Signals and Linear and Time-Invariant Systems in Discrete Time

- Properties of signals and systems (difference equations)
- Time-domain analysis
  - ZIR, system characteristic values and modes
  - ZSR, unit-pulse response and convolution
  - Stability, eigenresponse and transfer function
- Frequency-domain analysis

Time-domain analysis of discrete-time LTI systems

- Discrete-time signals
- Difference equation single-input, single-output systems in discrete time
- The zero-input response (ZIR): characteristic values and modes
- The zero (initial) state response (ZSR): the unit-pulse response, convolution
- System stability
- The eigenresponse and (zero state) system transfer function
Discrete-time signal by sampling a continuous-time signal

- Consider a continuous-time signal $x : \mathbb{R} \rightarrow \mathbb{R}$ sampled every $T > 0$ seconds
  
  \[ x(kT + t_0) = x[k] \quad \text{for} \quad k \in \mathbb{Z}, \]

  where
  - $t_0$ is the sampling time of the $0^{\text{th}}$ sample, and
  - $T$ is assumed less than the Nyquist sampling period of $x$, and
  - $x[k]$ (with square brackets) is the $k^{\text{th}}$ sample itself.

- Here $x[\cdot]$ is a discrete-time signal defined on $\mathbb{Z}$.

Example of sampling with $t_0 = 0$ and positive signal $x$
Introduction to signals and systems in discrete time

• A discrete-time function (or signal) \( x : A \rightarrow B \) is one with countable (time) domain \( A \).
• We will take the range \( B = \mathbb{R} \) or \( B = \mathbb{C} \).
• Typically, we will herein take domain \( A = \mathbb{Z} \) or \( \mathbb{Z}^{\geq n} \) for some (finite) integer \( n \geq 0 \).
• Some properties of signals are as in continuous time: e.g., periodic, causal, bounded, even or odd.
• Similarly, some signal operations are as in continuous time: e.g., spatial shift/scale, super-position, time reflection, and (integer valued) time shift.

Time scaling: decimation and interpolation

• Time scaling can be implemented in continuous time prior to sampling at a fixed rate, or the sampling rate itself could be varied (again recall the Nyquist sampling rate).
• In discrete time, a signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) can be decimated (subsampled) by an integer factor \( L \neq 0 \) to create the signal \( x_L \) defined by

\[
x_L[k] = x[kL], \quad \forall k \in \mathbb{Z},
\]

i.e., \( x_L \) is defined only by every \( L^{th} \) sample of \( x \).
• A discrete-time signal \( x \) can also be interpolated by an integer factor \( L > 0 \) to create \( x_L \) satisfying

\[
x_L[kL] = x[k], \quad \forall k \in \mathbb{Z}.
\]
• For an interpolated signal \( x_L \), the values of \( x_L[r] \) for \( r \) not a multiple of \( L \) (i.e., \( \forall k \in \mathbb{Z} \) s.t. \( r \neq kL \)) can be set in different ways, e.g., between consecutive samples:
  - (piecewise constant) hold: \( x_L[r] = x_L[L \lfloor r/L \rfloor] = x[r/L] \)
  - linear interpolation:

\[
x_L[r] = x[r/L] + \frac{r - L \lfloor r/L \rfloor}{L}(x[r/L] + 1) - x[r/L])
\]
Questions

- Is the functional mapping \( x \rightarrow x_L \) causal for linear interpolation?
- Is the hold causal?
- **Exercise:** Show that if a periodic, continuous-time signal \( x(t) \), with period \( T_0 \), is periodically sampled every \( T \) seconds, then the resulting discrete-time signal \( x[k] \) is periodic if and only if \( T/T_0 \) is rational.

Unit pulse \( \delta \), unit step \( u \), unit delay \( \Delta \), and convolution *

- Some important signals in discrete time are as those in continuous time, e.g., polynomials, exponentials, unit step.
- In discrete time, rather than the (unit) impulse, there is unit pulse (Kronecker delta):
  \[
  \delta[k] = \begin{cases} 
  1 & \text{if } k = 0 \\
  0 & \text{else} 
  \end{cases}
  \]
- Any discrete-time signal \( x \) can thus be written as
  \[
  x[k] = \sum_{r=-\infty}^{\infty} x[r] \delta[k - r] = \sum_{r=-\infty}^{\infty} x[k - r] \delta[r]
  = (x * \delta)[k]
  \]
- or just \( x = x * \delta \), i.e., the unit pulse \( \delta \) is the identity of discrete-time convolution.
- Define the operator \( \Delta \) as unit delay (time-shift), i.e., \( \forall \) signals \( y \) and \( \forall k, r \in \mathbb{Z} \),
  \[
  (\Delta^r y)[k] := y[k - r].
  \]
- The discrete-time unit step \( u \) satisfies \( \delta = u - \Delta u \), equivalently: \( \forall k \in \mathbb{Z} \),
  \[
  \delta[k] = u[k] - u[k - 1] \quad \text{and} \quad u[k] = \sum_{r=0}^{\infty} (\Delta^r \delta)[k] = \sum_{r=0}^{\infty} \delta[k - r].
  \]
Unit pulse and unit step functions

- **Exercise**: For any signal causal \( f (\{f[k], k \geq 0\}) \), show that
  \[
  \forall k \geq 0, \ (f \ast u)[k] = \sum_{r=0}^{k} f[r].
  \]

Exponential signals in discrete time

- Real-valued exponential (geometric) signals have the form \( x[k] = A\gamma^k, k \in \mathbb{Z} \), where \( A, \gamma \in \mathbb{R} \).
- Consider the scalar \( z = \gamma e^{j\Omega} \in \mathbb{C} \) with \( \gamma > 0, \Omega \in \mathbb{R} \), where again \( j := \sqrt{-1} \).
- Generally, complex-valued exponential signals have the (polar) form
  \[
  x[k] = Ae^{j\phi}z^k = A\gamma^k e^{j(\Omega k + \phi)}, \ k \in \mathbb{Z},
  \]
  where w.l.o.g. we can take
  \[
  -\pi < \Omega, \phi \leq \pi \text{ and real } \ A > 0.
  \]
- **Exercise**: Show this complex-valued exponential is periodic if and only if \( \Omega/\pi \) is rational.
- By the Euler-De Moivre identity,
  \[
  x[k] = A\gamma^k e^{j(\Omega k + \phi)} = A\gamma^k \cos(\Omega k + \phi) + jA\gamma^k \sin(\Omega k + \phi), \ k \in \mathbb{Z}.
  \]
Systems - single input, single output (SISO)

- In the figure, $f$ is an input signal that is being transformed into an output signal, $y$, by the depicted system (box).

- To emphasize this functional transformation, and clarify system properties, we will write the output signal (i.e., system “response” to the input $f$) as

$$ y = Sf, $$

where, again, we are making a statement about functional equivalence:

$$ \forall k \in \mathbb{Z}, \ y[k] = (Sf)[k]. $$

- Again, $Sf$ is not $S$ “multiplied by” $f$, rather a functional transformation of $f$.

SISO systems (cont)

- The $n$ signals $\{x_1, x_2, \ldots, x_n\}$ are the internal states of the system.

- The states can be taken as outputs of unit-delay operators, $\Delta$, i.e.,

$$ \forall k \in \mathbb{Z}, \ (\Delta y)[k] = y[k-1]. $$

- Some properties of systems are as in continuous time: e.g., linear, time invariant, causal, memoryless, stable (with different conditions for stability as we shall see).
Difference equation for an discrete time, LTI, SISO system

- For linear and time-invariant systems in discrete time, relate output $y$ to input $f$ via difference equation in standard (time-advance operator) form:

$$\forall k \geq -n, \quad y[k + n] + a_{n-1}y[k + n - 1] + \ldots + a_1y[k + 1] + a_0y[k] = b_mf[k + m] + b_{m-1}f[k + m - 1] + \ldots + b_1f[k + 1] + b_0f[k],$$

given
- scalars $a_k$ for $0 \leq k \leq n$, with $a_n := 1$, and scalars $b_k$ for $0 \leq k \leq m$,
- $a_0 \neq 0$ or $b_0 \neq 0$ (so that $P, Q$ are of minimal degree), and
- initial conditions $y[-n], y[-n + 1], \ldots, y[-2], y[-1]$.

- Compact representation of the above difference equation:

$$Q(\Delta^{-1})y = P(\Delta^{-1})f,$$

where polynomials

$$Q(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad P(z) = \sum_{k=0}^{m} b_k z^k,$$

$\Delta^{-1}$ is the unit time-advance operator: $(\Delta^{-1}y)[k] \equiv y[k+1], (\Delta^{-r}y)[k] \equiv y[k+r]$

Discussion: conditions for causality and difference equation in $\Delta$

- Exercise: Show that the difference equation $Q(\Delta^{-1})y = P(\Delta^{-1})f$ is not causal if $\deg(P) = m > n = \deg(Q)$, i.e., the system is not proper.

- A not anti-causal difference equation can be implemented simply using memory to store a sliding window of prior values of the input $f$ and delaying the output.

