \in \mathbb{C} \) repeated \( r - 1 \) times are
\[ k^{r-1}z^k, \ k^{r-2}z^k, \ldots, \ k^2z^k, \ k^zk, \quad \text{for} \ k \geq 0. \]

• Also, if \( v \pm jq \) are characteristic values repeated \( r - 1 \) times, with \( v, q \in \mathbb{R} \) and \( q \neq 0 \),
we can use the \( 2k \) real-valued modes
\[ k^a|z|^k \cos(k\angle z), k^a|z|^k \sin(k\angle z), \quad \text{for} \ a \in \{0, 1, 2, \ldots, r - 1\}, \]
where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).
• Again let $n' \leq n$ be the number of non-zero roots of $Q$ (characteristic values),

• *i.e.*, $r := n - n' \geq 0$ is the order ($1+$repetition) of the characteristic value 0, and

• $r \geq 0$ is the smallest index such that the coefficient of $Q$ $a_r \neq 0$.

• So, there is a polynomial $\tilde{Q}$ such that $Q(z) = z^r \tilde{Q}(z)$ and $\tilde{Q}(0) \neq 0$.

• Because the constant signal zero cannot be a characteristic mode, we add $r = n - n'$ time-advanced unit-pulses:

$$\forall k \geq -n, \ y_{ZI}[k] = \sum_{i=1}^{r} C_i \delta[k + i] + y_N[k]$$

$$= C_r \delta[k + r] + C_{r-1} \delta[k + r - 1] + \ldots + C_0 \delta[k + 1] + y_N[k]$$

where $y_N$ is a "natural response" (linear combination of $n'$ characteristic modes).

• The $n$ initial conditions are then met by the $r$ coefficients $C_i$ of the advanced unit pulses together with the $n' = n - r$ coefficients of the characteristic modes in $y_N$.

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**Zero State Response - the unit-pulse response**

• Recall the LTIC system

$$\sum_{r=0}^{n} a_r \Delta^{-r} y =: Q(\Delta^{-1}) y = P(\Delta^{-1}) f := \sum_{r=0}^{m} b_r \Delta^{-r} f$$

with $a_n \equiv 1$, $a_0 \neq 0$ or $b_0 \neq 0$, $m \leq n$.

• We can express any input signal

$$f[k] = \sum_{r=0}^{\infty} f[r] \delta[k - r] \ \forall k \geq 0, \ i.e., \ \forall f, \ f = f \ast \delta.$$ 

• So the unit pulse $\delta$ is the identity of the convolution operator in discrete time.

• Thus, by LTI, the ZSR $y_{ZS}$ is the convolution of input $f$ and ZSR $h$ to unit pulse $\delta$,

$$y_{ZS}[k] = \sum_{r=0}^{\infty} f[r] h[k - r] = (f \ast h)[k], \ \forall k \geq 0,$$

• $h$ is called the **unit-pulse response** of the LTIC system, *i.e.,*

$$Q(\Delta^{-1}) h = P(\Delta^{-1}) \delta \ s.t. \ h[k] = 0 \ \forall k < 0.$$
Computing an LTIC system’s unit-pulse response, \( h \)

- For the LTIC system in standard form, if \( a_0 \neq 0 \) then
  \[
  h = \left( \frac{b_0}{a_0} \right) \delta + y_N u
  \]
  where \( y_N \) is a natural response of the system (linear combination of characteristic modes).

- Note that \( h[k] = 0 \) for all \( k < 0 \) owing to the unit step \( u \).

- The \( n \) scalars of the natural response \( y_N \) component of \( h \) are solved using
  \[
  (Q(\Delta^{-1})h)[k] = (P(\Delta^{-1})\delta)[k] \quad \text{for} \quad k \in \{-n, -n + 1, \ldots, -2, -1\}
  \]

Unit-pulse response when zero is a characteristic value

- If \( r \geq 0 \) is the smallest index such that \( a_r \neq 0 \) (\( 0 \) is a char. mode of order \( r \)), then may need to add \( r \) delayed unit-pulse terms to \( h \):
  \[
  h = \sum_{i=0}^{r-1} A_i \Delta^i \delta + \left( \frac{b_0}{a_r} \right) \Delta^r \delta + y_N u,
  \]
  where
  - by definition of the standard form of the difference equation, if \( r > 0 \), \( a_0 = 0 \) so \( b_0 \neq 0 \), and
  - \( r \leq n \) since \( 0 \neq a_n := 1 \).

- So if \( r = 0 \) (i.e., \( a_0 \neq 0 \)), then \( A_0 = b_0/a_0 \) as above, where \( \sum_{i=0}^{r-1}(\ldots) := 0 \).

- Exercise: Prove \( A_r = b_0/a_r \) for \( 0 \leq r \leq n \).

- Thus, zero is a characteristic value of degree \( r \geq 0 \), and

- there are \( r \) characteristic modes that will all be zero.

- The additional unit-pulse terms introduce \( r \) degrees of freedom in the form of the coefficients \( A_0, A_1, \ldots, A_{r-1} \) to accommodate the \( n = r + n' \) initial conditions of the unit-pulse response: \( h[-n] = h[-n + 1] = \ldots = h[-2] = h[-1] = 0 \).
Computing the ZSR - example 1

- Recall that the difference equation \( y = 7f - 3\Delta y \) corresponds to the above system; in standard form:
  \[
  \forall k \geq -1, \quad y[k + 1] + 3y[k] = 7f[k + 1].
  \]
  with \( Q(z) = z + 3 \), \( P(z) = 7z \) and \( n = 1 = m \).

- Since the system characteristic value is \( -3 \) and \( b_0 = 0 \), the (zero state) unit-pulse response has the form \( h[k] = c(-3)^k u[k] \).

- The scalar \( c \) is solved by evaluating the above difference equation at time \( k = -1 \):
  \[
  (Q(\Delta^{-1})h)[-1] = (P(\Delta^{-1})\delta)[-1]
  \]
  \[i.e., \quad h[0] + 3h[-1] = 7\delta[0] \]
  \[\Rightarrow \quad c + 3 \cdot 0 = 7 \cdot 1, \quad c = 7 \]

---

Computing the ZSR - example 1 (cont)

- So, \( h[k] = 7(-3)^k u[k] \).

- If the input is \( f[k] = 4(0.5)^k u[k] \), the system ZSR is, for all \( k \geq 0 \),
  \[
  y_{ZS}[k] = \sum_{r=0}^{k} h[r] f[k-r] = \sum_{r=0}^{k} 7(-3)^r 4(0.5)^{k-r}
  \]
  \[= 28(0.5)^k \sum_{r=0}^{k} (-6)^r = 28(0.5)^k \frac{(-6)^{k+1} - 1}{-6 - 1} u[k] \]
  \[= (24(-3)^k + 4(0.5)^k)u[k] \].

- Note how the ZIR \( y_{ZI} \) has a term that is a characteristic mode (excited by the input \( f \)) and a term that is proportional to the input \( f \) (this forced response is an eigenresponse).

- **Exercise**: For the difference equation, \( y[k + 1] + 3y[k] = 7f[k] \ \forall k \geq -1 \): draw the block diagram, show that \( h[k] = 21(-3)^{k-1} u[k] + (7/3)\delta[k] \), and find the ZSR to the above input \( f \).

- **Exercise**: Read "sliding tape" method to compute convolution in Lathi, p. 595.
Computing the unit pulse response - example 2

• Find the ZSR of the following system to input \( f[k] = 2(-5)^k u[k] \):

![System Diagram]

**Exercise:** show the difference equation for this system (in direct canonical form) is:

\[
\forall k \geq 0, \quad y[k + 2] - 5y[k + 1] + 6y[k] = 1.5f[k + 1]
\]

That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, P(z) = 1.5z \) and \( m = 1 \).

• So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and \( b_0 = 0 \) so the unit-pulse response

\[
h[k] = (c_13^k + c_22^k)u[k].
\]

To find the constants, evaluate the difference equation at \( k = -1 \):

\[
2h[1] - 10h[0] + 12h[-1] = 3\delta[0]
\]

\[
\Rightarrow 2h[1] - 10h[0] = 3
\]

\[
\Rightarrow (2 \cdot 3 - 10 \cdot 1)c_1 + (2 \cdot 2 - 10 \cdot 1)c_2 = 3
\]

\[
\Rightarrow -4c_1 - 6c_2 = 3
\]

and at \( k = -2 \):

\[
2h[0] - 10h[-1] + 12h[-2] = 3\delta[-1]
\]

\[
\Rightarrow 12h[0] = 0 \Rightarrow h[0] = 0
\]

\[
\Rightarrow c_1 + c_2 = 0.
\]

Thus, \( c_2 = -1.5 = -c_1 \) so that \( h[k] = (-1.5(3)^k + 1.5(2)^k)u[k] \) and for \( k \geq 0 \)

\[
y_{ZS}[k] = (h \ast f)[k] = \sum_{r=0}^{k} h[r]f[k-r].
\]

**Exercise:** Write the ZSR as a sum of system modes \( 2^k \) and \( 3^k \) and a (force) term like the input, here taken as \( f[k] = 4(-5)^k u[k] \).
Convolution - other important properties

- Again, for a LTI system with impulse response $h$ and input $f$, the ZSR is $y_{ZS} = f \star h$, where
  \[(f \star h)[k] = \sum_{r=-\infty}^{\infty} f[r]h[k-r]\]

- By simply changing the dummy variable of summation to $r' = h - r$, can show convolution is commutative: $f \star h = h \star f$.

- One can directly show that convolution $f \star h$ is a bi-linear mapping from pairs of signals $(f, h)$ to signals $(y_{ZS})$, consistent with convolution’s commutative property and the (zero state) system with impulse response $h$ being LTI;

  - that is, $\forall$ signals $f, g, h$ and scalars $\alpha, \beta \in \mathbb{C}$,
    \[
    (\alpha f + \beta g) \star h = \alpha (f \star h) + \beta (g \star h)
    \]

- By changing order of summation (Fubini’s theorem), one can easily show that convolution is associative, i.e., $\forall$ signals $f, g, h$,
  \[
  (f \star g) \star h = f \star (g \star h).
  \]

Convolution - other important properties (cont)

- We’ll use these properties when composing more complex systems from simpler ones.

- By just changing variables of integration, we can show how to exchange time-shift with convolution, i.e., $\forall$ signals $f, h : \mathbb{Z} \rightarrow \mathbb{C}$ and times $k \in \mathbb{Z}$,
  \[
  (\Delta^k f) \star h = \Delta^k (f \star h);
  \]
  recall how convolution represents the ZSR of linear and time-invariant systems.

- By the ideal sampling property, recall that the identity signal for convolution is the unit pulse $\delta$, i.e., $\forall$ signals $f$,
  \[
  f \star \delta = \delta \star f = f
  \]

- **Exercise**: Adapt the proofs of these properties in continuous time to this discrete-time case.

- **Exercise**: In particular, show that if $f$ and $h$ are causal signals, then $y = f \star h$ is causal; i.e., if the unit-pulse response $h$ of a system is a causal signal, then the system is causal.
System stability - ZIR - asymptotically stable

- Consider a SISO system with input $f$ and output $y$.

- Recall that the ZIR $y_{ZI}$ is a linear combination of the system’s characteristic modes, where the coefficients depend on the initial conditions, possibly including some initial unit-pulse terms if zero is a characteristic value (system pole).

- A system is said to be asymptotically stable if for all initial conditions,
  $$\lim_{k \to \infty} y_{ZI}[k] = 0.$$  

- So, a system is asymptotically stable if and only if all of its characteristic values have magnitude less than 1.