- Example: Decoding B (bidirectional) frames of MPEG video.
Numerical solution to difference equation by recursive substitution

- Given the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \), the input \( f[k] \) for \( k \geq 0 \), and initial conditions \( y[-n], \ldots, y[-1] \),
- one can recursively solve for \( y \) (\( y[k] \) for \( k \geq 0 \)) by rewriting the system equation as
  \[
  y[k + n] = -\sum_{r=0}^{n-1} a_r y[k + r] + \sum_{r=0}^{m} b_r f[k + r] \quad \text{for } k \geq -n
  \]
  \[
  \Rightarrow y[k] = -\sum_{r=0}^{n-1} a_r y[k + r - n] + \sum_{r=0}^{m} b_r f[k + r - n] \quad \text{for } k \geq 0.
  \]
- For example, the difference equation in standard form,
  \[
  y[k + 1] + 3y[k] = 7f[k + 1] \quad \text{for } k \geq -1,
  \]
can be rewritten as
  \[
  y[k] = -3y[k - 1] + 7f[k] \quad \text{for } k \geq 0.
  \]
- So, given \( f \) and \( y[-1] \) we can recursively compute
  \[
  y[0] = -3y[-1] + 7f[0], \quad y[1] = -3y[0] + 7f[1], \quad y[2] = -3y[1] + 7f[2], \quad \text{etc.}
  \]
- Exercise: If \( f = u \) and \( y[-1] = 7 \) then find \( y[3] \) for this example.

Approach to closed-form solution: ZIR and ZSR

- The total response \( y \) of \( P(\Delta^{-1})f = Q(\Delta^{-1})y \) to the given initial conditions and input \( f \) is a sum of two parts:
  - the ZSR, \( y_{ZS} \), which solves
    \[
    P(\Delta^{-1})f = Q(\Delta^{-1})y_{ZS} \quad \text{with zero i.c.'s, i.e., with } 0 = y[-n] = \ldots = y[-1];
    \]
  - the ZIR, \( y_{ZI} \), which solves
    \[
    0 = Q(\Delta^{-1})y_{ZI} \quad \text{with the given initial conditions.}
    \]
- The total response \( y \) of the system to \( f \) and the given initial conditions is, by linearity,
  \[
  y = y_{ZI} + y_{ZS}.
  \]
- We will determine the ZIR by finding the characteristic modes of the system.
- We will determine the ZSR by convolution of the input with the (zero state) unit-pulse response, the latter also in terms of characteristic modes.
Consider again the difference equation:
\[ \forall k \geq -1, \quad y[k + 1] + 3y[k] = 7f[k + 1], \]

i.e., \( Q(z) = z + 3 \) with degree \( n = 1 \), and \( P(z) = 7z \) with degree \( m = 1 \).

Exercise: Show that the following system corresponds to this difference equation.

\[ \begin{align*}
f & \quad \rightarrow \quad 7 \\
\Delta & \quad \rightarrow \quad -3 \Delta y \\
\rightarrow \quad + \quad \rightarrow \quad y \\
-3 & \quad \rightarrow \quad -3 \\
\end{align*} \]

By recursive substitution, the total response is, \( \forall k \geq -1 \):
\[ \begin{align*}y[k] &= -3y[k - 1] + 7f[k] \\
&= -3(-3y[k - 2] + 7f[k - 1]) + 7f[k] \\
&= (-3)^2y[k - 2] - 3 \cdot 7f[k - 1] + 7f[k] \\
&= \ldots \\
&= (-3)^{k+1}y[-1] + \sum_{r=0}^{k} (-3)^{k-r} f[r] \\
&=: (-3)^{k+1}y[-1] + \sum_{r=0}^{\infty} h[k - r] f[r] =: (-3)^{k+1}y[-1] + (h * f)[k], \end{align*} \]

where \( h[k] := 7(-3)^{k}u[k] \) is the (zero state) unit-pulse response,
\( y[-1] \) is the given \( (n = 1) \) initial condition, and
we have defined the discrete-time convolution operator with \( \sum_{r=0}^{\infty}(...) := 0 \).
Total response - example (cont)

- **Exercise**: Prove by induction this expression for $y[k]$ for all $k \geq -1$.

- **Exercise**: Prove convolution is commutative: $h \ast f = f \ast h$.

- So, we can write the total response $y = y_{ZI} + y_{ZS}$ starting from the time of oldest initial condition:

  $$\forall k \geq -1, \quad y_{ZI}[k] = (-3)^{k+1}y[-1]$$

  $$\forall k \geq -1, \quad y_{ZS}[k] = u[k] \sum_{r=0}^{k} 7(-3)^{k-r}f[r] = u[k](h \ast f)[k]$$

  where $y_{ZS}[k] = 0$ when $k < 0$.

- Obviously, this example involves a linear, time-invariant and causal system as described by the difference equation above.

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Total response - discussion

- Note that in CMPSC 360, we don’t restrict our attention to linear and time-invariant difference equations.

- We use recursive substitution to guess at the form of the solution and then verify our guess by an inductive proof.

- In this course, we will describe a systematic approach to solve any LTIC difference equation,

  - *i.e.*, to solve for the output of a DT-LTIC system given the input and initial conditions.

- And again as in continuous time, we will see important insights about discrete-time signals and LTIC systems through frequency-domain representations and analysis.
ZIR - the characteristic values

• Note that \( \forall k, \Delta^{-r}z^k = z^{k+r} = z^rz^k \), i.e., the \( r \)-units time-advance operator, \( \Delta^{-r} \), is replaced by the scalar \( z^r \) for all \( r \in \mathbb{Z} \).

• Our objective is to solve for the ZIR, i.e., solve
  \[
  Q(\Delta^{-1})y \equiv 0 \text{ given } y[-n], y[-n+1], \ldots, y[-2], y[-1].
  \]

• Note that exponential (or "geometric") functions, \( \{z^k \mid k \in \mathbb{Z}\} \) for \( z \in \mathbb{C} \), are eigenfunctions of time-shift operators of the form \( Q(\Delta^{-1}) \) for a polynomial \( Q \).

• That is, for any non-zero scalar \( z \in \mathbb{C} \), if we substitute \( y[k] = z^k \forall k \in \mathbb{Z} \) we get:
  \[
  \forall k \in \mathbb{Z}, \ (Q(\Delta^{-1})y)[k] = Q(\Delta^{-1})z^k = Q(z)z^k.
  \]

• So, to solve \( Q(z)z^k \equiv 0 \) for all time \( k \geq 0 \), when \( z \neq 0 \) we require
  \[
  Q(z) = 0, \text{ the characteristic equation of the system.}
  \]

ZIR - the characteristic values (cont)

• If \( z \) is a root of the characteristic polynomial \( Q \) of the system, then
  - \( z \) would be a characteristic value of the system, and
  - the signal \( \{z^k\}_{k \geq 0} \) is a characteristic mode of the system when \( z \neq 0 \), i.e.,
    \[
    Q(\Delta^{-1})z^k = 0, \ \forall k \geq 0.
    \]

• Since \( Q \) has degree \( n \), there are \( n \) roots of \( Q \) in \( \mathbb{C} \), each a system characteristic value.
Let $n' \leq n$ be the number of non-zero roots of $Q$, i.e., $\hat{Q}(z) = Q(z)/z^{n-n'}$ is a polynomial satisfying $\hat{Q}(0) \neq 0$.

Though there may be some repeated roots of the characteristic polynomial $Q$, there will always be $n'$ different, linearly independent characteristic modes, $\mu_k$, i.e.,

$$\forall k \geq -n, \sum_{r=1}^{n'} c_r \mu_r[k] = 0 \iff \forall r, \text{ scalars } c_r = 0.$$

When $n = n'$, by system linearity, we will be able to write

$$\forall k \geq -n, y_{ZI}[k] = \sum_{r=1}^{n} c_r \mu_r[k],$$

for scalars $c_r \in \mathbb{C}$ that are found by considering the given initial conditions

$$y[k] = \sum_{r=1}^{n} c_r \mu_r[k] \text{ for } k \in \{-n, \ldots, -2, -1\},$$

i.e., $n$ equations in $n$ unknowns ($c_r$).

The linear independence of the modes implies linear independence of these $n$ equations in $c_r$, and so they have a unique solution.

ZIR - the case of different, non-zero, real characteristic values

If there are $n$ different non-zero roots of $Q$ in $\mathbb{R}$, $z_1, z_2, \ldots, z_n$, then there are $n$ characteristic modes: for $r \in \{1, 2, \ldots, n\}$,

$$\forall \text{ time } k, \mu_r[k] = z_r^k.$$

Therefore,

$$\forall k \geq -n, y_{ZI}[k] = \sum_{r=1}^{n} c_r z_r^k.$$

The $n$ unknown scalars $c_r \in \mathbb{R}$ can be solved using the $n$ equations:

$$y[k] = \sum_{r=1}^{n} c_r z_r^k, \text{ for } k \in \{-n, -n+1, \ldots, -2, -1\}.$$
**Example:** Consider the difference equation:
\[ \forall k \geq -3, \quad 2y[k + 3] - 10y[k + 2] + 12y[k + 1] = 3f[k + 2], \]
with \( y[-2] = 1 \) and \( y[-1] = 3 \).

- That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, P(z) = (3/2)z \) and \( m = 1 \).

- So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and the ZIR is
\[ \forall k \geq -n = -2, \quad y_{ZI}[k] = c_1 3^k + c_2 2^k \]

- Using the initial conditions to find the scalars \( c_1, c_2 \):
\[ 1 = y[-2] = c_1 3^{-2} + c_2 2^{-2} \quad \text{and} \quad 3 = y[-1] = c_1 3^{-1} + c_2 2^{-1}. \]

- **Exercise:** Now solve for \( c_1 \) and \( c_2 \).