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System stability - ZIR - asymptotically stable: Example

- If the characteristic polynomial $Q(z) = (z - 0.5)(z^2 + 0.0625)$, then

- the system’s characteristic values (roots of $Q$) are $0.5, \pm 0.25j$ each with magnitude less than one,

- and the ZIR is of the form,
  $$y_{ZI}[k] = \left( c_1 (0.5)^k + c_2 (0.25j)^k + c_2^* (-0.25j)^k \right) u[k]$$  
  $$= \left( c_1 (0.5)^k + 2\text{Re}\{c_2\} (0.25)^k \cos(k\pi/2) - 2\text{Im}\{c_2\} (0.25)^k \sin(k\pi/2) \right) u[k],$$

- recalling that $j^k = e^{jk\pi/2}$.

- So, $y_{ZI}[k] \to 0$ as $k \to \infty$ for all $c_1, c_2$ (i.e., for all initial conditions), and

- hence is asymptotically stable.
System stability - bounded signals

- A signal $y$ is said to be bounded if
  $$\exists M < \infty \text{ s.t. } \forall k \in \mathbb{Z}, \ |y[k]| \leq M;$$
oindent otherwise $y$ is said to be unbounded.

- For example, $y[k] = 0.25(\frac{1+j\sqrt{3}}{2})^k u[k]$ is bounded (can use $M = 0.25$).

- Also, $3\cos(5k)$ is bounded (can use $M = 3$).

- But both $2^k \cos(5k)$ and $3 \cdot (-2)^k$ are unbounded.

System stability - ZIR - marginally stable

- A system is said to be marginally stable if it is not asymptotically stable but $y_{ZI}$ is always (for all initial conditions) bounded.

- A system is marginally stable if and only if
  - it has no characteristic values with magnitude strictly greater than 1,
  - it has at least one characteristic value with magnitude exactly 1, and
  - all magnitude-1 characteristic values are not repeated.

- That is, a marginally stable system has
  - some characteristic modes of the form $\cos(\Omega k)$ or $\sin(\Omega k)$,
  - while the rest of the modes are all of the form $k^r |z|^k \cos(\Omega k)$ or $k^r |z|^k \sin(\Omega k)$, with $|z| < 1$ and integer degree $r \geq 0$.

- Exercise: Explain why we can take the frequency $\Omega \in (-\pi, \pi]$ without loss of generality.
System stability - ZIR - marginally stable: Example

- The characteristic polynomial is \( Q(z) = z(z^2 + 1)(z - 0.25) \) gives characteristic values 0, 0.25, ±j,
- then the system is marginally stable with modes \((0.25)^k, \cos(k\pi/2), \sin(k\pi/2)\),
- the last two of which are bounded but do not tend to zero as time \( k \to \infty \).

System stability - ZIR - unstable

- A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.
- For example, the system with \( Q(z) = (z^2 - 0.5)(z + 3) \) is unstable owing to the characteristic value −3 with unbounded mode \((-3)^k\).
- For another example, if the characteristic polynomial is \( Q(z) = (z^2 + 1)^2(z - 0.5) \) then the purely imaginary characteristic values ±j are repeated, and hence the two additional modes \( k\sin(k\pi/2), k\cos(k\pi/2) \) are unbounded, so this system is unstable.
- Similarly, if \( Q(z) = (z^2 - 1)^2(z - 0.5) \) then the characteristic values ±1 are repeated and the modes \( k \) and \( k(-1)^k \) are unbounded, so this system is unstable too.
System stability - ZSR - BIBO stable

- A SISO system is said to be *Bounded Input, Bounded Output* (BIBO) stable if \( \forall \) bounded input signals \( f \), the ZSR \( y_{ZS} \) is bounded.

- A sufficient condition for BIBO stability is absolute summability of the unit-pulse response,
  \[
  \sum_{k=0}^{\infty} |h[k]| < \infty.
  \]

- To see why: If the input \( f \) is bounded (by \( M_f \) with \( 0 \leq M_f < \infty \)) then \( \forall k \geq 0 \):
  \[
  |y_{ZS}[k]| = |(f * h)[k]| = \left| \sum_{r=0}^{k} f[k-r]h[r] \right| \\
  \leq \sum_{r=0}^{k} |f[k-r]h[r]| \quad \text{(by the triangle inequality)} \\
  \leq \sum_{r=0}^{k} M_f |h[r]| \\
  \leq M_f \sum_{r=0}^{\infty} |h[r]| =: M_y < \infty,
  \]
System stability - ZSR - BIBO stable

- The condition of absolute summability of the unit-pulse response,
\[ \sum_{r=0}^{\infty} |h[r]| < \infty, \]
is also necessary for, and hence equivalent to, BIBO stability.

- If any component characteristic mode of \( h \) is unbounded, then \( h \) will not to be absolutely summable.

- Thus, if the system (ZIR) is asymptotically stable it will be BIBO stable; the converse is also true.

ZSR - the transfer function, \( H \)

- Recall that for any polynomial \( Q \) and \( z \in \mathbb{C} \) (including \( s = jw, \ w \in \mathbb{R} \)),
\[ Q(\Delta^{-1})z^k = Q(z)z^k, \ \forall k \geq 0. \]

- So, if we guess that a "particular" solution of the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \) with input \( f[k] = Az^ku[k] \) is of the form \( y_0[k] = AH(z)z^k = H(z)f[k], k \geq 0, \) then we get by substitution that \( \forall k \geq 0, z \in \mathbb{C}, \)
\[ (Q(\Delta^{-1})y_0)[k] = AH(z)Q(z)z^k = (P(\Delta^{-1})f)[k] = AP(z)z^k \]
\[ \Rightarrow H(z) = P(z)/Q(z). \]

- The "rational polynomial" \( H = P/Q \) is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

- So, the ZSR (forced response + characteristic modes) would be of the form:
\[ y_{ZS}[k] = (H(z)z^k + \text{linear combination of char. modes})u[k]. \]

- Recall that for the example with \( Q(z) = z + 3 \) and \( P(z) = 7z \), we computed the unit-pulse response \( h[k] = 7(-3)^ku[k] \) and the ZSR to input \( f[k] = 4(0.5)^ku[k] \) as \( y_{ZS}[k] = (24(-3)^k + 4(0.5)^k)u[k] \).

- Here, note that \( H(0.5) = P(0.5)/Q(0.5) = 1 \), i.e., the forced response component of \( y_{ZS} \) is \( H(0.5)f[k] = 1 \cdot 4(0.5)^ku[k] = 4(0.5)^ku[k] \).
ZSR - unit-pulse response \( h \), transfer function \( H \), and eigenresponse

- \( y_{ZS}[k] = (H(z)Az^k + \text{linear combination of char. modes})u[k] \) is the ZSR to input \( f[k] = Az^k u[k] \), where \( H(z) = P(z)/Q(z) \).

- The eigenresponse is a special case of the forced response for exponential inputs.

- If \( |z| = 1 \), i.e., \( z = e^{j\Omega} \) for some \( \Omega \in (-\pi, \pi) \) (w.l.o.g.), and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:
  \[
y[k] \to H(e^{j\Omega})Ae^{j\Omega k} \quad \text{as} \quad k \to \infty.
  \]

- Since \( y = h * f \), we get that as \( k \to \infty \) for a LTIC and asymptotically stable system,
  \[
y_{ZS}[k] = \sum_{r=0}^{k} h[r]Ae^{j\Omega(k-r)}
  \]
  \[
  = Ae^{j\Omega k} \sum_{r=0}^{k} h[r]e^{-j\Omega r} \to Ae^{j\Omega k}H(e^{j\Omega}),
  \]
  \[
  \Rightarrow \sum_{r=0}^{\infty} h[r]e^{-j\Omega r} = H(e^{j\Omega}), \quad \forall \Omega \in (-\pi, \pi).
  \]

ZSR - transfer function \( H \) and eigenresponse (cont)

- The LT1 system transfer function \( H \) is the Discrete-Time Fourier Transform (DTFT) of the system unit-pulse response \( h \):
  \[
  \forall \Omega \in \mathbb{R}, \quad H(e^{j\Omega}) = \sum_{r=0}^{\infty} h[r]e^{-j\Omega r}.
  \]

- Note that \( H(e^{j\Omega}) \) is periodic since \( H(e^{j\Omega}) = H(e^{j\Omega + 2\pi k}) \) for any integer \( k \).

- For the z-transform (and DTFS) we will use this notation for \( H \), but for the DTFT we will instead write \( H(\Omega) \).
Frequency-domain methods for discrete-time signals

- Discrete-Time Fourier Series (DTFS) of periodic signals
- Discrete-Time Fourier Transform (DTFT)
- Sampled data systems
- DFT & FFT
- $z$-transform for (complete) transient response
- Eigenresponse
- Canonical system realization of a difference equation

Discrete-time Fourier series of periodic signals

- For all $r \in \mathbb{Z}$, note that the signal $\{\exp(j2\pi kr/N) \mid k \in \mathbb{Z}\}$ "repeats itself" every $N$ units of (discrete) time, in particular
  $$\forall r, \quad \exp(j2\pi kr/N)|_{k=0} = 1 = \exp(j2\pi kr/N)|_{k=N}$$
- Also the signals $\{\exp(j2\pi kr/N) \mid k \in \mathbb{Z}\} \equiv \{\exp(j2\pi kr'/N) \mid k \in \mathbb{Z}\}$ whenever $r' = r \mod N$.
- Suppose $N$ is the period of periodic signal $x = \{x[k] \mid k \in \mathbb{Z}\}$.
- We can write $x$ as a Discrete-Time Fourier Series (DTFS):
  $$\forall k \in \mathbb{Z}, \quad x[k] = \sum_{r=0}^{N-1} D_r e^{j2\pi kr/N}.$$
Discrete-time Fourier series of periodic signals (cont)

- Consider the $N$ signals $\xi_r[k] := e^{j2\pi kr/N}$ over any time-interval $A$ of length $N$.

- Equivalently consider these $N$ signals $\xi_r$ as $N$-vectors in $\mathbb{R}^N$, i.e., the $k^{th}$ entry of vector $\xi_r$ is $\xi_r[k]$.

- If these signals/vectors $\{\xi_r\}_{r=1}^N$ are linearly independent, then they will form a basis spanning all other signals $x : A \rightarrow \mathbb{R}$, equivalently all other vectors $x \in \mathbb{R}^N$.

- i.e., any such $x$ can be written as a linear combination of the $\{\xi_r\}_{r=1}^N$ giving the DTFS of $x$: $x_r = \sum_{r=0}^{N-1} D_r \xi_r$.

- If we show that these signals/vectors $\{\xi_r\}_{r=1}^N$ are orthogonal then
  - linear independence follows
  - the coordinate $D_r$ (DTFS coefficients) is found by simply projecting $x$ onto the vector $\xi_r$: $D_r = \langle x, \xi_r \rangle / ||\xi_r||^2$.

DTFS - coefficients (cont)

- Consider any period of $x : \mathbb{Z} \rightarrow \mathbb{R}$, say $\{0, 1, 2, ..., N-1\}$.

- First note that for any $v \in \mathbb{Z}$ that is not a multiple of $N$ ($N \not| v$ so that $e^{j2\pi v/N} \neq 1$), the geometric series

$$\sum_{k=0}^{N-1} e^{j2\pi kr/N} = \sum_{k=0}^{N-1} \left(e^{j2\pi v/N}\right)^k = \frac{e^{j2\pi N v/N} - e^{j2\pi 0 v/N}}{e^{j2\pi v/N} - 1} = 0.$$

- Thus, for any $r \neq v \in \mathbb{Z}$ such that $N \not| (v - r)$, the inner product $\langle \xi_r, \xi_v \rangle =$

$$\langle \{e^{j2\pi kr/N}\}, \{e^{j2\pi v/N}\} \rangle := \sum_{k=0}^{N-1} e^{j2\pi kr/N} e^{-j2\pi v/N} = \sum_{k=0}^{N-1} e^{j2\pi (r - v)/N} = 0,$$

recalling that the inner product is conjugate-linear in the second argument so that $\langle x, x \rangle = ||x||^2$ when $x$ is $\mathbb{C}$-valued.