- Note: When a coefficient \( c \) is worked out to be zero, it may not be exactly zero in practice, and the corresponding characteristic mode \( z^k \) will increasingly contribute to ZIR \( y_{ZI} \) over time if \( |z| > 1 \) (i.e., an "unstable" mode in discrete time).

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**ZIR - the case of not-real characteristic values**

- The characteristic polynomial \( Q \) may have non-real roots, but such roots come in complex-conjugate pairs because \( Q \)'s coefficients \( a_k \) are all real.

- For example, if the characteristic polynomial is
\[ Q(z) = (z - 1)(z^2 - 2z - 2) \]
then the characteristic values (\( Q \)'s roots) are
\[ -1, \quad 1 \pm j\sqrt{3} \quad \text{again recalling} \quad j = \sqrt{-1}. \]

- Because we have three different characteristic values \( \in \mathbb{C} \), we can specify three corresponding characteristic modes,
\[ (-1)^k, (1 + j\sqrt{3})^k, (1 - j\sqrt{3})^k, \quad \forall k \geq 0, \]
and construct the ZIR as
\[ \forall k \geq -n = -3, \quad y_{ZI}[k] = c_1(-1)^k + c_2(1 + j\sqrt{3})^k + c_3(1 - j\sqrt{3})^k \]
\[ = c_1(-1)^k + c_2 e^{k\pi j/3} + c_3 e^{-k\pi j/3} \]
where
- \( c_1 \in \mathbb{R} \) and \( c_2 = c_3 \in \mathbb{C} \) so that \( y_{ZI} \) is real-valued, and again,
- these scalars are determined by the \( n = 3 \) given (real) initial conditions: \( y[-3], y[-2], y[-1] \).
By the Euler-De Moivre identity for the previous example,
\[
\forall k \geq -3, \ y_{ZI}[k] = c_1(-1)^k + (c_2 + c_3)2^k \cos(k\pi/3) + j(c_2 - c_3)2^k \sin(k\pi/3)
\]
\[
= c_1(-1)^k + 2\Re\{c_2\}2^k \cos(k\pi/3) - 2\Im\{c_2\}2^k \sin(k\pi/3)
\]

Again, because all initial conditions are real and \(Q\) has real coefficients, \(y_{ZI}\) is real valued and so \(c_3 = \overline{c_2} \Rightarrow c_2 + c_3, j(c_2 - c_3) \in \mathbb{R}\).

In general, consider two complex conjugate characteristic values \(v \pm jq\) corresponding to two complex-valued characteristic modes \(|z|^k e^{\pm jk\angle z}\), where \(|z| = \sqrt{v^2 + q^2}\) and \(\angle z = \arctan(q/v)\).

One can use Euler’s identity to show that the corresponding real-valued characteristic modes are
\[
|z|^k \cos(k\angle z), \ |z|^k \sin(k\angle z)
\]

Consider the case where at least one characteristic value is of order > 1, i.e., there are repeated roots of the characteristic polynomial, \(Q\).

For example, \(Q(z) = (z + 0.75)^3(z - 0.5)\) has a triple (twice repeated) root at \(-0.75\) and a single root at \(0.5\).

Again, \((-0.75)^k\) is a characteristic mode because \(Q(\Delta^{-1})(-.75)^k \equiv 0\) follows from
\[
(\Delta^{-1} + .75)(-.75)^k = \Delta^{-1}(-.75)^k + .75(-.75)^k
\]
\[
= (-.75)^{k+1} + .75(-.75)^k
\]
\[
= 0.
\]

Similarly, \((0.5)^k\) is a characteristic mode since \((\Delta^{-1} - 0.5)(0.5)^k \equiv 0\).

Also, \((k-.75)^k\) is a characteristic mode because \(Q(\Delta^{-1})k(-.75)^k \equiv 0\) follows from
\[
(\Delta^{-1} + .75)^2k(-.75)^k
\]
\[
= (\Delta^{-2} + 1.5\Delta^{-1} + (.75)^2)k(-.75)^k
\]
\[
= \Delta^{-2}k(-.75)^k + 1.5\Delta^{-1}k(-.75)^k + (.75)^2k(-.75)^k
\]
\[
= (k + 2)(-.75)^{k+2} + 1.5(k + 1)(-.75)^{k+1} + (.75)^2k(-.75)^k
\]
\[
= (-.75)^{k+2}(k + 2) - 2(k + 1) + k
\]
\[
= 0.
\]
Similarly, \( \{k^2(-.75)^k\} \) is also a characteristic mode because
\((\Delta^{-1} + .75)^3k^2(-.75)^k = 0\).

Note that without three such linearly independent characteristic modes
\(\{(-.75)^k, k(-.75)^k, k^2(-.75)^k; \ k \geq 0\}\)
for the twice-repeated (triple) characteristic value -.75, the initial conditions will create an
"overspecified" set of \(n\) equations involving fewer than \(n\) "unknown" coefficients \(c_k\) of
the linear combination of modes forming the ZIR.

For this example,
\[y_{ZI}[k] = c_0(-0.75)^k + c_1k(-0.75)^k + c_2k^2(-0.75)^k + c_3(0.5)^k, \ k \geq -4.\]

If the given initial conditions are, say,
\[y[-4] = 12, \ y[-3] = 6, \ y[-2] = -5, \ y[-1] = 10,\]
the four equations to solve for the four unknown coefficients \(c_k\) are:
\[y_{ZI}[-4] = (-.75)^{-4}c_0 + (-4)(-.75)^{-4}c_1 + (-4)^2(-.75)^{-4}c_2 + (.5)^{-4}c_3 = 12\]
\[y_{ZI}[-3] = (-.75)^{-3}c_0 + (-3)(-.75)^{-3}c_1 + (-3)^2(-.75)^{-3}c_2 + (.5)^{-3}c_3 = 6\]
\[y_{ZI}[-2] = (-.75)^{-2}c_0 + (-2)(-.75)^{-2}c_1 + (-2)^2(-.75)^{-2}c_2 + (.5)^{-2}c_3 = -5\]
\[y_{ZI}[-1] = (-.75)^{-1}c_0 + (-1)(-.75)^{-1}c_1 + (-1)^2(-.75)^{-1}c_2 + (.5)^{-1}c_3 = 10\]

ZIR - general case of repeated, non-zero characteristic values

In general, a set of \(r\) linearly independent modes corresponding to a non-zero characteristic
value \(z \in \mathbb{C}\) repeated \(r - 1\) times are
\[k^{r-1}z^k, k^{r-2}z^k, ..., kz^k, z^k, \ \text{for} \ k \geq 0.\]

Also, if \(v \pm jq\) are characteristic values repeated \(r - 1\) times, with \(v, q \in \mathbb{R}\) and \(q \neq 0\),
we can use the \(2k\) real-valued modes
\[k^a|z|^k \cos(k\angle z), k^a|z|^k \sin(k\angle z), \ \text{for} \ a \in \{0, 1, 2, ..., r - 1\},\]
where \(|z| = \sqrt{v^2 + q^2}\) and \(\angle z = \arctan(q/v)\).
• Again let \( n' \leq n \) be the number of non-zero roots of \( Q \) (characteristic values),
• i.e., \( r := n - n' \geq 0 \) is the order \((1+\)repetition\) of the characteristic value 0, and
• \( r \geq 0 \) is the smallest index such that \((the \ coefficient \ of \ Q) a_r \neq 0\).
• So, there is a polynomial \( \bar{Q} \) such that \( Q(z) = z^r \bar{Q}(z) \) and \( \bar{Q}(0) \neq 0 \).
• Because the constant signal zero cannot be a characteristic mode, we add \( r = n - n' \) time-advanced unit-pulses:

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{i=1}^{r} C_i \delta[k + i] + y_N[k] \\
= C_r \delta[k + r] + C_{r-1} \delta[k + r - 1] + ... + C_1 \delta[k + 1] + y_N[k]
\]
where \( y_N \) is a “natural response” (linear combination of \( n' \) characteristic modes).
• The \( n \) initial conditions are then met by the \( r \) coefficients \( C_i \) of the advanced unit pulses together with the \( n' = n - r \) coefficients of the characteristic modes in \( y_N \).

---

**ZIR - when some characteristic values are zero - example**

• Consider a fourth-order system with characteristic polynomial \( Q(z) = z^2(z + 1)^2 \).
• Thus the poles are \( 0, -1 \) each repeated and the (non-zero) characteristic modes are \((-1)^k, k(-1)^k\).
• So, the ZIR is, for \( k \geq -4 \):

\[
y_{ZI}[k] = C_2 \delta[k + 2] + C_1 \delta[k + 1] + c_1(-1)^k + c_2 k(-1)^k
\]
• That is, the ZIR has four unknown coefficients \( C_2, C_1, c_1, c_2 \) to account for the four (given) initial conditions \( y[-4], y[-3], y[-2], y[-1] \).
Zero State Response - the unit-pulse response

- Recall the LTIC system

\[ \sum_{r=0}^{n} a_r \Delta^{-r} y = Q(\Delta^{-1}) y = P(\Delta^{-1}) f : = \sum_{r=0}^{m} b_r \Delta^{-r} f \]

with \( a_n \equiv 1, a_0 \neq 0 \) or \( b_0 \neq 0, m \leq n \).

- We can express any input signal

\[ f[k] = \sum_{r=0}^{\infty} f[r] \delta[k - r] \quad \forall k \geq 0, \quad \text{i.e., } \forall f, f = f \ast \delta. \]

- So the unit pulse \( \delta \) is the identity of the convolution operator in discrete time.