- So, these signals are orthogonal and the DTFS coefficients of $N$-periodic $x$ are

$$D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j2\pi kr/N} = \frac{\langle x, \{e^{j2\pi kr/N}\} \rangle}{||\{e^{j2\pi kr/N}\}||^2}.$$
• Let’s now compute the inner product of $\xi_v$, for any $v \in \{0, 1, \ldots, N-1\}$, with the DTFS of $N$-periodic $x$:

$$
\langle x, \{e^{j2\pi v/N}\} \rangle = \sum_{k=0}^{N-1} x[k] e^{-j2\pi kv/N} = \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} D_r e^{j2\pi kr/N} e^{-j2\pi kv/N}
$$

$$
= \sum_{r=0}^{N-1} D_r \sum_{k=0}^{N-1} e^{j2\pi (r-v)/N} = \sum_{r=0}^{N-1} D_r N\delta(r - v) = D_v N
$$

• Again, we have verified the DTFS coefficients is

$$
D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j2\pi kr/N} = \frac{\langle x, \{e^{j2\pi kr/N}\} \rangle}{||\{e^{j2\pi kr/N}\}||^2}
$$

### DTFS - example

• Problem: Identify the DTFS coefficients (if they exist) for

$$
x[k] = 7 \sin(5.7\pi k) + 2 \cos(3.2\pi k), \quad k \in \mathbb{Z}.
$$

• Solution: Since $\sin$ and $\cos$ have period $2\pi$, we can subtract integer multiples of $2\pi$ to get

$$
x[k] = 7 \sin(1.7\pi k) + 2 \cos(1.2\pi k).
$$

• $1.7\pi k$ is an integer multiple of $2\pi$ when (integer) $k = 20$, and when $k = 5$ for $1.2\pi k$, so least common multiple of these periods is $k = 20$.

• Thus, the period of $x$ is $N = 20$ and the fund. frequ. is $\Omega_0 = 2\pi/N = 0.1\pi$.

• By Euler’s identity & adding $2\pi k$ to the negative exponents,

$$
x[k] = \frac{7}{2j} e^{j1.7\pi k} - \frac{7}{2j} e^{-j1.7\pi k} + e^{j1.2\pi k} + e^{-j1.2\pi k}
$$

$$
= -3.5j e^{j1.7\pi k} + 3.5j e^{j0.3\pi k} + e^{j1.2\pi k} + e^{j0.8\pi k}.
$$

• So, the DTFS of $x[k] = \sum_{r=0}^{19} D_r e^{jr0.1\pi k}$ with $D_{17} = -3.5j = 3.5e^{-j\pi/2}$, $D_3 = 3.5j = 3.5e^{j\pi/2}$, $D_{12} = 1$ and $D_8 = 1$; else $D_r = 0$ (incl. the fundamental $r \in \{1, 19\}$ & DC $r = 0$ components).
DTFS - example and exercise

- **Example:** The DTFS of an even rectangle wave with period \( N = 6 \) and duty cycle 3:

\[
x[k] = \sum_{k=-\infty}^{\infty} \Delta^6 k (\Delta^{-1} u - \Delta^2 u) \quad \text{is}
\]

\[
= \sum_{r=0}^{5} D_r e^{j2\pi kr/6}, \quad \text{where the fund. freq. } \Omega_o = 2\pi/6 \text{ and, } \forall r \in \mathbb{Z},
\]

\[
D_r = \frac{1}{6} \sum_{k=-3}^{2} x[k] e^{j2\pi kr/6} = \frac{1}{6} \sum_{k=-1}^{1} 1 \cdot e^{j2\pi kr/6} = \frac{1}{6} (1 + 2 \cos(2\pi kr/6)).
\]

- **Exercise:** Plot \( x[k] \) as a function of time \( k \) and plot its (periodic) spectrum:

\[
\forall r \in \{0, 1, 2, ..., 5\}, k \in \mathbb{Z},
\]

\[
\tilde{X}(r2\pi/6 + 2\pi k) = D_r.
\]

DTFS - Parseval’s theorem

- The average power of the \( N \)-periodic discrete-time signal \( x \) is

\[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \overline{x[k]};
\]

equivalently, the sum could be taken over any interval of length \( N \in \mathbb{Z}^{>0} \).

- Substituting the Fourier series of \( x \) separately for \( x[k] \) and \( \overline{x[k]} \) (using a different summation-index variable for each substitution), leads to **Parseval’s theorem**

\[
P_x = \sum_{r=0}^{N-1} |D_r|^2.
\]

- Parseval’s theorem can be used to determine the amount of periodic signal \( x \)'s power resides in a given frequency band \([a, b] \subset [0, 2\pi] \) radians/s:

1. determine the harmonics \( r\Omega_o \) of \( x \) that reside in this band, \( i.e. \), integers \( r \in [a/\Omega_o, b/\Omega_o] \)

where \( x \)'s fundamental frequency \( \Omega_o = 2\pi/N \).

2. sum just over these harmonics to get the answer, \( \sum_{[a/\Omega_o] \leq r \leq [b/\Omega_o]} |D_r|^2 \).
DTFS - Parseval's theorem example

- Find the fraction of $x$'s average power in the frequency band $[0.4\pi, 1.1\pi]$ radians/s where

$$\forall k \in \mathbb{Z}, \ x[k] = \sum_{v=-\infty}^{\infty} (3\delta[k - 4v] - 4\delta[k - 1 - 4v])$$

- **Solution:** $x$ has period $N = 4$ and average power

$$P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{4} \sum_{k=0}^{3} |x[k]|^2 = \frac{1}{4} (3^2 + (-4)^2 + 0^2 + 0^2) = 25/4$$

- $x$ has fundamental frequency $\Omega_o = 2\pi/N = \pi/2$ radians/s and discrete-time Fourier coefficients

$$D_r = \frac{1}{N} \sum_{r=0}^{N-1} x[r] e^{-j(r\Omega_o)k} = \frac{1}{4} \left( 3 - 4e^{-jr\pi/2} \right), \ 0 \leq r \leq N - 1 = 3.$$  

- The harmonics $r$ of $x$ that reside in $[0.4\pi, 1.1\pi]$ satisfy $0.4\pi \leq r\Omega_o = r\pi/2 \leq 1.1\pi$, i.e., $r \in \{1, 2\}$.

- So, by Parseval’s theorem, the answer is $(|D_1|^2 + |D_2|^2)/P_x$

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### Periodic extensions

- Consider signal $x : \mathbb{Z} \to \mathbb{R}$ having finite support $\{-M, -M + 1, ..., 0, ..., M - 1, M\}$ for $0 < M < \infty$; i.e., $\forall |k| > M, \ x[k] = 0$.

- For $N \geq M$, define $2N$-periodic $x^{(N)}$ such that

$$x^{(N)}[k] = \begin{cases} 
  x[k] & \text{if } |k| \leq M \\
  0 & \text{if } M < |k| \leq N
\end{cases}$$

- $x^{(N)}$ is a periodic extension of the finite-support signal $x$, where again $x^{(N)}$'s period is $2N$ and

$$\lim_{N \to \infty} x^{(N)} = x.$$
For \( r \in \{-N+1,-N+2,\ldots,N-1,N\} \), the DTFS of \( x^{(N)} \) has coefficients

\[
D_r^{(N)} = \frac{1}{2N} \sum_{k=-N+1}^{N} x^{(N)}[k] e^{-j2\pi kr/(2N)} = \frac{1}{2N} \sum_{k=-M}^{M} x[k] e^{-j2\pi kr/(2N)} = \frac{1}{2N} \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi kr/(2N)} =: \frac{1}{2N} X\left(\frac{2\pi r}{2N}\right),
\]

where the Discrete-Time Fourier Transform (DTFT) of (aperiodic) \( x : \mathbb{Z} \to \mathbb{R} \) is

\[
X(\Omega) := \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} =: (\mathcal{F}x)(\Omega).
\]

Note that Fourier integrals (spectra of discrete-time signals) are periodic, repeating themselves every \( 2\pi \) radians: \( \forall \Omega \in \mathbb{R}, \ell \in \mathbb{Z}, \)

\[
X(\Omega) = X(\Omega + \ell 2\pi).
\]

### Inverse DTFT by Fourier Integral

Thus, \( \forall k \in \mathbb{Z}, \)

\[
x[k] = \lim_{N \to \infty} x^{(N)}[k] = \lim_{N \to \infty} \sum_{r=-N+1}^{N} D_r^{(N)} e^{j2\pi kr/(2N)} = \lim_{N \to \infty} \sum_{r=-N+1}^{N} X\left(\frac{2\pi r}{2N}\right) e^{j2\pi kr/(2N)} \frac{1}{2N} \frac{2\pi}{2N} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{jk\Omega} d\Omega
\]

where the last equality is by Riemann integration with \( 2\pi/(2N) \to d\Omega. \)

Thus, we have derived the inverse DTFT by Fourier integral of \( X \) giving (aperiodic) \( x, \)

\[
\forall k \in \mathbb{Z}, \quad x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{jk\Omega} d\Omega =: (\mathcal{F}^{-1}X)[k].
\]
DTFT Examples - exponential signal

• If \( x = \delta \) then obviously \( X \equiv 1 \).

• The geometric signal \( x[k] = \gamma^k u[k] \) for scalar \( \gamma \) s.t. \( |\gamma| < 1 \) has DTFT

\[
X(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k = \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{1}{(1 - \gamma) \cos(\Omega) + j\gamma \sin(\Omega)}
\]

• Note that

\[
|X(\Omega)| = \frac{1}{(1 - \gamma \cos(\Omega))^2 + \gamma^2 \sin^2(\Omega)} = \frac{1}{1 + \gamma^2 - 2\gamma \cos(\Omega)}
\]

\[
\angle X(\Omega) = -\arctan \left( \frac{\gamma \sin(\Omega)}{1 - \gamma \cos(\Omega)} \right)
\]

• Exercise: What are the maximum and minimum values of \( |X| \), i.e., how would this plot depend on \( \gamma > 0 \)? Plot \( x \) and \( \angle X \). How do these plots differ when \(-1 < \gamma < 0\)?

• Exercise: Find the DTFT of anticausal signal \( x[k] = \gamma^k u[-k] \) for scalar \( \gamma \) s.t. \( |\gamma| > 1 \).

• Exercise: Find the DTFT of \( x[k] = \gamma^{|k|}, k \in \mathbb{Z} \), for scalar \( \gamma \) s.t. \( |\gamma| < 1 \).

DTFT Examples - exponential signal (cont)

• The plots above are for \( \gamma = 0.5 \).

• Note how \( X \) has period \( 2\pi \).

• Exercise: Find the DTFT of anticausal signal \( x[k] = \gamma^k u[-k] \) for scalar \( \gamma \) s.t. \( |\gamma| > 1 \).
DTFT Examples - Square and Triangle Pulse

• For $T \in \mathbb{Z}^\geq 0$, the even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u \ (i.e., \ x[k] = u[k + T] - u[k - (T + 1)])$, has DTFT
  \[ X(\Omega) = \sum_{k=-T}^{T} 1 e^{-jk\Omega} = 1 + 2\sum_{k=1}^{T} \cos(k\Omega), \ \Omega \in \mathbb{R}. \]

• Exercise (even rectangle pulse in frequency domain):
  Show that for fixed $\Omega'$ s.t. $0 < \Omega' < \pi$,
  \[ \mathcal{F}^{-1}\{\Delta_{-\Omega} u - \Delta_{\Omega} u\}[k] = \frac{\Omega'}{\pi}\text{sinc}(\Omega' k), \ k \in \mathbb{Z}. \]

• For $T \in \mathbb{Z}^\geq 0$, the odd triangle pulse with support $2T + 1$, $x[k] \equiv k(\Delta^{-T}u[k] - \Delta^{T+1}u[k])$ has DTFT
  \[ X(\Omega) = \sum_{k=-T}^{T} ke^{-jk\Omega} = -2j\sum_{k=1}^{T} k \sin(k\Omega), \ \Omega \in \mathbb{R}. \]

DTFT Examples - exponential sinusoid

• For fixed time $K_0$, clearly
  \[ \mathcal{F}\{\delta[k - K_0]\}(\Omega) = e^{jK_0\Omega}, \]
  where here $\delta$ is the unit pulse.

• Note that $e^{jK_0\Omega}$ is a sinusoidal function of $\Omega$ with period $2\pi/K_0$ (frequency $K_0$).

• Exercise: For fixed frequency $\Omega_0$, show that
  \[ \mathcal{F}\{e^{-jK_0}\}(\Omega) = 2\pi \sum_{v=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi v), \]
  where here $\delta$ is the Dirac impulse (in the frequency domain $\Omega \in \mathbb{R}$). Hint: work with $\mathcal{F}^{-1}$.

• So, the DTFT of a $N$-periodic signal with Fourier series
  \[ \sum_{r=0}^{N-1} D_r e^{j2\pi kr/N} \xrightarrow{\mathcal{F}} 2\pi \sum_{v=-\infty}^{\infty} \sum_{r=0}^{N-1} D_r \delta(\Omega - \frac{2\pi rv}{N} + 2\pi v) \]
DTFT - Time shift and frequency shift properties

- If fixed $K_0 \in \mathbb{Z}$ and $X = \mathcal{F}\{x\}$ then
  \[
  \mathcal{F}\{\Delta^{K_0}x\}(\Omega) = \sum_{k=-\infty}^{\infty} (\Delta^{K_0}x)[k]e^{-jk\Omega} \\
  = \sum_{k=-\infty}^{\infty} x[k-K_0]e^{-jk\Omega} \\
  = \sum_{k'=-\infty}^{\infty} x[k']e^{-j(k+K_0)\Omega} \\
  = e^{-jK_0\Omega}X(\Omega),
  \]
  i.e., shift in time by $K_0$ corresponds to product with sinusoid of period $2\pi/K_0$ (linear phase shift) in frequency domain.

- Exercise: Prove the dual property that if fixed $\Omega_0 \in \mathbb{R}$ and $X = \mathcal{F}\{x\}$ then
  \[
  \mathcal{F}\{xe^{\Omega_0 k}\}(\Omega) = X(\Omega - \Omega_0),
  \]
  i.e., modulation (multiplication by a sinusoid) in time domain results in frequency shift.

- Exercise: Use the convolution properties to prove the time and frequency shift properties. Hint: $(\Delta^{K_0}\delta) * x = \Delta^{K_0}x$.

DTFT - Convolution properties

- Let $X_r = \mathcal{F}\{x_r\}$ for $r \in \{1, 2\}$.
  \[
  \mathcal{F}\{x_1 * x_2\}(\Omega) := \sum_{k=-\infty}^{\infty} (x_1 * x_2)[k]e^{-jk\Omega} \\
  := \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k-l]e^{-j(k-l)\Omega}e^{-jl\Omega} \quad \text{where we've } x e^{j\Omega} e^{-j\Omega} = 1 \\
  = \sum_{k'=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k']e^{-jk'\Omega}e^{-jl\Omega} \quad \text{where } k' = k - l \\
  = \sum_{l=-\infty}^{\infty} x_1[l]e^{-jl\Omega} \sum_{k=-\infty}^{\infty} x_2[k']e^{-jk'\Omega} =: X_1(\Omega)X_2(\Omega)
  \]

- Exercise: Prove the dual property that
  \[
  \mathcal{F}\{x_1x_2\}(\Omega) = \frac{1}{2\pi}(X_1 * X_2)(\Omega) := \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(v)X_2(\Omega - v)dv.
  \]
DTFT - Parseval’s Theorem

- The energy of a signal DT $x$ is
  
  $$E_x := \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} x[k] \overline{x[k]} = \sum_{k=-\infty}^{\infty} (F^{-1}X)[k] \overline{(F^{-1}X)[k]}$$

  $$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} |X(\Omega)e^{jk\Omega}d\Omega| \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{jk\Omega}d\Omega$$

  $$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{jk\Omega}d\Omega \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{-jk\Omega}d\Omega$$

  $$= \frac{1}{2\pi} \int_{2\pi} X(\Omega) \overline{X(\Omega)} \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{j\Omega} \overline{e^{-jk\Omega}} \right) d\Omega d\Omega'$$

  $$= \frac{1}{2\pi} \int_{2\pi} X(\Omega) \overline{X(\Omega)} d\Omega d\Omega' = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 d\Omega d\Omega', \text{ recalling that for fixed } \Omega': F^{-1}\{2\pi\delta(\Omega - \Omega')\}[k] = e^{-jk\Omega} \text{ & } \int_{2\pi} X(\Omega)\delta(\Omega - \Omega')d\Omega = \overline{X(\Omega')}.$$
DTFT - Parseval’s Theorem - exercises

- **Exercise:** Repeat this calculation using \( X(\Omega) = 1 + 2 \sum_{k=1}^{T} \cos(k\Omega) \).

- **Exercise:** Compute amount of energy of \( x \) in the frequency band \([-\pi/6, \pi/6]\), i.e.,

\[
\frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} |X(\Omega)|^2 d\Omega
\]

Analysis of Stable DT LTI Systems in Steady-State

- Consider a SISO, DT-LTIC system described by the difference equation

\[
Q(\Delta^{-1})y = P(\Delta^{-1})f,
\]

where \( f \) is the input and \( y \) is the ZSR (output).

- Recall that by the time-shift property,

\[
Q(e^{j\Omega})Y_{ZS}(\Omega) = P(e^{j\Omega})F(\Omega) \Rightarrow Y_{ZS}(\Omega) = H(\Omega)F(\Omega).
\]

- We now re-derive from first principles the eigenresponse by first recalling that the ZSR \( y_{ZS} = f * h \) where \( h \) is the unit-pulse response.

- Taking DTFTs, \( Y_{ZS} = HF \) where \( H = \mathcal{F}h \) is the transfer function.

- Suppose the system is BIBO/asymptotically stable, i.e., the \( n \) roots of \( Q \) (system char. modes/poles) \( z \) all have modulus \( |z| < 1 \).

- The ZSR will consist of a forced response plus characteristic modes, where the latter will \( \to 0 \) over time (our stability assumption) so that the forced response becomes the steady-state response.
Analysis of Stable DT LTI Systems in Steady-State (cont)

- The forced response to a persistent sinusoidal input
  \[ f[k] = A_f e^{j(\Omega k + \phi)} \]
  will be of the form
  \[ y_{ss}[k] = A_y e^{j(\Omega k + \phi_y)} \]
  where (for \( k \geq 0 \)),
  \[ Q(e^{j\Omega})y_{ss}[k] = (Q(\Delta^{-1})y_{ss})[k] = (P(\Delta^{-1})f)[k] = P(e^{j\Omega})f[k]. \]
  \[ \Rightarrow y_{ss}[k] = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})} f[k] \]

- Also,
  \[ y_{zs}[k] = \sum_{v=0}^{k} h[v] A_f e^{j(\Omega(k-v) + \phi)} = f[k] \sum_{v=0}^{k} h[v] e^{-j\Omega v} \]
  \[ \Rightarrow f[k] H(\Omega_n) =: y_{ss}[k] \text{ as } k \to \infty. \]

Transfer Function and Eigenresponse in Discrete Time (cont)

- Equating the forced responses (steady-state response for a stable system), we again get that the system transfer function is
  \[ H(\Omega) = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})} = (Fh)(\Omega). \]

- Note that \( \forall k \in \mathbb{Z}, H(\Omega) = H(\Omega + 2\pi k). \)

- Also, we write \( H(\Omega) \) not \( H(e^{j\Omega}) \) for the DTFT.

- So, the eigenresponse of a BIBO/asymptotically stable SISO, DT-LTIC system is the steady-state response to a sinusoid:
  \[ f[k] = A_f e^{j(\Omega k + \phi)} \rightarrow H(\Omega_n) f[k] = A_y e^{j(\Omega k + \phi_y)} =: y_{ss}[k] \]

- The system magnitude response (gain) is \( |H(\Omega)| = |P(e^{j\Omega})|/|Q(e^{j\Omega})| \),
  i.e., \( A_y = A_f |H(\Omega_0)| \).

- The system phase response is \( \angle H(\Omega) = \angle P(e^{j\Omega}) - \angle Q(e^{j\Omega}) \),
  i.e., \( \phi_y = \phi_f + \angle H(\Omega_0) \). 

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• **Problem:** For the system $2y[k] = 0.6y[k - 1] - 7f[k]$ find the steady-state response (if it exists) to $f[k] = 4 \cos(5k)u[k]$.

• **Solution:** The difference equation in standard form is $(Q(\Delta^{-1})y)[k] = y[k + 1] - 0.3y[k] = -3.5f[k + 1] = (P(\Delta^{-1})f)[k]$, where $Q(z) = z - 0.3$ and $P(z) = -3.5z$.

The sole system characteristic value (root of $Q$, system pole) is 0.3, hence the system is BIBO/asymptotically stable.

By Euler’s identity $f[k] = (2e^{j5k} + 2e^{j(-5)k})u[k]$.

By linearity, the eigenresponse is therefore $2H(5)e^{j5k} + 2H(-5)e^{j(-5)k}$, where $H(\Omega) = P(e^{j\Omega})/Q(e^{j\Omega}) = -3.5e^{j\Omega}/(e^{j\Omega} - 0.3) = \overline{H(-\Omega)}$, so that $|H(\Omega)| = \frac{3.5}{\sqrt{(\cos(\Omega) - .3)^2 + \sin^2(\Omega)}}$, $\angle H(\Omega) = \pi + \Omega - \arctan\left(\frac{\sin(\Omega)}{\cos(\Omega) - .3}\right)$.

• **Exercise:** Show that the eigenresponse is also simply $|H(5)|4\cos(5k + \angle H(5))$.

---

**2D Image Processing Example**

• Apply 1-dimensional filtering to a 2-dimensional (2D) image by separately performing row and column operations.

• For $256 \times 256$ pixel (2D) image,

$$ f = \begin{bmatrix} f[1, 1] & f[1, 2] & \cdots & f[1, 256] \\ f[2, 1] & f[2, 2] & \cdots & f[2, 256] \\ \vdots & \vdots & \ddots & \vdots \\ f[256, 1] & f[256, 2] & \cdots & f[256, 256] \end{bmatrix} $$

• If $f[k, i]$ represents the 8-bit (grey) intensity of the pixel in row $k$ and column $i$ (i.e., 8 bits per pixel or bpp), then the "raw" image size will be $256^2 \text{bits} = 16 \text{Mb} = 2 \text{MB}$.

• Each of $f$’s rows of pixels can be processed by a system with unit-pulse response $h$ to obtain a new row of pixels, and thus a new image $y$:

$$ \forall k, \ f[k, \cdot] \rightarrow [h] \rightarrow y[k, \cdot] $$

• Alternatively, each of $f$’s columns of pixels can be processed by a system with unit-pulse response $h$ to obtain a new column of pixels, and thus a new image $y$:

$$ \forall i, \ f[\cdot, i] \rightarrow [h] \rightarrow y[\cdot, i] $$
Image Processing: High-Pass and Low-Pass Filtering

- The system $h$ may have a specific signal processing objective.
- The output pixels $y[k, i]$ may be quantized to fewer bpp than those of the input, thus achieving image compression.
- The simple low-pass filter (L)
  \[ h[k] = \frac{1}{2}(\delta[k] + \delta[k-1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] + f[k-1]) \]
  can capture shading and texture in the image.
- The simple high-pass filter (H)
  \[ h[k] = \frac{1}{2}(\delta[k] - \delta[k-1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] - f[k-1]) \]
  can capture edges in the image.
- Typically more compression possible in higher-frequency bands (H).