- Thus, by LTI, the ZSR \( y_{ZS} \) is the convolution of input \( f \) and ZSR \( h \) to unit pulse \( \delta \),

\[ y_{ZS}[k] = \sum_{r=0}^{\infty} f[r] h[k - r] = (f \ast h)[k], \quad \forall k \geq 0, \]

- \( h \) is called the unit-pulse response of the LTIC system, i.e.,

\[ Q(\Delta^{-1}) h = P(\Delta^{-1}) \delta \quad \text{s.t. } h[k] = 0 \quad \forall k < 0. \]

Computing an LTIC system's unit-pulse response, \( h \)

- For the LTIC system in standard form, if \( a_0 \neq 0 \) then

\[ h = \left( b_0 / a_0 \right) \delta + y_N u \]

where \( y_N \) is a natural response of the system (linear combination of characteristic modes).

- Note that \( h[k] = 0 \) for all \( k < 0 \) owing to the unit step \( u \).

- The \( n \) scalars of the natural response \( y_N \) component of \( h \) are solved using

\[ (Q(\Delta^{-1}) h)[k] = (P(\Delta^{-1}) \delta)[k] \quad \text{for } k \in \{-n, -n+1, ..., -2, -1\} \]
Unit-pulse response when zero is a characteristic value

- If \( r \geq 0 \) is the smallest index such that \( a_r \neq 0 \) (0 is a char. mode of order \( r \)), then may need to add \( r \) delayed unit-pulse terms to \( h \):

\[
h = \sum_{i=0}^{r-1} A_i \Delta^i \delta + (b_0/a_r) \Delta^r \delta + y_0 u,
\]

where

- by definition of the standard form of the difference equation, if \( r > 0 \), \( a_0 = 0 \) so \( b_0 \neq 0 \), and
- \( r \leq n \) since \( 0 \neq a_n := 1 \).

- So if \( r = 0 \) (i.e., \( a_0 \neq 0 \)), then \( A_0 = b_0/a_0 \) as above, where \( \sum_{i=0}^{r-1} (...) := 0 \).

- Exercise: Prove \( A_r = b_0/a_r \) for \( 0 \leq r \leq n \).

- Thus, zero is a characteristic value of degree \( r \geq 0 \), and

- there are \( r \) characteristic modes that will all be zero.

- The additional unit-pulse terms introduce \( r \) degrees of freedom in the form of the coefficients \( A_0, A_1, ..., A_{r-1} \) to accommodate the \( n = r + n' \) initial conditions of the unit-pulse response: \( h[-n] = h[-n+1] = ... = h[-2] = h[-1] = 0 \).

Computing the ZSR - example 1

- Recall that the difference equation \( y = 7f - 3\Delta y \) corresponds to the above system; in standard form:

\[
\forall k \geq -1, \quad y[k+1] + 3y[k] = 7f[k+1].
\]

with \( Q(z) = z + 3 \), \( P(z) = 7z \) and \( n = 1 = m \).

- Since the system characteristic value is \(-3\) and \( b_0 = 0 \), the (zero state) unit-pulse response has the form \( h[k] = c(-3)^k u[k] \).

- The scalar \( c \) is solved by evaluating the above difference equation at time \( k = -1 \):

\[
(Q(\Delta^{-1})h)[-1] = (P(\Delta^{-1})\delta)[-1]
\]

\[
i.e., \quad h[0] + 3h[-1] = 7\delta[0]
\]

\[
\Rightarrow c + 3 \cdot 0 = 7 \cdot 1, \quad c = 7
\]
Computing the ZSR - example 1 (cont)

• So, \( h[k] = 7(-3)^k u[k] \).

• If the input is \( f[k] = 4(0.5)^k u[k] \), the system ZSR is, for all \( k \geq 0 \),

\[
\begin{align*}
y_{ZS}[k] &= \sum_{r=0}^{k} h[r]f[k-r] = \sum_{r=0}^{k} 7(-3)^r 4(0.5)^{k-r} \\
&= 28(0.5)^k \sum_{r=0}^{k} (-6)^r = 28(0.5)^k \frac{(-6)^{k+1} - 1}{-6-1} u[k] \\
&= (24(-3)^k + 4(0.5)^k)u[k].
\end{align*}
\]

• Note how the ZIR \( y_{ZI} \) has a term that is a characteristic mode (excited by the input \( f \)) and a term that is proportional to the input \( f \) (this forced response is an eigenresponse).

• Exercise: For the difference equation, \( y[k+1] + 3y[k] = 7f[k] \ \forall k \geq -1 \): draw the block diagram, show that \( h[k] = 21(-3)^{k-1}u[k] + (7/3)\delta[k] \), and find the ZSR to the above input \( f \).

• Exercise: Read “sliding tape” method to compute convolution in Lathi, p. 595.

Computing the unit pulse response - example 2

• Find the ZSR of the following system to input \( f[k] = 2(-5)^k u[k] \):

\begin{align*}
\text{Exercise}: \text{show the difference equation for this system (in direct canonical form) is:} \\
\forall k \geq 0, \quad y[k+2] - 5y[k+1] + 6y[k] &= 1.5f[k+1] \\
\text{That is, } Q(z) = z^2 - 5z + 6 = (z-3)(z-2) \text{ and } n = 2, \ P(z) = 1.5z \text{ and } m = 1. \\
\text{So, the } n = 2 \text{ characteristic values are } z = 3, 2 \text{ and } b_0 = 0 \text{ so the unit-pulse response} \\
h[k] &= (c_13^k + c_22^k)u[k].
\end{align*}
Computing the unit pulse response - example 2 (cont)

• To find the constants, evaluate the difference equation at $k = -1$:

\[
2h[1] - 10h[0] + 12h[-1] = 3\delta[0] \\
\Rightarrow 2h[1] - 10h[0] = 3 \\
\Rightarrow (2 \cdot 3 - 10 \cdot 1)c_1 + (2 \cdot 2 - 10 \cdot 1)c_2 = 3 \\
\Rightarrow -4c_1 + -6c_2 = 3
\]

and at $k = -2$:

\[
2h[0] - 10h[-1] + 12h[-2] = 3\delta[-1] \\
\Rightarrow 12h[0] = 0 \\
\Rightarrow h[0] = 0
\]

• Thus, $c_2 = -1.5 = -c_1$ so that $h[k] = (-1.5(3)^k + 1.5(2)^k)u[k]$ and for $k \geq 0$

\[
y_{ZS}[k] = (h * f)[k] = \sum_{r=0}^{k} h[r]f[k-r].
\]

• Exercise: Write the ZSR as a sum of system modes $2^k$ and $3^k$ and a (force) term like the input, here taken as $f[k] = 4(-5)^k u[k]$.

---

Convolution - other important properties

• Again, for a LTI system with impulse response $h$ and input $f$, the ZSR is $y_{ZS} = f * h$, where

\[
(f * h)[k] = \sum_{r=-\infty}^{\infty} f[r]h[k-r]
\]

• By simply changing the dummy variable of summation to $r' = h - r$, can show convolution is commutative: $f * h = h * f$.

• One can directly show that convolution $f * h$ is a bi-linear mapping from pairs of signals $(f, h)$ to signals $(y_{ZS})$, consistent with convolution’s commutative property and the (zero state) system with impulse response $h$ being LTI;

• that is, $\forall$ signals $f, g, h$ and scalars $\alpha, \beta \in \mathbb{C}$,

\[
(\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h)
\]

• By changing order of summation (Fubini’s theorem), one can easily show that convolution is associative, i.e., $\forall$ signals $f, g, h$,

\[
(f * g) * h = f * (g * h).
\]
Convolution - other important properties (cont)

- We’ll use these properties when composing more complex systems from simpler ones.
- By just changing variables of integration, we can show how to exchange time-shift with convolution, i.e., \( \forall \) signals \( f, h : \mathbb{Z} \rightarrow \mathbb{C} \) and times \( k \in \mathbb{Z} \),
  \[
  \left( \Delta^k f \right) \ast h = \Delta^k (f \ast h);
  \]
  recall how convolution represents the ZSR of linear and time-invariant systems.
- By the ideal sampling property, recall that the identity signal for convolution is the unit pulse \( \delta \), i.e., \( \forall \) signals \( f \),
  \[
  f \ast \delta = \delta \ast f = f
  \]
- Exercise: Adapt the proofs of these properties in continuous time to this discrete-time case.
- Exercise: In particular, show that if \( f \) and \( h \) are causal signals, then \( y = f \ast h \) is causal; i.e., if the unit-pulse response \( h \) of a system is a causal signal, then the system is causal.

System stability - ZIR - asymptotically stable

- Consider a SISO system with input \( f \) and output \( y \).
- Recall that the ZIR \( y_{ZI} \) is a linear combination of the system’s characteristic modes, where the coefficients depend on the initial conditions, possibly including some initial unit-pulse terms if zero is a characteristic value (system pole).
- A system is said to be asymptotically stable if for all initial conditions,
  \[
  \lim_{k \rightarrow \infty} y_{ZI}[k] = 0.
  \]
- So, a system is asymptotically stable if and only if all of its characteristic values have magnitude less than 1.
System stability - ZIR - asymptotically stable: Example

- If the characteristic polynomial \( Q(z) = (z - 0.5)(z^2 + 0.0625) \), then
- the system’s characteristic values (roots of \( Q \)) are 0.5, \( \pm 0.25j \) each with magnitude less than one,
- and the ZIR is of the form,
  \[
  y_{ZI}[k] = (c_1(0.5)^k + c_2(0.25j)^k + c_2(-0.25j)^k) u[k] \\
  = (c_1(0.5)^k + 2\text{Re}\{c_2\}(0.25)^k \cos(k\pi/2) - 2\text{Im}\{c_2\}(0.25)^k \sin(k\pi/2)) u[k],
  \]
- recalling that \( j^k = e^{jk\pi/2} \).
- So, \( y_{ZI}[k] \to 0 \) as \( k \to \infty \) for all \( c_1, c_2 \) (i.e., for all initial conditions), and
- hence is asymptotically stable.

System stability - bounded signals

- A signal \( y \) is said to be bounded if
  \[
  \exists M < \infty \text{ s.t. } \forall k \in \mathbb{Z}, \ |y[k]| \leq M;
  \]
  otherwise \( y \) is said to be unbounded.
- For example, \( y[k] = 0.25(\frac{1+j\sqrt{3}}{2})^k u[k] \) is bounded (can use \( M = 0.25 \)).
- Also, \( 3\cos(5k) \) is bounded (can use \( M = 3 \)).
- But both \( 2^k \cos(5k) \) and \( 3 \cdot (-2)^k \) are unbounded.
System stability - ZIR - marginally stable

- A system is said to be marginally stable if it is not asymptotically stable but \( y_{ZI} \) is always (for all initial conditions) bounded.
- A system is marginally stable if and only if
  - it has no characteristic values with magnitude strictly greater than 1,
  - it has at least one characteristic value with magnitude exactly 1, and
  - all magnitude-1 characteristic values are not repeated.
- That is, a marginally stable system has
  - some characteristic modes of the form \( \cos(\Omega k) \) or \( \sin(\Omega k) \),
  - while the rest of the modes are all of the form \( k^r|z|^k \cos(\Omega k) \) or \( k^r|z|^k \sin(\Omega k) \),
    with \(|z| < 1\) and integer degree \( r \geq 0 \).
- Exercise: Explain why we can take \( \Omega \in (\pi, \pi] \) without loss of generality.
- Note: The dimension of frequency \( \Omega \) is \([\Omega]=[\text{radians per unit time}]\).

System stability - ZIR - marginally stable: Example

- The characteristic polynomial is \( Q(z) = z(z^2 + 1)(z - 0.25) \) gives characteristic values 0, 0.25, \( \pm j \).
- then the system is marginally stable with modes \((0.25)^k, \cos(k\pi/2), \sin(k\pi/2)\).
- the last two of which are bounded but do not tend to zero as time \( k \to \infty \).
System stability - ZIR - unstable

- A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.

- For example, the system with $Q(z) = (z^2 - 0.5)(z + 3)$ is unstable owing to the characteristic value $-3$ with unbounded mode $(-3)^k$.

- For another example, if the characteristic polynomial is $Q(z) = (z^2 + 1)^2(z - 0.5)$ then the purely imaginary characteristic values $\pm j$ are repeated, and hence the two additional modes $k\sin(k\pi/2), k\cos(k\pi/2)$ are unbounded, so this system is unstable.

- Similarly, if $Q(z) = (z^2 - 1)^2(z - 0.5)$ then the characteristic values $\pm 1$ are repeated and the modes $k$ and $k(-1)^k$ are unbounded, so this system is unstable too.

ZIR stability - stability of poles

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• A SISO system is said to be \textit{Bounded Input, Bounded Output} (BIBO) stable if \( \forall \) bounded input signals \( f \), the ZSR \( y_{ZS} \) is bounded.

• A sufficient condition for BIBO stability is absolute summability of the unit-pulse response,

\[
\sum_{k=0}^{\infty} |h[k]| < \infty.
\]

• To see why: If the input \( f \) is bounded (by \( M_f \) with \( 0 \leq M_f < \infty \)) then \( \forall k \geq 0 \):

\[
|y_{ZS}[k]| = |(f * h)[k]| = \left| \sum_{r=0}^{k} f[k-r]h[r] \right| \\
\leq \sum_{r=0}^{k} |f[k-r]h[r]| \quad \text{(by the triangle inequality)} \\
\leq \sum_{r=0}^{k} M_f|h[r]| \\
\leq M_f \sum_{r=0}^{\infty} |h[r]| =: M_y < \infty,
\]

System stability - ZSR - BIBO stable

• The condition of absolute summability of the unit-pulse response,

\[
\sum_{r=0}^{\infty} |h[r]| < \infty,
\]

is also necessary for, and hence equivalent to, BIBO stability.

• If any component characteristic mode of \( h \) is unbounded, then \( h \) will not to be absolutely summable.

• Thus, if the system (ZIR) is asymptotically stable it will be BIBO stable; the converse is also true.
ZSR - the transfer function, $H$

- Recall that for any polynomial $Q$ and $z \in \mathbb{C}$ (including $s = jw$, $w \in \mathbb{R}$),
  \[
  Q(\Delta^{-1})z^k = Q(z)z^k, \quad \forall k \geq 0.
  \]

- So, if we guess that a "particular" solution of the system $Q(\Delta^{-1})y = P(\Delta^{-1})f$ with input $f[k] = Az^ku[k]$ is of the form $y_0[k] = AH(z)z^k = H(z)f[k]$, $k \geq 0$, then we get by substitution that $\forall k \geq 0, z \in \mathbb{C}$,
  \[
  (Q(\Delta^{-1})y_0)[k] = (P(\Delta^{-1})f)[k] \Rightarrow AH(z)Q(z)z^k = AP(z)z^k \\
  \Rightarrow H(z) = P(z)/Q(z).
  \]

- The "rational polynomial" $H = P/Q$ is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

- So, the ZSR (forced response + characteristic modes) would be of the form:
  \[
  y_{ZS}[k] = (AH(z)z^k + \text{linear combination of char. modes})u[k].
  \]

- Recall that for the example with $Q(z) = z + 3$ and $P(z) = 7z$, we computed the unit-pulse response $h[k] = 7(-3)^ku[k]$ and the ZSR to input $f[k] = 4(0.5)^ku[k]$ as $y_{ZS}[k] = (24(-3)^k + 4(0.5)^k)u[k]$.

- Here, note that $H(0.5) = P(0.5)/Q(0.5) = 1$, i.e., the forced response component of $y_{ZS}$ is $H(0.5)f[k] = 1 \cdot 4(0.5)^ku[k] = 4(0.5)^ku[k]$.

---

ZSR - unit-pulse response $h$, transfer function $H$, and eigenresponse

- $y_{ZS}[k] = (H(z)Az^k + \text{linear combination of char. modes})u[k]$ is the ZSR to input $f[k] = Az^ku[k]$, where $H(z) = P(z)/Q(z)$.

- The eigenresponse is a special case of the forced response for exponential inputs.

- If $|z| = 1$, i.e., $z = e^{j\Omega}$ for some $\Omega \in (-\pi, \pi]$ (w.l.o.g.), and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:
  \[
  y[k] \rightarrow AH(e^{j\Omega})e^{j\Omega k} \quad \text{as } k \rightarrow \infty.
  \]

- Since $y = h * f$, we get that as $k \rightarrow \infty$ for a LTIC and asymptotically stable system,
  \[
  y_{ZS}[k] = \sum_{r=0}^{k} h[r]Ae^{j\Omega(k-r)} = Ae^{j\Omega k} \sum_{r=0}^{k} h[r]e^{-j\Omega r} \rightarrow Ae^{j\Omega k}H(e^{j\Omega}),
  \]
  \[
  \Rightarrow \sum_{r=0}^{\infty} h[r]e^{-j\Omega r} = H(e^{j\Omega}), \quad \forall \Omega \in (-\pi, \pi].
  \]
ZSR - transfer function $H$ and eigenresponse (cont)

- The LTIC system transfer function $H$ is the $z$-transform of the system unit-pulse response $h$:
  
  $$ H(z) = \sum_{k=0}^{\infty} h[k] z^{-k}, $$

  where $z \in \mathbb{C}$ is in $H$’s “region of convergence”.

- Note that $H(e^{j\Omega})$ is periodic since $H(e^{j\Omega}) = H(e^{j(\Omega+2\pi k)})$ for any integer $k$.

- For the $z$-transform (and DTFS) we will use this notation for $H$, but for the DTFT we will write $H(\Omega)$ instead of $H(e^{j\Omega})$.

---

Frequency-domain methods for discrete-time signals

- Discrete-Time Fourier Series (DTFS) of periodic signals
- Discrete-Time Fourier Transform (DTFT)
- sampled data systems
- DFT & FFT
- $z$-transform for (complete) transient response
- eigenresponse
- canonical system realization of a difference equation
Discrete-time Fourier series of periodic signals

- For all \( r, N \in \mathbb{Z} \), note that the signal \( \{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \} \) “repeats itself” every \( N > 0 \) units of (discrete) time \( k \), in particular
  \[
  \forall r \in \mathbb{Z}, \; e^{jr \frac{2\pi}{N} k}|_{k=0} = e^{jr \frac{2\pi}{N} k}|_{k=N} = 1
  \]
- Also the signals \( \{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \} \equiv \{ \exp(jr' \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \} \) whenever \( r' = r \) mod \( N \).
- Suppose \( N \) is the period of periodic signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) and \( \Omega_o = 2\pi/N \) be the fundamental frequency of \( x \) (recall \( [\Omega_o] = \) radians/unit-time).
- We can write \( x \) as a Discrete-Time Fourier Series (DTFS):
  \[
  \forall k \in \mathbb{Z}, \; x[k] = \sum_{r=0}^{N-1} D_r e^{j r \Omega_o k}.
  \]
  where \( r \) indexes \( x \)’s \( N \) harmonics.
- Note that the DTFS can also be written for any discrete-time signal \( x : A \to \mathbb{R} \) defined over any finite interval of time, e.g., \( A = \{0, 1, 2, ..., N-1\} \) or \( A = \{-N, -N+1, ..., -1\} \) for integer \( N < \infty \).