Image Processing: Tandem Row and Column Filtering

- Define $y_{LH}$ as the output of
  \[ f \rightarrow \text{row filtering} \rightarrow \text{column filtering} \rightarrow y \]
- Similarly define $y_{LL}$, $y_{HH}$ and $y_{HL}$.
- The $y$ images are downsampled by a factor of four (two in each direction).
- The $y_{LL}$ image will have a lot of energy while $y_{HH}$ will have the least energy.
- This motivates non-uniform quantization (bit allotment per pixel) of these images.
- Together with a coding strategy for the quantized images (particularly for the regions of zero pixel-values), this is the basic approach used in JPEG leading to very good compression, e.g., from 8 bpp to 0.2-0.5 bpp.
Sampling Continuous-Time Signals (A/D)

- Consider continuous-time signal $x$ with $X = \mathcal{F}x$.
- Recall that by sampling at period $T$ with impulses in continuous time $t \in \mathbb{R}$, we get
  \[
  x_T(t) := \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} x(kT)e^{-jkTw} =: X_T(w),
  \]
equivalently, $X_T(w) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( w - v\frac{2\pi}{T} \right)$.
- Now define the sampled process in discrete-time $k \in \mathbb{Z}$ and its DTFT,
  \[
  \hat{x}[k] := x(kT) \xrightarrow{\mathcal{F}} \hat{X}(\Omega) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{-jk\Omega}.
  \]
- Substituting $w = \Omega/T$ we get
  \[
  \hat{X}(\Omega) = X_T \left( \frac{\Omega}{T} \right) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( \frac{\Omega - v2\pi}{T} \right).
  \]
- Exercise: Read decimation (downsampling) and interpolation (upsampling) of Lathi Figs. 8.17 & 10.9.

Sampling Continuous-Time Signals - example

- We are particularly interested in the case where
  - the continuous-time signal $x$ is band-limited, i.e., $\exists w' > 0$ s.t. $X(w) = 0$ for $|w| > w'$, and
  - the sampling frequency is greater than Nyquist’s, i.e., $2\pi/T > 2w' \Rightarrow w'T < \pi$.
- Example: For fixed $w' > 0$, consider the cts-time signal $x(t) = \text{Asinc}(w't)$ with FT
  \[
  X(w) = \frac{A\pi}{w'}(u(w + w') - u(w - w')).
  \]
- Sampling $x$ at period $T < \pi/w'$ we get the discrete-time signal $x[k] = \text{Asinc}(w'kT)$.
- Using inverse DTFT, recall that we can easily check that the DTFT of $x$ is,
  \[
  \hat{X}(\Omega) = \sum_{v=-\infty}^{\infty} \frac{A\pi}{w'T}(u(\Omega + w'T - 2\pi v) - u(\Omega - w'T - 2\pi v))
  \]
  \[
  = \sum_{v=-\infty}^{\infty} \frac{1}{T}X \left( \frac{\Omega - 2\pi v}{T} \right),
  \]
  noting $\forall T > 0$, $u(\frac{\Omega}{T} \pm w') = u(\frac{\Omega}{T}(\Omega \pm w'T)) = u(\tilde{\Omega} \pm w'T), \tilde{\Omega} := \Omega - 2\pi v$. 

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Sampled Data Systems: A/D (analog-to-digital conversion)

- Suppose the signal $f$ is sampled every $T_s$ seconds, i.e., at sampling frequency $w_s := 2\pi/T_s$.

- Recall Poisson’s identity (the Fourier series of the picket-fence function)
  \[ p_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_st} \]

- Let’s rederive the relationship between the spectrum of a sampled continuous-time signal and its discrete-time counterpart by first defining the discrete-time signal
  \[ \forall k \in \mathbb{Z}, \quad f[k] = f(kT_s). \]

- We want to relate the (continuous-time) Fourier transform of $f$ to the (discrete-time) Fourier transform of $\hat{f}$,
  \[ \mathcal{F}(\Omega) := \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega_0} = \sum_{k=-\infty}^{\infty} f(kT_s)e^{-j\Omega_0}. \]
Sampled Data Systems: A/D (cont)

• To this end, recall
  \[ f(t) \mapsto \frac{1}{T_s} \sum_{k=-\infty}^{\infty} f(t)e^{jk\omega_s} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s), \]
  and also
  \[ f(t) \mapsto \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s) \mapsto \sum_{k=-\infty}^{\infty} f(kT_s)e^{-jk\omega_s} = F(wT_s). \]

• Equating these two expressions for \(\mathcal{F}\{f_{T_s}\}\) we get,
  \[ \hat{F}(wT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s). \]

• Substituting \(w = \Omega/T_s\) we get,
  \[ \hat{F}(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F \left( \frac{\Omega - k2\pi}{T_s} \right). \]

Sampled Data Systems: D/A (digital to analog conversion)

• Now consider a discrete time signal \(\hat{y}[k]\).

• We implement at D/A with a \(T_s\)-second hold, i.e., construct the continuous-time signal
  \[ y(t) := \sum_{k=-\infty}^{\infty} \hat{y}[k]r_{T_s}(t - kT_s), \]
  where
  \[ r_{T_s}(t) := u(t) - u(t - T_s) \mapsto T_s\text{sinc}(wT_s/2)e^{-jwT_s/2} =: R_{T_s}(w). \]

• Note that \(y\) is in the form of a convolution, so:
  \[ Y(w) = \sum_{k=-\infty}^{\infty} \hat{y}[k]R_{T_s}(w)e^{-jkwT_s} = R_{T_s}(w)\hat{Y}(wT_s). \]
Consider a digital system $\hat{H}(\Omega)$ (or $\hat{H}(e^{j\Omega})$ depending on notation), whose (ZS) output is $\hat{y}$ when the input is $\hat{f}$, i.e., $\hat{Y} = \hat{H}\hat{F}$.

The equivalent continuous-time transformation of the tandem system

$f \rightarrow A/D (T_s\text{-sample}) \rightarrow \hat{H}(\Omega) \rightarrow D/A (T_s\text{-hold}) \rightarrow y$

with input $f$ has (ZS) output

$$Y(w) = R_T(w)\hat{Y}(wT_s) = R_T(w)\hat{H}(wT_s)\hat{F}(wT_s)$$

$$= R_T(w)\hat{H}(wT_s)\frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s).$$

Exercise: Show that if $f$ is band-limited by $w_s/2$ (i.e., $w_s$ is greater than $f$’s Nyquist frequency) and the previous sampled data system is followed by an ideal low-pass filter with bandwidth $w_s/2$, then the equivalent (continuous-time) transfer function is

$$H(w) = \hat{H}(wT_s)T_s^{-1}R_T(w)(u(w + w_s/2) - u(w - w_s/2))$$

Note that the term in the transfer function $H$,

$$T_s^{-1}R_T(w)(u(w + w_s/2) - u(w - w_s/2)) = \text{sinc}(\Omega/2)(u(\Omega + \pi) - u(\Omega - \pi))$$

is not a constant function of $\Omega = wT_s$.

This distortion due to the hold function $R$ can be reduced by putting in tandem with $\hat{H}$ an equalizer system with transfer function approximately

$$\hat{R}^{-1}(\Omega) := \sum_{k=-\infty}^{\infty} \frac{u(\Omega + \pi - k2\pi) - u(\Omega - \pi - k2\pi)}{\text{sinc}((\Omega - k2\pi)/2)}$$

i.e.,

$$\hat{H}(\Omega) \rightarrow \hat{R}^{-1}(\Omega)$$
Sampled Data Systems: equalization of hold $\text{sinc}(\Omega/2)$ by $\widetilde{R}^{-1}(\Omega)$

- the hold (at left, $R$) distorts the signal by attenuating its higher frequency components
- the equalizer (at right, $R^{-1}$) amplifies at the higher frequencies to cancel out this distortion

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**DFT and FFT - Reading Exercise on Computational Issues**

- Read Lathi Sec. 5.2 and 5.3 re. continuous-time FS, FT
- Read Lathi Sec. 10.6 re. DTFS, DTFT
• Z-transform definition and region of convergence.

• Basic z-transform pairs and properties.

• Inverse z-transform of rational polynomials by Partial Fraction Expansion (PFE).

• Total transient response of SISO DT LTIC systems $Q(\Delta^{-1})y = P(\Delta^{-1})f$.

• The steady-state eigenresponse revisited.

• System composition and canonical realizations.

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**The unilateral z-transform & region of convergence**

• The z-transform of a signal $x = \{x[k]\}_{k \geq 0}$ is

$$X(z) = (Zx)(z) = \sum_{k=0}^{\infty} x[k]z^{-k} := \lim_{K \to \infty} \sum_{k=0}^{K} x[k]z^{-k},$$

where $z \in \mathbb{C}$.

• If the signal $x$ is bounded by an exponential (geometric), i.e.,

$\exists M, \gamma \in \mathbb{R}^{>0}$ such that $\forall k \in \mathbb{Z}^{\geq 0}$, $|x[k]| \leq M\gamma^k$ (i.e., $-M\gamma^k \leq x[k] \leq M\gamma^k$)

then the series $X(z)$ converges in the region outside of a disk centered $0 \in \mathbb{C}$,

$$\{z \in \mathbb{C} \mid |z| > \gamma\}.$$

• To see why bounded by an exponential suffices, recall absolute convergence $\Rightarrow$ convergence:

$$\forall k \geq 0, \quad |x[k]z^{-k}| = |x[k]| \cdot |z|^{-k} \leq M\gamma^k |z|^{-k} = M(\gamma/|z|)^k$$

$$\Rightarrow \sum_{k=0}^{\infty} |x[k]z^{-k}| \leq M \sum_{k=0}^{\infty} (\gamma/|z|)^k$$

which converges if $\gamma/|z| < 1$. 

---
Basic $z$-transform pairs and RoCs

\[ \delta[k] \xrightarrow{Z} 1, \ z \in \mathbb{C} \]
\[ u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \ |z| > 1 \]
\[ \beta^k u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{1}{1-\beta z^{-1}} = \frac{z}{z-\beta}, \ |z| > |\beta| \]
\[ \{\beta^{-1} u[k-1]\}(z) \xrightarrow{Z} \sum_{k=1}^{\infty} \beta^{-k-1} z^{-k} = z^{-1} \sum_{k=0}^{\infty} \beta^k z^{-k'} = z^{-1} \frac{1}{1-\beta z^{-1}} = \frac{1}{z-\beta}, \ |z| > |\beta| \]
\[ e^{j\Omega k} u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} e^{j\Omega k} z^{-k} = \frac{1}{1-e^{j\Omega} z^{-1}}, \ |z| > 1 \ (\beta = e^{j\Omega}) \]
\[ k \beta^k u[k] \xrightarrow{Z} \sum_{k=0}^{\infty} k \beta^k z^{-k} = \beta \frac{d}{d\beta} \sum_{k=0}^{\infty} \beta^k z^{-k} = \beta \frac{d}{d\beta} \frac{1}{1-\beta z^{-1}} = \frac{\beta z^{-1}}{(1-\beta z^{-1})^2}, \ |z| > |\beta| \]

**Exercise:** Find $\mathcal{Z}\{A \cos(\Omega_0 k + \phi) u[k]\}$ and $\mathcal{Z}\{A \sin(\Omega_0 k + \phi) u[k]\}$.

---

**Basic $z$-transform properties: linearity**

- The $z$-transform is a linear operator: for all scalars $a_1, a_2 \in \mathbb{C}$ and all signals $x_1, x_2 : \mathbb{Z}^0 \to \mathbb{C}$ with respective ROCs $C_1, C_2 \subset \mathbb{C}$,
  \[ (\mathcal{Z}(a_1 x_1 + a_2 x_2))(z) = a_1 (\mathcal{Z}x_1)(z) + a_2 (\mathcal{Z}x_2)(z), \ z \in C_1 \cap C_2. \]
- Note that
  \[ \{z \mid |z| > \gamma_1\} \bigcap \{z \mid |z| > \gamma_2\} = \{z \mid |z| > \max\{\gamma_1, \gamma_2\}\} \subset \mathbb{C}. \]
Basic $z$-transform properties: advance time shift

- **Advance time shift** (no change in RoC): Let $X = \mathcal{Z}x$.

\[
\Delta^{-1}x \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k+1]z^{-k} = -zx[0] + \sum_{k=-1}^{\infty} x[k+1]z^{-k}
\]

\[
= -zx[0] + z \sum_{k=-1}^{\infty} x[k+1]z^{-(k+1)}
\]

\[
= -zx[0] + z \sum_{k=0}^{\infty} x[k]z^{-k'}
\]

\[
= -zx[0] + zX(z)
\]

- **Exercise:** For $v \in \mathbb{Z}^{>0}$ show by induction that

\[
(Z\{\Delta^{-v}x\})(z) = -\sum_{k=1}^{\nu} z^k x[v-k] + z^\nu X(z)
\]

Basic $z$-transform properties: delay time shift

- **Delay time shift** (no change in RoC): For $v \in \mathbb{Z}^{>0}$,

\[
\Delta^v(xu) \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k-v]u[k-v]z^{-k}
\]

\[
= \sum_{k=v}^{\infty} x[k-v]z^{-k} = \sum_{k=0}^{\infty} x[k']z^{-k'-v}
\]

\[
= z^{-v}X(z).
\]

- **So in the “zero-state” (input-output) context (i.e., $x[k]u[k] = 0$ for $k < 0$), we identify multiplying by $z^{-1}$ in complex-frequency domain with the unit delay $\Delta$ in the time domain.**

- **Delay $v \in \mathbb{Z}^{>0}$ of non-causal $x$:**

\[
\Delta^v x \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k-v]z^{-k} = \sum_{k'=v}^{\infty} x[k']z^{-k'-v}
\]

\[
= \sum_{k'=-v}^{\infty} x[k']z^{-k'-v} + z^{-v}X(z).
\]
Basic $z$-transform properties: frequency shift & convolution

- Let $X = Zx$ with $\text{RoC } C(\gamma) := \{z \in \mathbb{C} \mid |z| > \gamma\}$.

$$
\beta^k x[k] \overset{Z}{\rightarrow} \sum_{k=0}^{\infty} \beta^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k] (z/\beta)^{-k} = X(z/\beta), \quad z \in C(\gamma/|\beta|).
$$

- For signals $x_1, x_2 : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}$ ($x_1[k], x_2[k] = 0$ for $k < 0$), with respective ROCs $C_1, C_2 \subset \mathbb{C}$,

$$
x_1 \ast x_2 \overset{Z}{\rightarrow} \sum_{k=0}^{\infty} (x_1 \ast x_2)[k] z^{-k} = \sum_{v=0}^{\infty} \sum_{k=0}^{k} x_1[v] x_2[k-v] z^{-(k-v)} z^{-v} = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=0}^{\infty} x_2[k-v] z^{-(k-v)} = X_1(z) X_2(z), \quad z \in C_1 \cap C_2.
$$

Basic $z$-transform properties: convolution, IVT & FVT

- So convolution in the time-domain is multiplication in the frequency domain.

- The converse is also true.

- Directly by definition of $X = Zx$, we get the initial value theorem

$$
\lim_{z \rightarrow \infty} X(z) = x[0].
$$

- There is also a “final value” theorem for $\lim_{k \rightarrow \infty} x[k]$. 

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• We now study transient analysis of LTI difference equations using $z$-transforms.

• Recall our system is defined given polynomials $P, Q$, input $f$ and initial conditions:
  - $Q(\Delta^{-1})y = P(\Delta^{-1})f$, where $y$ is the (total) output and
  - input $f[k] = 0$ for $k < 0$,
  - degree of polynomial $Q = n \geq m$ = degree of polynomial $P$ (causal system),
  - $Q(z) = z^n + \sum_{v=0}^{n-1} a_v z^v$ (i.e., $a_n = 1$) and $P(z) = \sum_{v=0}^{m} b_v z^v$,
  - $a_n \neq 0$ or $b_n \neq 0$ for pol’ls $Q, P$ of minimum degree,
  - $n$ initial conditions $y[-n], y[-n+1], ..., y[-2], y[-1]$.

• We can restate the difference equation in terms of delays by delaying both sides by $n$ time-units (i.e., apply with $\Delta^n$), to get

\[
\Delta^n Q(\Delta^{-1})y = \Delta^n P(\Delta^{-1})f
\]

\[
\Rightarrow \tilde{Q}(\Delta)y := \sum_{v=0}^{n} a_v \Delta^{n-v} y = \sum_{v=0}^{m} b_v \Delta^{n-v} f =: \tilde{P}(\Delta)f
\]

Total response of SISO LTIC systems (cont)

• So, taking the $z$-transform of the (delay) difference equation, we get by the (delay) time-shift and linearity properties that

\[
\sum_{v=0}^{n} a_v \sum_{k=-v}^{1-n} y[k] z^{-k-v} + \tilde{Q}(z^{-1})Y(z) = \tilde{P}(z^{-1})F(z)
\]

• So, solving for the total response $Y$ we get

\[
Y(z) = \frac{\tilde{P}(z^{-1})}{\tilde{Q}(z^{-1})} F(z) - \sum_{v=0}^{n} a_v \sum_{k=-v}^{1-n} y[k] z^{-k-v} = Y_{ZS}(z) + Y_{ZI}(z)
\]

where the ZIR and ZSR in the complex-frequency ($z$) domain respectively are

\[
Y_{ZI}(z) := -\sum_{v=0}^{n} a_v \sum_{k=-v}^{1-n} y[k] z^{-k-v} = -\sum_{v=0}^{n} a_v \sum_{k=-v}^{1-n} y[k] z^{-n-k-v}
\]

\[
Y_{ZS}(z) := \frac{\tilde{P}(z^{-1})}{\tilde{Q}(z^{-1})} F(z) = \frac{P(z)}{Q(z)} F(z) = H(z) F(z)
\]

• $H$ is the system transfer function, and $Y$’s RoC is the intersection of those of its characteristic modes, i.e., determined by the characteristic value(s) of largest modulus.
Total response of SISO LTIC systems - example

- Suppose i.c. $y[-1] = -1$, input $f[k] = 2(-3)^k u[k]$ and output $y$ s.t.
  \[ \forall k \geq -1, \quad 2y[k+1] + 2y[k] = 3f[k+1] + 2f[k]. \]

- To find the total response $y$, we take the $z$-transform of the equivalent system: \( \forall k \geq 0, \)
  \[ 2y[k] + 2y[k-1] = 3f[k] + 2f[k-1] \]
  \[ \Rightarrow 2Y(z) + 2(z^{-1}Y(z) + y[-1]) = 3F(z) + 2z^{-1}F(z). \]

- So by the delay property for non-causal signals ($y$), the total response
  \[ Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} F(z) + \frac{-2y[-1]}{2 + 2z^{-1}} \]
  \[ = \frac{H(z)F(z) - y[-1]}{1 + z^{-1}} \]
  \[ = Y_{ZS}(z) + Y_{ZI}(z) \]
  with RoC for $Y$ being the intersection of those of $F$ and $H$.

- Here $F(z) = Z\{2(-3)^k\} = 2/(1 + 3z^{-1})$, $y[-1] = -1$, so
  \[ Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} \cdot \frac{2}{1 + 3z^{-1}} + \frac{1}{1 + z^{-1}} \]
  \[ = \left( \frac{3.5}{1 + 3z^{-1}} + \frac{-0.5}{1 + z^{-1}} \right) + \frac{1}{1 + z^{-1}} \]
  where for the last equality see PFE below (here in $z^{-1}$).

- Understanding that the ZIR begins at $k = -1$ (initial condition) and the ZSR at time $k = 0$, we get:
  \[ \forall k \geq -1, \quad y[k] = (3.5(-3)^k - 0.5(-1)^k)u[k] + (-1)^k = y_{ZS}[k] + y_{ZI}[k], \]
  where we minded the ambiguity $Zx = Zxu$.

- Exercise: Verify this solution using time-domain methods, i.e.,
  \[ y = y_{ZI} + y_{ZS} = y_{ZI} + h * f, \]
  where $h$ and $y_{ZI}$ consist of char. modes.

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Inverse $z$-transform of proper rational polynomials

- We now describe how to find $Z^{-1}X$ of causal signal $X$ that is rational polynomial in $z$, i.e., $X(z) = M(z)/N(z)$ where $M(z)$ and $N(z)$ are polynomials in $z$.

- If $\deg(M) = \deg(N) + 1$, we perform long division to write $X = c + \tilde{M}/N$ where $\deg(N) = \deg(\tilde{M})$ and $Z^{-1}X = c\delta + Z^{-1}\{\tilde{M}/N\}$.

- If $\deg(M) = \deg(N)$ and $M(0) = 0$ (so $z^{-1}M(z)$ is a polynomial), we can factor $z$ from $M$ to get
  \[ X(z) = z^{-1}M(z)/N(z). \]

- We will find $Z^{-1}X$ using PFE of the strictly proper rational polynomial $z^{-1}M(z)/N(z)$.

- Alternatively, we could apply PFE on strictly proper rational polynomials in $z^{-1}$, $z^{-k}M(z)/(z^{-K}N(z))$ where $K := \deg(N)$, as in the previous example.

Partial Fraction Expansion (PFE) example in $z$ (not $z^{-1}$)

- For example, suppose
  \[ X(z) := \frac{z(3z + 2)}{z^2 - 0.64} = z\frac{(3z + 2)}{(z + 0.8)(z - 0.8)} = z\left(\frac{0.25}{z + 0.8} + \frac{2.75}{z - 0.8}\right) = 0.25\frac{z}{z + 0.8} + 2.75\frac{z}{z - 0.8} \]
  where PFE (below) gave the numerators (residues) 0.25 and 2.75.

- So,
  \[ (Z^{-1}X)[k] = 0.25(-0.8)^k u[k] + 2.75(0.8)^k u[k] \]

- Note that the associated RoC of $X$ is $\{z \in \mathbb{C} | |z| > 0.8\}$. 
Partial Fraction Expansion (PFE) - preliminaries

- Let $K = \deg(N) = \deg(M)$ so that we can factor
  \[ N(z) = \prod_{k=1}^{K} (z - p_k), \]
  where the $p_k$ are the roots of $N$ (poles of $M/N$).

- We assume $M$ and $N$ have no common roots, i.e., no “pole-zero cancellation” issue to consider, so that the $p_k$ are the poles of $M/N$.

- Again, we assume $M(0) = 0$ ($0$ is a zero of $M/N$) and so $z^{-1}M(z)$ is a polynomial of degree $K - 1$.

- Note that the RoC for $M(z)/N(z)$ is \( \{ z \in \mathbb{C} | |z| > \max_k |p_k| \} \).

PFE - the case of no repeated poles

- Suppose there are no repeated poles for $M/N$, i.e., $\forall k \neq l, \ p_k \neq p_l$.

- In this case, we can write the PFE of $z^{-1}M(z)/N(z)$ as
  \[
  \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} = \frac{M(z)}{N(z)} = \sum_{l=1}^{K} c_l \frac{z}{z - p_l} = \sum_{l=1}^{K} c_l \frac{1}{1 - p_l z^{-1}}
  \]
  where the scalars (Heaviside coefficients) $c_l \in \mathbb{C}$ are
  \[
  c_l = \lim_{z \to p_l} \frac{z^{-1}M(z)}{N(z)}(z - p_l) = \frac{z^{-1}M(z)}{N(z)}(z - p_l) \bigg|_{z=p_l}.
  \]

- That is, to find the Heaviside coefficient $c_k$ over the term $z - p_k$ in the PFE, we have removed (covered up) the term $z - p_k$ from the denominator $N(z)$ and evaluated the remaining rational polynomial at $z = p_k$.