Discrete-time Fourier series of periodic signals (cont)

- Consider the \( N \) signals \( \xi_r[k] := e^{jr \Omega_o k} \) over any time-interval \( A \) of length \( N \).
- Equivalently consider these \( N \) signals \( \xi_r \) as \( N \)-vectors in \( \mathbb{R}^N \), i.e., the \( k^{th} \) entry of vector \( \xi_r \) is \( \xi_r[k] \).
- If these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are linearly independent, then they will form a basis spanning all other signals \( x : A \to \mathbb{R} \), equivalently all other vectors \( x \in \mathbb{R}^N \),
- i.e., any such \( x \) can be written as a linear combination of the \( \{ \xi_r \}_{r=0}^{N-1} \) giving the DTFS of \( x \):
  \[
  x_r = \sum_{r=0}^{N-1} D_r \xi_r.
  \]
- If we show that these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are orthogonal then
  - linear independence follows
  - the \( r^{th} \) coordinate \( D_r \) (DTFS coefficients) is found by simply projecting \( x \) onto the vector \( \xi_r \):
    \[
    D_r = \langle x, \xi_r \rangle / ||\xi_r||^2.
    \]
• Consider any period of $x : \mathbb{Z} \to \mathbb{R}$, say $\{0, 1, 2, \ldots, N-1\}$.

• First note that for any $v \in \mathbb{Z}$ that is not a multiple of $N$ (so $e^{jv\Omega_o} = e^{j(v/2N)} \neq 1$), the geometric series

$$\sum_{k=0}^{N-1} e^{jv\Omega_0 k} = \sum_{k=0}^{N-1} \left(e^{j2\pi/N}\right)^k = \frac{e^{jv(2\pi/N)} - e^{jv(2\pi/N)0}}{e^{j2\pi/N} - 1} = 0.$$ 

• Thus, for any $r \neq v \in \mathbb{Z}$ such that $N \not| (v - r)$, the inner product $\langle \xi_r, \xi_v \rangle = \langle \{e^{jv(2\pi/N)k}\}, \{e^{jr(2\pi/N)k}\} \rangle := \sum_{k=0}^{N-1} e^{jr(2\pi/N)k} e^{jv(2\pi/N)k} = \sum_{k=0}^{N-1} e^{j(r-v)(2\pi/N)k} = 0$, recalling that the inner product is conjugate-linear in the second argument so that $\langle x, x \rangle = \|x\|^2$ when $x$ is $\mathbb{C}$-valued.

• So, these signals are orthogonal and the DTFS coefficients of $N$-periodic $x$ are

$$D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-jr\Omega_0 k} = \frac{\langle x, \{e^{jr\Omega_0 k}\} \rangle}{\|\{e^{jr\Omega_0 k}\}\|^2}, \quad \Omega_o = \frac{2\pi}{N}.$$ 

---

DTFS - checking coefficients

• Let’s now compute the inner product of $\xi_v$, for any $v \in \{0, 1, \ldots, N-1\}$, with the DTFS of $N$-periodic $x$:

$$\langle x, \{e^{jv\Omega_0 k}\} \rangle = \sum_{k=0}^{N-1} x[k] e^{-jr\Omega_0 k} = \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} D_r e^{jr\Omega_0 k} e^{-jr\Omega_0 k} = \sum_{r=0}^{N-1} D_r e^{j(r-v)\Omega_0 k} \sum_{k=0}^{N-1} D_r N \delta(r - v) = D_v N$$

• Again, we have verified the DTFS coefficients is

$$D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-jr\Omega_0 k} = \frac{\langle x, \{e^{jr\Omega_0 k}\} \rangle}{\|\{e^{jr\Omega_0 k}\}\|^2}, \quad \Omega_o = \frac{2\pi}{N}.$$ 

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DTFS - summary for $N$-periodic signal $x$

$$\forall k \in \mathbb{Z}, \quad x[k] = \sum_{r=0}^{N-1} D_r e^{j r \Omega_0 k}$$

- **period $N$**
- **harmonic index $r$**
- **fundamental freq. $\Omega_0$**
- **harmonic freq. $r \Omega_0$**
- **time $k$**

Fourier coef. $D$

$$\forall r \in \{0,1,2,\ldots,N-1\}, \quad D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j r \Omega_0 k}$$

$$\langle x, \{e^{j r \Omega_0 k}\} \rangle = \frac{1}{||\{e^{j r \Omega_0 k}\}||^2}, \quad \Omega_0 = \frac{2\pi}{N}$$

---

**DTFS - example**

**Problem:**
Identify the DTFS coefficients (if they exist) for

$$x[k] = 7 \sin(5.7\pi k) + 2 \cos(3.2\pi k), \quad k \in \mathbb{Z}.$$  

**Solution:**

- First note that the two components of $x$ are periodic, so their sum is periodic. *(Why is this so in discrete time?)*

- Since $\sin$ and $\cos$ have period $2\pi$, we can subtract integer multiples of $2\pi$ to get

  $$x[k] = 7 \sin(1.7\pi k) + 2 \cos(1.2\pi k).$$

- $1.7\pi k$ is an integer multiple of $2\pi$ when (integer) $k = 20$, and when $k = 5$ for $1.2\pi k$, so *least common multiple* of these periods is $k = 20$.

- *(Show that one can alternatively find the greatest common divisor of the component frequencies.)*
DTFS - example (cont)

• Thus, the period of \( x \) is \( N = 20 \) and the fund. frequ. is \( \Omega_o = 2\pi/N = 0.1\pi \).

• By Euler’s identity and adding \( 2\pi k \) to the negative exponents,
\[
x[k] = \frac{7}{2}e^{j1.7\pi k} - \frac{7}{2}e^{-j1.7\pi k} + e^{j1.2\pi k} + e^{-j1.2\pi k} = -3.5je^{j1.7\pi k} + 3.5je^{0.3\pi k} + e^{j1.2\pi k} + e^{j0.8\pi k}.
\]

• So, the DTFS of \( x[k] = \sum_{r=0}^{19} D_r e^{jr\Omega_0 k} \) with
\[
D_{17} = -3.5j = 3.5e^{-j\pi/2}, \quad D_3 = 3.5j = 3.5e^{j\pi/2}, \quad D_{12} = 1, \quad \text{and} \quad D_8 = 1;
\]
else \( D_r = 0 \) (incl. the fundamental \( r \in \{1, 19\} \) & DC \( r = 0 \) components).

DTFS - example and exercise

\[\text{Example:} \quad \text{The DTFS of an even rectangle wave with period } N = 6 \text{ and duty cycle } 3: \]
\[
x[k] = \sum_{\ell=-\infty}^{\infty} \Delta^6(\Delta^{-1}u - \Delta^2u)[k] = \sum_{\ell=-\infty}^{\infty} (u[k + 1 - 6\ell] - u[k - 2 - 6\ell])
\]
is
\[
\sum_{r=0}^{5} D_r e^{jr\Omega_0 k},
\]

where the fund. freq. \( \Omega_o = 2\pi/6 \) and, \( \forall r \in \mathbb{Z} \),
\[
D_r = \frac{1}{6} \sum_{k=-3}^{2} x[k]e^{-j\Omega_0 k} = \frac{1}{6} \sum_{k=-1}^{1} e^{-j\ell(2\pi/6)k} = \frac{1}{6}(1 + 2\cos(r(2\pi/6)k)).
\]

\[\text{Exercise:} \quad \text{Plot } x[k] \text{ as a function of time } k \text{ and plot its (periodic) spectrum:} \]
\[\forall r \in \{0, 1, 2, \ldots, 5\}, \ell \in \mathbb{Z}, \]
\[
\hat{X}(r2\pi/6 + 2\pi\ell) = D_r.
\]
DTFS - Parseval’s theorem

• The average power of the $N$-periodic discrete-time signal $x$ is

$$P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \overline{x[k]};$$

equivalently, the sum could be taken over any interval of length $N \in \mathbb{Z}^>0$.