- This approach, called the Heaviside cover-up method, works even when $p$ is $\mathbb{C}$-valued.

- Given the PFE of $z^{-1}M/N$, \((Z^{-1}M/N)[k] = \sum_{l=1}^{K} c_l p_l^k u[k] \).
To prove that the above formula for the Heaviside coefficient $c_l$ is correct, note that the claimed PFE of $z^{-1}M(z)/N(z)$ is

$$\sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) / N(z)$$

Thus, the PFE equals $z^{-1}M(z)/N(z)$ if and only if the numerator polynomials are equal, i.e., iff

$$z^{-1}M(z) = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) =: \tilde{M}(z).$$

Again, two polynomials are equal if their degrees, $L$, are equal and either:

- their coefficients are the same, or
- they agree at $L + 1$ (or more) different points, e.g., two lines ($L = 1$) are equal if they agree at 2 ($\approx L + 1$) points.

Since $z^{-1}M(z)$ is a polynomial of degree $< K$, it suffices to check that whether $z^{-1}M(z) = \tilde{M}(z)$ for all $z = p_k, k \in \{1, 2, ..., K\}$, i.e., this would create $K$ equations in $< K$ unknowns (the coefficients of $M$).

PFE - proof of Heaviside cover-up method (cont)

But note that any pole $p_r$ of $z^{-1}M(z)/N(z)$ is a root of all but the $r^{th}$ term in $\tilde{M}$, so that

$$\tilde{M}(p_r) = c_r \prod_{k \neq r} (p_r - p_k)$$

$$= \left. \frac{z^{-1}M(z)}{\prod_{k \neq r} (z - p_k)} \right|_{z = p_r} \prod_{k \neq r} (p_r - p_k)$$

$$= p_r^{-1}M(p_r) \prod_{k \neq r} (p_r - p_k)$$

$$= p_r^{-1}M(p_r).$$

Q.E.D.
PFE - the case of no repeated poles - example

• To find the inverse $z$-transform of a proper rational polynomial $X = M/N$ with $M(0) = 0$, first factor its denominator $N$ and factor $z$ from $M$, e.g.,

$$X(z) = \frac{z^3 + 5z^2}{z^3 + 9z^2 + 26z + 24} = \frac{z^2 + 5z}{(z + 4)(z + 3)(z + 2)}, \quad \text{for } |z| > 4.$$

• So, by PFE

$$X(z) = \frac{z}{(z + 4)(z + 3)(z + 2)} = \frac{z^{-1}M(z)}{N(z)} = z^{-1}M(z) = 1z^2 + 5z + 0 = c_4(z + 3)(z + 2) + c_3(z + 4)(z + 2) + c_2(z + 4)(z + 3) =: \hat{M}(z).$$

• We can solve for the 3 constants $c_k$ by comparing the 3 coefficients of quadratic $M$ and $\hat{M}$.

• The Heaviside cover-up method suggests we try $z = -2, -3, -4$ to solve for $c_2, c_3, c_4$:

$$c_4 = \frac{z^2 + 5z}{(z + 3)(z + 2)} \bigg|_{z=-4} = -2, \quad c_3 = \frac{z^2 + 5z}{(z + 4)(z + 2)} \bigg|_{z=-3} = 6, \quad c_2 = \frac{z^2 + 5z}{(z + 4)(z + 3)} \bigg|_{z=-2} = -3$$

• Thus, $x[k] = (Z^{-1}X)[k] = (-2(-4)^k + 6(-3)^k - 3(-2)^k)u[k]$. 

PFE - the case of a non-repeated, complex-conjugate pair of poles

• Again, recall that for polynomials with all coefficients $\in \mathbb{R}$, all complex poles will come in complex-conjugate pairs, $p_1 = \overline{p_2}$.

• The case of non-repeated poles $p_1, p_2 = \alpha \pm j\beta$ ($\alpha, \beta \in \mathbb{R}, j := \sqrt{-1}$) that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients $c_1, c_2$, i.e., $c_1 = \overline{c_2}$.

• In the PFE, we can alternatively combine the terms

$$\frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} = \frac{r_1z + r_0}{(z - \alpha)^2 + \beta^2}$$

where by equating the two numerator polynomials' coefficients,

$$r_0 = -c_1p_2 - c_2p_1 = -2\text{Re}\{c_1p_2\} \in \mathbb{R} \quad \text{and} \quad r_1 = c_1 + c_2 = 2\text{Re}\{c_1\} \in \mathbb{R}.$$

• Exercise: Show that

$$2|c| \cdot |p|^k \cos(k \angle p + \angle c) \overset{Z}{\to} \frac{cz}{z - p} + \frac{\overline{cz}}{z - \overline{p}}.$$

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• To find the inverse $z$-transform of

\[ X(z) = \frac{3z^2 + 2z}{z^3 + 5z^2 + 10z + 12}, \]

first factor the denominator and divide the numerator by $z$ to get

\[ X(z) = \frac{3z + 2}{(z^2 + 2z + 4)(z + 3)}. \]

• Note that the poles of $X$ are $-3$ and $-1 \pm j\sqrt{3}$ (so $X$’s RoC is $|z| > 3$).

• So, we can expand $X$ to

\[ X(z) = \frac{r_1 z + r_0}{z + 2z + 4} + \frac{c_3}{z + 3}, \]

where by the Heaviside cover-up method,

\[ c_3 = \frac{3z + 2}{z^2 + 2z + 4} \bigg|_{z=-3} = -1. \]

To find $r_1, r_0$, we will compare coefficients of the numerator polynomials of $X$ (actually $z^{-1}X$) and its PFE, i.e.,

\[ 0z^2 + 3z + 2 = (r_1 z + r_0)(z + 3) + c_3(z^2 + 2z + 4) \quad (\ast) \]

\[ = (r_1 - 1)z^2 + (3r_1 + r_0 - 2)z + 3r_0 - 4. \]

• Thus, by comparing coefficients

\[ 0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4 \]

we get

\[ r_0 = 2 \quad \text{and} \quad r_1 = 1. \]

• Note how $z = -3$ in $(\ast)$ gives $c_3 = -1$ as Heaviside cover-up did.
Thus by substituting, we get
\[ X(z) = \frac{z + 2}{z^2 + 2z + 4} + \frac{-1}{z + 3} \]

Exercise: Show that
\[ x[k] = (z^{-1}X)[k] = \left( \sqrt{4/3} \ 2^k \cos(k2\pi/3 - \pi/6) - (-3)^k \right) u[k]. \]

If a particular pole \( p \) of \( z^{-1}M(z)/N(z) \) is of order \( r \geq 1 \), i.e., \( N(z) \) has a factor \( (z-p)^r \), then the PFE of \( z^{-1}M(z)/N(z) \) has the terms
\[
\frac{c_1}{z-p} + \frac{c_2}{(z-p)^2} + \ldots + \frac{c_r}{(z-p)^r} = \sum_{k=1}^{r} \frac{c_k}{(z-p)^k} = \frac{z^{-1}M(z)}{N(z)} - \Phi(z)
\]
with \( c_k \in \mathbb{C} \ \forall k \in \{1, 2, \ldots, r\} \), where \( \Phi(z) \) represents the other PFE terms of \( z^{-1}M(z)/N(z) \) (i.e., corresponding to poles \( \neq p \)).

Note that equating \( z^{-1}M(z)/N(z) \) to its PFE and multiplying by \((z-p)^r\) gives
\[
\frac{z^{-1}M(z)}{N(z)}(z-p)^r = c_r + \sum_{k=1}^{r-1} c_k(z-p)^{r-k} + \Phi(z)(z-p)^r
\]
\[
\Rightarrow \frac{z^{-1}M(z)}{N(z)}(z-p)^r \bigg|_{z=p} = c_r,
\]
i.e., Heaviside cover-up (of the entire term \((z-p)^r\)) works for \( c_r \).
\begin{itemize}
  \item To find \( c_{r-1} \), we differentiate the previous display to get
  \[
  \frac{d}{dz} \frac{z^{-1}M(z)}{N(z)}(z-p)^r = \sum_{k=1}^{r-1} c_k (r-k) (z-p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z-p)^r
  \]
  \[
  = c_{r-1} + \sum_{k=1}^{r-2} c_k (r-k) (z-p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z-p)^r
  \]
  \[
  \Rightarrow c_{r-1} = \left( \frac{d}{dz} \frac{z^{-1}M(z)}{N(z)}(z-p)^r \right) \bigg|_{z=p}
  \]

  \item If we differentiate the original display \( k \in \{0, 1, 2, \ldots, r-1\} \) times and then substitute \( z = p \), we get (with \( 0! := 1 \))
  \[
  \left( \frac{d^k}{dz^k} \frac{z^{-1}M(z)}{N(z)}(z-p)^r \right) \bigg|_{z=p} = k! c_{r-k}
  \]
  \[
  \Rightarrow c_{r-k} = \frac{1}{k!} \left( \frac{d^k}{dz^k} \frac{z^{-1}M(z)}{N(z)}(z-p)^r \right) \bigg|_{z=p}
  \]
\end{itemize}

---

\textbf{PFE - the general case of repeated poles - example}

\begin{itemize}
  \item To find the inverse \( z \)-transform of
  \[
  X(z) = \frac{z(3z+2)}{(z+1)(z+2)^3}
  \]
  write the PFE of \( X \) as
  \[
  X(z) = z \frac{c_1}{z+1} + z \frac{c_{2,1}}{z+2} + z^2 \frac{c_{2,2}}{(z+2)^2} + z^3 \frac{c_{2,3}}{(z+2)^3},
  \]
  so clearly the RoC of causal \( X \) is \( |z| > 2 \).

  \item By Heaviside cover-up
  \[
  c_1 = \frac{3z+2}{(z+2)^3} \bigg|_{z=-1} = -1 \quad \text{and} \quad c_{2,3} = \frac{3z+2}{z+1} \bigg|_{z=-2} = 4.
  \]
\end{itemize}
PFE - the general case of repeated poles - example (cont)

• Also,

\[
\begin{align*}
    c_{2,2} &= \left. \frac{1}{1!} \left( \frac{d}{dz} \frac{3z + 2}{z + 1} \right) \right|_{z = -2} = \left. \frac{1}{1!} \left( \frac{1}{(z + 1)^3} \right) \right|_{z = -2} = 1 \\
    c_{2,1} &= \left. \frac{1}{2!} \left( \frac{d^2}{dz^2} \frac{3z + 2}{z + 1} \right) \right|_{z = -2} = \left. \frac{1}{2!} \left( \frac{-2}{(z + 1)^3} \right) \right|_{z = -2} = 1
\end{align*}
\]

• Thus,

\[
X(z) = \frac{1}{z + 1} + \frac{1}{z + 2} + \frac{1}{(z + 2)^2} + \frac{4}{(z + 2)^3} \quad \forall |z| > 2
\]

\[
\Rightarrow x[k] = (Z^{-1}X)[k] = \left( -(-1)^k + (-2)^k + k(-2)^{k-1} + \frac{4k(k-1)}{2} \right) u[k]
\]

• Exercise: Show by induction and integration by parts that: \( \forall m \in \mathbb{Z}^{>0}, \)

\[
\left( \frac{k}{m} \right)^{k-m} u[k] \xrightarrow{Z} \frac{z}{(z-\gamma)^m}
\]

• Exercise: Find the ZSR \( y \) to input \( f[k] = 2^k u[k] = 2e^{j\pi/2} u[k] \) of the marginally stable system \( H(z) = 4/(z^2 + 1) \).

PFE of \( M/N \) when \( M(0) \neq 0 \)

• If \( M(0) \neq 0 \) (so cannot factor \( z \) from \( M(z) \)), then just perform long division if \( \deg(M) \geq \deg(N) \) to get a strictly proper rational polynomial, factor \( N \) to find the poles, and find the PFE as before.

• When taking inverse \( z \)-transform, recall the \( z \)-transform pair

\[
\beta^{k-1} u[k-1] \xrightarrow{z} \frac{1}{z-\beta}, \quad |z| > |\beta|
\]

• For example, to find the ZSR to \( f[k] = 2(-1)^k u[k] \) of the system \( y[k+2] - 4y[k+1] = 5f[k+1] \), take \( z \)-transforms of the proper form \( y[k+1] - 4y[k] = 5f[k] \) to get

\[
\begin{align*}
Y(z) &= H(z)F(z) = \frac{5}{z-4}F(z) = \frac{10z}{(z-4)(z+1)} \\
&= \frac{8}{z-4} + \frac{2}{z+1} \quad \text{(by PFE)} \\
\Rightarrow y[k] &= 8(4)^{k-1} u[k-1] + 2(-1)^{k-1} u[k-1]
\end{align*}
\]

• Exercise: Repeat for the system \( y[k+1] - 4y[k] = 5f[k+1] \).
PFE and eigenresponse for asymptotically stable systems

- The total response of a SISO LTI system to input \( f \) is of the form
  \[
  Y(z) = H(z)F(z) + \frac{P_1(z)}{Q(z)} = \frac{P(z)}{Q(z)}F(z) + \frac{P_1(z)}{Q(z)} = Y_{ZS}(z) + Y_{ZI}(z).
  \]
  where \( P_1 \) depends on the initial conditions and the RoC is the intersection of that of input \( F = Zf \) and the system characteristic modes.

- Unlike for DTFT notation, we here write \( H(z) = \frac{Q(z)}{P(z)} = (Zh)(z) \).

- Suppose the system is BIBO/asymptotically stable and the input is a sinusoid at frequency \( \Omega_o \), \[ f[k] = Ae^{j\Omega ok}u[k] = Ae^{j\phi}(e^{j\Omega o}k)u[k] \text{ with } A > 0 \]
  \[ F(z) = Ae^{j\phi}z/(z - e^{j\Omega o}) \text{ with RoC } |z| > 1. \]

- Since \( e^{j\Omega o} \) cannot be a system pole (owing to asymptotic stability all poles have modulus strictly less than one), we can use Heaviside cover-up on
  \[
  Y_{ZS}(z) = H(z)F(z) = \frac{P(z)}{Q(z)}(z - e^{j\Omega o})Ae^{j\phi} \text{ to get}
  \]
  \[
  Y_{ZS}(z) = \frac{z H(e^{j\Omega o})}{z - e^{j\Omega o}}Ae^{j\phi} + \text{char. modes} = H(e^{j\Omega o})F(z) + \text{char. modes}.
  \]

PFE and eigenresponse for asymptotically stable systems (cont)

- Thus, the total response of an asymptotically stable system to a sinusoidal input \( f \) at frequency \( \Omega_o \) is
  \[ y[k] = H(e^{j\Omega o})f[k] + \text{ linear combination of characteristic modes}.
  \]
- So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as \( k \to \infty \),
  \[ y[k] \to H(e^{j\Omega o})f[k] = H(e^{j\Omega o})Ae^{j(\Omega ok + \phi)} = |H(e^{j\Omega o})|Ae^{j(\Omega ok + \phi + \angle H(e^{j\Omega o}))}, \]
  where again,

- \( H = P/Q \) is the system’s transfer function,
- \( |H(e^{j\Omega o})| \) is the system’s magnitude response at freq. \( \Omega_o \), and
- \( \angle H(e^{j\Omega o}) \) is the system’s phase response at freq. \( \Omega_o \).
PFE and eigenresponse for asymptotically stable systems (cont)

- Laplace’s approximation: the rate at which the total response converges to the eigenresponse response is according to the characteristic value of largest modulus,
  - which will be $< 1$ owing to the stability assumption,
  - i.e., giving the modes(s) that $\to 0$ slowest.

- In continuous-time systems, it’s the characteristic value of largest real part, which will be negative owing to stability assumption.

Canonical system-realizations - direct form

- Consider the proper ($m \leq n$) transfer function
  \[ H(z) = \frac{P(z)}{Q(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{Y(z)}{F(z)} \]

- The direct-form realization employs the interior system state $X := F/Q$, i.e., $F = QX$ and $Y = PX$ where the former implies (with $a_n = 1$),
  \[ F(z) = \sum_{r=0}^{n} a_r z^r X(z) \Rightarrow z^n X(z) = F(z) - \sum_{r=0}^{n-1} a_r z^r X(z). \]

- For $n = 2$, there are two “system states” (outputs of unit delays), $X$ and $zX$ (respectively, $x[k]$ and $(\Delta^{-1} x)[k] = x[k+1])$:

![Diagram of canonical system-realization - direct form]
• Now adding $Y = PX$, we finally get the direct-form canonical system-realization of $H$:

![Diagram of direct-form canonical realization]

• Again, state variables taken as outputs of unit delays, here: $x, \Delta^{-1}x, \ldots, \Delta^{-(n-1)}x$.

• If $b_n = b_2 \neq 0$, there is direct coupling of input and output, $H$ is proper but not strictly so, $h = Z^{-1}H$ has a unit-pulse component $b_2\delta$.

---

Note that this $n = 2$ example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for $n = 2$, $H(z) = \frac{P(z)}{Q(z)}$ where

$$Q(z) = z^2 + a_1z + a_0 = (z - \alpha)^2 + \beta^2$$

for $\alpha, \beta \in \mathbb{R}$, so the poles are $\alpha \pm j\beta$.

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Canonical system realizations by PFE

- In the general case of a proper transfer function, we can use partial-fraction expansion
  - grouping the terms corresponding to a complex-conjugate pair of poles, i.e., a second-order denominator, and
  - using a direct-form realization for these terms.
- Besides the PFE-based and direct-form realizations, there are other (zero-state) system realizations, e.g., “observer” canonical.
- For proper rational-polynomial transfer functions $H = P/Q$, all of these realizations involve $n$ (degree of $Q$) unit delays, the output of each being a different interior state variable of the system.

$H(z) = \frac{.3z^2 - .1}{z^2 - 0.1z - .3} = .3 + \frac{.3z - .01}{(z - 0.6)(z + 0.5)} = .3 + \frac{17/1.1}{z - 0.6} + \frac{16/1.1}{z + 0.5}$

Note that one cannot factor $z$ from the numerator of $H$.

Exercise: Find a realization for this transfer function $H$ by

1. splitting/forking the input signal $F$,
2. using the direct canonical form for each of these 3 terms of $H$ found by long division and PFE, and
3. summing three resulting output signals to get the (ZS) output $Y = HF$. 

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Digital Proportional-Integral (PI) system

- Consider a continuous-time signal \( x \) sampled every \( T \) seconds,
  \[
  \forall k \in \mathbb{Z}^+, \quad x[k] = x(kT),
  \]
  and its integral \( y(t) = \int_0^t x(\tau) d\tau \).
- The sampled integral can be approximated, \( y(kT) \approx y[k] \), by the trapezoid rule,
  \[
  y[k] = y[k-1] + \frac{x[k-1] + x[k]}{2} T.
  \]
- In the complex-frequency domain,
  \[
  Y(z) = Y(z)z^{-1} + \frac{X(z)z^{-1} + X(z)}{2}\frac{T}{1 - z^{-1}}.
  \]
  \[
  \Rightarrow Y(z) = T \frac{1 + z^{-1}}{2} X(z).
  \]

Digital PI system (cont)

- So, a digital PI transfer function would be of the form,
  \[
  G(z) = K_p + \frac{K_i T}{2} \frac{1 + z^{-1}}{1 - z^{-1}}.
  \]
  for constants \( K_p, K_i \).
- In practice, PID or PI systems \( G \) are commonly used to control a plant \( H \), where \( G \) may be in series with \( H \) or in the feedback branch.

- Exercises:
  - Draw the direct-form canonical realization for \( G \).
  - Draw the block diagram for the closed-loop system with negative feedback:
    \( Y = HX \) and \( X = F - GY \) where \( H \) is the (open-loop) system.
  - Find the closed-loop transfer function \( Y/F \).
• Consider a LTI system with input \( f \) and output \( y \),

\[
y[k] = \sum_{r=0}^{K} h[k-r]f[r] + v[k], \quad k \in \mathbb{Z},
\]

where \( v \) is an additive noise process and \( K \) is the maximum system order.

• The system (unit-pulse response) \( h \) is not known.

• Past values of the output \( y \) are observed (known).

• At time \( k \), the objective is to forecast the next output \( \hat{y}[k+1] \), based on the assumed known/observed quantities:
  - the next input \( f[k+1] \),
  - the past \( R \) input-output pairs \( \{f[r], y[r]\}_{k-R+1 \leq r \leq k} \).

**RLS objective and \( R^{th} \)-order linear tap filter**

• The output of an \( R^{th} \)-order RLS tap-filter at time \( k \) is,

\[
\hat{y}_k[i] = \sum_{r=i-R+1}^{i} \eta_k[i-r]f[r], \quad i \leq k + 1.
\]

• The objective of this filter at time \( k \) is to accurately estimate the system output \( y[k+1] \) with \( \hat{y}_k[k+1] \) by choosing the \( R \) filter coefficients

\[
\eta_k[k-R+1], ..., \eta_k[k-1], \eta_k[k]
\]

that minimize the following sum-of-square-error objective:

\[
\mathcal{E}_k = \sum_{r=k-R+1}^{k} \lambda^{k-r} |y[r] - \hat{y}_k[r]|^2 = \sum_{r=k-R+1}^{k} \lambda^{k-r} |e_k[r]|^2
\]

where

- \( \lambda > 0 \) is a forgetting factor and
- error \( e_k[r] := y[r] - \hat{y}_k[r] \).
RLS filter

• So, to minimize $E_k$, substitute $\hat{y}_k[r]$ into $E_k$ and solve

$$0 = \frac{\partial E_k}{\partial \eta_k[i]} \quad \text{for} \quad i \in \{k - R + 1, \ldots, k - 1, k\}.$$ 

• That is, $R$ equations in $R$ unknowns: for $i \in \{k - R + 1, \ldots, k - 1, k\}$,

$$0 = \sum_{r=k-R+1}^{k} 2\lambda^{k-r}e_k[r] \frac{\partial e_k[r]}{\partial \eta_k[i]}$$

$$= \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(y[r] - \hat{y}_k[r]) \left( -\frac{\partial \hat{y}_k[r]}{\partial \eta_k[i]} \right)$$

$$= \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(\hat{y}_k[r] - y[r])f[r - i]$$

• Exercise: Prove the last equality.

RLS filter (cont)

• Substituting $\hat{y}_k[r]$, rewrite these equations to get the following $R$ equations in $R$ unknowns $\eta_k[i]$ that are $E_k$-minimizing: for $i \in \{k - R + 1, \ldots, k - 1, k\}$,

$$\sum_{r=k-R+1}^{k} \lambda^{k-r}f[r - i] \sum_{\ell=r-R+1}^{r} f[\ell]\eta_k[r - \ell] = \sum_{r=k-R+1}^{k} \lambda^{k-r}y[r]f[r - i]$$

• Exercise: Prove the last equality and write it in matrix form.

• Exercise: Research how the $E_k$-minimizing filter parameters $\eta_k$ can be computed recursively, i.e., using $\eta_{k-1}$.

• The filter order $R$ can also be “trial adapted” to discover the system order $K$ so that the error-minimizing filter parameters $\eta_k$ “track” the system unit-pulse response $h$ over time $k$.

• Note the required initial “warm-up” period of $R$ time-units where the outputs of system $h$ are simply observed and recorded and no estimates are made.

• Exercise: If there was no additive noise process $\nu$ and the system unit-pulse response $h$ had finite support (i.e., a FIR system with $K < \infty$), show how $h$ can be deduced from input-output $(f, y)$ observations.