• Substituting the Fourier series of $x$ separately for $x[k]$ and $\overline{x[k]}$ (using a different summation-index variable for each substitution), leads to Parseval’s theorem

$$P_x = \sum_{r=0}^{N-1} |D_r|^2.$$  

• Parseval’s theorem can be used to determine the amount of periodic signal $x$‘s power resides in a given frequency band $[\Omega_1, \Omega_2] \subset [0, 2\pi]$ radians/unit-time:

1. determine the harmonics $r \Omega_o$ of $x$ that reside in this band, i.e., integers $r \in [\Omega_1/\Omega_o, \Omega_2/\Omega_o]$ where $x$’s fundamental frequency $\Omega_o = 2\pi/N$.
2. sum just over these harmonics to get the answer, $\sum_{\Omega_r / \Omega_o \leq r \leq \Omega_2 / \Omega_o} |D_r|^2.$

DTFS - Parseval’s theorem example

• Find the fraction of $x$‘s average power in the frequency band $[0.4\pi, 1.1\pi]$ where

$$\forall k \in \mathbb{Z}, \ x[k] = \sum_{v=-\infty}^{\infty} (3\delta[k - 4v] - 4\delta[k - 1 - 4v]).$$

• Solution: $x$ has period $N = 4$ and average power

$$P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{4} \sum_{k=0}^{3} |x[k]|^2 = \frac{1}{4} (3^2 + (-4)^2 + 0^2 + 0^2) = \frac{25}{4};$$

• $x$ has fundamental frequency $\Omega_o = 2\pi/N = \pi/2$ radians/unit-time and discrete-time Fourier coefficients

$$D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j(r\Omega_o)k} = \frac{1}{4} \left(3 - 4e^{-j\pi/2}\right), \ 0 \leq r \leq N - 1 = 3.$$  

• The harmonics $r$ of $x$ that reside in $[0.4\pi, 1.1\pi]$ satisfy $0.4\pi \leq r\Omega_o = r\pi/2 \leq 1.1\pi$, i.e., $r \in \{1, 2\}$.

• So, by Parseval’s theorem, the answer is $(|D_1|^2 + |D_2|^2)/P_x.$

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Periodic extensions

- Consider signal \( x : \mathbb{Z} \to \mathbb{R} \) having finite support \( \{-M, -M + 1, \ldots, 0, \ldots, M - 1, M\} \) for \( 0 < M < \infty \); i.e., \( \forall |k| > M, x[k] = 0 \).

- For \( N \geq M \), define \( 2N \)-periodic \( x^{(N)} \) such that
  \[
  x^{(N)}[k] = \begin{cases} 
  x[k] & \text{if } |k| \leq M \\
  0 & \text{if } M < |k| \leq N
  \end{cases}
  \]

- \( x^{(N)} \) is a periodic extension of the finite-support signal \( x \), where again \( x^{(N)} \)'s period is \( 2N \) and
  \[
  \lim_{N \to \infty} x^{(N)} = x.
  \]

\[ x[k] \quad x^{N}[k] \]
\[ ... \quad -2N \quad -N \quad -M \quad M \quad N \quad 2N \quad ... \]

DTFS of periodic extension leading to DTFT

- For \( r \in \{-N + 1, -N + 2, \ldots, N - 1, N\} \), the DTFS of \( x^{(N)} \) has coefficients
  \[
  D_r^{(N)} = \frac{1}{2N} \sum_{k=-N+1}^{N} x^{(N)}[k] e^{-j2\pi \frac{r}{2N} k}
  \]
  \[
  = \frac{1}{2N} \sum_{k=-M}^{M} x[k] e^{-j2\pi \frac{r}{2N} k}
  \]
  \[
  = \frac{1}{2N} \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi \frac{r}{2N} k}
  \]
  \[
  = \frac{1}{2N} X \left( \frac{r 2\pi}{2N} \right),
  \]

  where the Discrete-Time Fourier Transform (DTFT) of (aperiodic) \( x : \mathbb{Z} \to \mathbb{R} \) is \( X : \mathbb{R} \to \mathbb{C} \):
  \[
  X(\Omega) := \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} =: (Fx)(\Omega), \ \Omega \in \mathbb{R}
  \]

- Note that Fourier integrals (spectra of discrete-time signals) are periodic, repeating themselves every \( 2\pi \) radians: \( \forall \Omega \in \mathbb{R}, \ \ell \in \mathbb{Z}, \)
  \[
  X(\Omega) = X(\Omega + \ell 2\pi).
  \]
Inverse DTFT by Fourier Integral

• Thus, \( \forall k \in \mathbb{Z}, \)
  \[
x[k] = \lim_{N \to \infty} x^{(N)}[k] = \lim_{N \to \infty} \sum_{r = -N+1}^{N} \frac{X(r)}{2N} e^{j \frac{2\pi}{2N} k} = \lim_{N \to \infty} \sum_{r = -N+1}^{N} X\left(r, \frac{2\pi}{2N}\right) e^{j \frac{2\pi}{2N} k} = \lim_{N \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega k} d\Omega.
\]
where the last equality is by Riemann integration with \(2\pi/(2N) \to d\Omega.\)

• Thus, we have derived the inverse DTFT by Fourier integral of \(X\) giving (aperiodic) \(x,\)
  \[
  \forall k \in \mathbb{Z}, \quad x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega k} d\Omega =: (\mathcal{F}^{-1}X)[k].
  \]

DTFT Examples - exponential signal

• If \(x = \delta\) then obviously \(X \equiv 1.\)

• The geometric signal \(x[k] = \gamma^k u[k]\) for scalar \(\gamma\) s.t. \(|\gamma| < 1\) has DTFT
  \[
  X(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k = \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{1}{(1 - \gamma) \cos(\Omega) + j\gamma \sin(\Omega)}
  \]

• Note that
  \[
  |X(\Omega)| = \frac{1}{(1 - \gamma \cos(\Omega))^2 + \gamma^2 \sin^2(\Omega)} = \frac{1}{1 + \gamma^2 - 2\gamma \cos(\Omega)}
  \]
  \[
  \angle X(\Omega) = -\arctan\left(\frac{\gamma \sin(\Omega)}{1 - \gamma \cos(\Omega)}\right)
  \]
DTFT Examples - exponential signal (cont)

- The plots above are for $\gamma = 0.5$.
- Note how $X$ has period $2\pi$.
- **Exercise:** What are the maximum and minimum values of $|X|$, i.e., how would this plot depend on $\gamma > 0$? Plot $x$ and $\angle X$. How do these plots differ when $-1 < \gamma < 0$?
- **Exercise:** Find the DTFT of anticausal signal $x[k] = \gamma^k u[-k]$ for scalar $\gamma$ s.t. $|\gamma| > 1$.
- **Exercise:** Find the DTFT of $x[k] = \gamma^{|k|}$, $k \in \mathbb{Z}$, for scalar $\gamma$ s.t. $|\gamma| < 1$.

DTFT Examples - Square and Triangle Pulse

- For $T \in \mathbb{Z}^+$, the even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ (i.e., $x[k] = u[k + T] - u[k - (T + 1)]$), has DTFT
  \[
  X(\Omega) = \sum_{k=-T}^{T} 1 - e^{-j\Omega k} = 1 + 2 \sum_{k=1}^{T} \cos(k\Omega), \ \Omega \in \mathbb{R}.
  \]
- **Exercise** (even rectangle pulse in frequency domain):
  Show that for fixed $\Omega'$ s.t. $0 < \Omega' \leq \pi$,
  \[
  \mathcal{F}^{-1}\{\Delta_{-\Omega} u - \Delta_{\Omega} u\}[k] = \frac{\Omega'}{\pi} \text{sinc}(\Omega' k), \ k \in \mathbb{Z}.
  \]
- For $T \in \mathbb{Z}^+$, the odd triangle pulse with support $2T + 1$, $x[k] \equiv k(\Delta^{-T}u[k] - \Delta^{T+1}u[k])$ has DTFT
  \[
  X(\Omega) = \sum_{k=-T}^{T} ke^{-j\Omega k} = -2j \sum_{k=1}^{T} k \sin(k\Omega), \ \Omega \in \mathbb{R}.
  \]
DTFT Examples - exponential sinusoid

- For fixed time $K_0$, clearly
  \[ \mathcal{F}\{\delta[k - K_0]\}(\Omega) = e^{jK_0\Omega}, \]
  where here $\delta$ is the unit pulse.

- Note that $e^{jK_0\Omega}$ is a sinusoidal function of $\Omega$ with period $2\pi/K_0$ radians.

- **Exercise:** For fixed frequency $\Omega_0$ radians/unit-time, show that
  \[ \mathcal{F}\{e^{-j\Omega_0k}\}(\Omega) = 2\pi \sum_{v=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi v), \]
  where here $\delta$ is the Dirac impulse (in the frequency domain $\Omega \in \mathbb{R}$). Hint: work with $\mathcal{F}^{-1}$.

- So, the DTFT of a $N$-periodic signal with Fourier series
  \[
  \sum_{r=0}^{N-1} D_r e^{j\frac{2\pi}{N}r} \xrightarrow{\mathcal{F}} \sum_{v=0}^{\infty} \sum_{r=0}^{N-1} D_r \delta(\Omega - v\frac{2\pi}{N} + 2\pi v)
  \]

DTFT - Time shift and frequency shift properties

- If fixed $K_0 \in \mathbb{Z}$ and $X = \mathcal{F}\{x\}$ then
  \[
  \mathcal{F}\{\Delta^{K_0}x\}(\Omega) = \sum_{k=-\infty}^{\infty} (\Delta^{K_0}x)[k]e^{-j\Omega k} \\
  = \sum_{k=-\infty}^{\infty} x[k - K_0]e^{-j\Omega k} \\
  = \sum_{k=-\infty}^{\infty} x[k']e^{-j(k + K_0)\Omega} \\
  = e^{-jK_0\Omega}X(\Omega),
  \]
  i.e., shift in time by $K_0$ corresponds to product with sinusoid of period $2\pi/K_0$ (linear phase shift) in frequency domain.

- **Exercise:** Prove the dual property that if fixed $\Omega_0 \in \mathbb{R}$ and $X = \mathcal{F}\{x\}$ then
  \[ \mathcal{F}\{x[k]e^{j\Omega_0k}\}(\Omega) = X(\Omega - \Omega_0), \]
  i.e., modulation (multiplication by a sinusoid) in time domain results in frequency shift.
DTFT - convolution properties

- Let $X_r = \mathcal{F}\{x_r\}$ for $r \in \{1, 2\}$.

$$\mathcal{F}\{x_1 \ast x_2\}(\Omega) \ := \ \sum_{k=-\infty}^{\infty} (x_1 \ast x_2)[k]e^{-j\Omega k}$$

$$:= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k-l]e^{-j(k-l)\Omega}e^{-jl\Omega} \ i.e., \ x_k e^{j\Omega} e^{-j\Omega} = 1$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k']e^{-j\Omega}e^{-jl\Omega} \ \text{where} \ k' = k - l$$

$$= \sum_{l=-\infty}^{\infty} x_1[l]e^{-jl\Omega} \sum_{k=-\infty}^{\infty} x_2[k']e^{-j\Omega} = : X_1(\Omega)X_2(\Omega)$$

- Exercise: Prove the dual property that
  
  $$\mathcal{F}\{x_1x_2\}(\Omega) = \frac{1}{2\pi} (X_1 \ast X_2)(\Omega) := \frac{1}{2\pi} \int_{2\pi} X_1(v)X_2(\Omega - v)dv.$$  

- Exercise: Use the convolution properties to prove the time and frequency shift properties. Hint: $(\Delta^K,\delta) \ast x = \Delta^{K,x}$.

- Exercise: Show that DTFT is a linear operator.

DTFT - Parseval’s Theorem

- The energy of a signal DT $x$ is

$$E_x := \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} x[k]\overline{x[k]} = \sum_{k=-\infty}^{\infty} (\mathcal{F}^{-1}X)[k](\mathcal{F}^{-1}X)[k]$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} X(\Omega')e^{j\Omega'k}d\Omega' \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{j\Omega k}d\Omega$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} X(\Omega')e^{j\Omega'k}d\Omega' \frac{1}{2\pi} \int_{2\pi} X(\Omega)e^{-j\Omega k}d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\Omega') \int_{2\pi} \overline{X(\Omega')} \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{j\Omega'k} e^{-j\Omega k} \right) d\Omega d\Omega'$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\Omega') \int_{2\pi} \overline{X(\Omega')} \frac{1}{2\pi} \left( 2\pi \delta(\Omega - \Omega') \right) d\Omega d\Omega'$$

$$= \frac{1}{2\pi} \int_{2\pi} X(\Omega') \overline{X(\Omega')} d\Omega' = \frac{1}{2\pi} \int_{2\pi} |X(\Omega')|^2 d\Omega', \ \text{recalling that for fixed} \ \Omega': \ \mathcal{F}^{-1}\{2\pi \delta(\Omega - \Omega')\}[k] = e^{-j\Omega'k} \ \& \ \int_{2\pi} \overline{X(\Omega)} \delta(\Omega - \Omega')d\Omega = X(\Omega').$$

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The even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ has energy

$$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-T}^{T} 1^2 = 2T + 1.$$

Recall its DTFT is $X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k}$, so

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\overline{X(\Omega)}d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-T}^{T} e^{-j\Omega k} \sum_{k'=-T}^{T} e^{j\Omega k'} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k=-T}^{T} 1 + \sum_{k<k'} e^{j(k-k')\Omega} \right) d\Omega$$

$$= \sum_{k=-T}^{T} \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\Omega + \sum_{k<k'} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(k-k')\Omega} d\Omega = \sum_{k=-T}^{T} 1 + 0 = 2T + 1.$$
• Consider a SISO, DT-LTIC system described by the difference equation
\[ Q(\Delta^{-1})y = P(\Delta^{-1})f, \]
where \( f \) is the input and \( y \) is the ZSR (output).

• Recall that by the time-shift property,
\[ Q(e^{j\Omega})Y_{ZS}(\Omega) = P(e^{j\Omega})F(\Omega) \Rightarrow Y_{ZS}(\Omega) = H(\Omega)F(\Omega). \]

• We now re-derive from first principles the eigenresponse by first recalling that the ZSR \( Y_{ZS} = f * h \) where \( h \) is the unit-pulse response.

• Taking DTFTs, \( Y_{ZS} = HF \) where \( H = \mathcal{F}h \) is the transfer function.

• Suppose the system is BIBO/asymptotically stable, \( \text{i.e.} \) the \( n \) roots of \( Q \) (system char. modes/poles) \( z \) all have modulus \( |z| < 1 \).

• The ZSR will consist of a forced response plus characteristic modes, where the latter will \( \to 0 \) over time (our stability assumption) so that the forced response becomes the steady-state response.

Analysis of Stable DT LTI Systems in Steady-State (cont)

• The forced response to a persistent sinusoidal input
\[ f[k] = A_f e^{j(\Omega_k + \phi_i)} \]
will be of the form
\[ y_{ss}[k] = A_y e^{j(\Omega_k + \phi_i)} \]
where (for \( k \geq 0 \),
\[ Q(e^{j\Omega_k})y_{ss}[k] = (Q(\Delta^{-1})y_{ss})[k] = (P(\Delta^{-1})f)[k] = P(e^{j\Omega_k})f[k]. \]
\[ \Rightarrow y_{ss}[k] = \frac{P(e^{j\Omega_k})f[k]}{Q(e^{j\Omega_k})} \]

• Also, the ZSR \( Y_{ZS} = h * f \), \( \text{i.e.} \) for all time \( k \geq 0 \):
\[ y_{ZS}[k] = \sum_{u=0}^{k} h[v]A_f e^{j(\Omega_{k-u} + \phi_i)} = f[k] \sum_{v=0}^{k} h[v]e^{-j\Omega_v} \]
\[ \to f[k]H(\Omega_o) =: y_{ss}[k] \text{ as } k \to \infty. \]
Transfer Function and Eigenresponse in Discrete Time (cont)

• Equating the forced responses (steady-state response for a stable system), we again get that the system transfer function is

\[ H(\Omega) = \frac{P(e^{i\Omega})}{Q(e^{i\Omega})} = (Fh)(\Omega). \]

• Note that \( \forall k \in \mathbb{Z}, H(\Omega) = H(\Omega + 2\pi k). \)

• Also, we write \( H(\Omega) \) not \( H(e^{j\Omega}) \) for the DTFT.

• So, the eigenresponse of a BIBO/asymptotically stable SISO, DT-LTIC system is the steady-state response to a sinusoid:

\[
    f[k] = A_f e^{j(\Omega k + \phi_f)} \rightarrow H(\Omega_0) f[k] = A_y e^{j(\Omega k + \phi_y)} =: y_{ss}[k]
\]

• The system magnitude response (gain) is \( |H(\Omega)| = |P(e^{j\Omega})|/|Q(e^{j\Omega})| \), i.e., \( A_y = A_f |H(\Omega_0)| \).

• The system phase response is \( \angle H(\Omega) = \angle P(e^{j\Omega}) - \angle Q(e^{j\Omega}) \), i.e., \( \phi_y = \phi_f + \angle H(\Omega_0) \).

Eigenresponse - example

• Problem: For the system \( 2y[k] = 0.6y[k - 1] - 7f[k] \) find the steady-state response (if it exists) to \( f[k] = 4 \cos(5k)u[k] \).

• Solution: The difference equation in standard form is \( (Q(\Delta^{-1})y[k] = y[k + 1] - 0.3y[k] = -3.5f[k + 1] = (P(\Delta^{-1})f)[k] \), where \( Q(z) = z - 0.3 \) and \( P(z) = -3.5z \).

• The sole system characteristic value (root of \( Q \), system pole) is 0.3, hence the system is BIBO/asymptotically stable.

• By Euler’s identity \( f[k] = (2e^{j5k} + 2e^{j(-5)k})u[k] \)

• By linearity, the eigenresponse is therefore

\[
    2H(5)e^{j5k} + 2H(-5)e^{j(-5)k},
\]

where \( H(\Omega) = P(e^{j\Omega})/Q(e^{j\Omega}) = -3.5e^{j\Omega}/(e^{j\Omega} - 0.3) = H(-\Omega) \), so that

\[
    |H(\Omega)| = \frac{3.5}{\sqrt{(\cos(\Omega) - 0.3)^2 + \sin^2(\Omega)}}, \quad \angle H(\Omega) = \pi + \Omega - \arctan\left(\frac{\sin(\Omega)}{\cos(\Omega) - 0.3}\right)
\]

• Exercise: Show that the eigenresponse is also simply \( |H(5)|4 \cos(5k + \angle H(5)) \).
2D Image Processing Example

- Apply 1-dimensional filtering to a 2-dimensional (2D) image by separately performing row and column operations.

- For 256 × 256 pixel (2D) image,

\[
    f = \begin{bmatrix}
        f[1, 1] & f[1, 2] & \ldots & f[1, 256] \\
        \vdots & \vdots & \ddots & \vdots \\
        f[256, 1] & f[256, 2] & \ldots & f[256, 256]
    \end{bmatrix}
\]

- If \( f[k, i] \) represents the 8-bit (grey) intensity of the pixel in row \( k \) and column \( i \) (i.e., 8 bits per pixel or bpp), then the "raw" image size will be \( 256^2 \text{bits} = 16 \text{Mb} = 2 \text{MB} \).

- Each of \( f \)’s rows of pixels can be processed by a system with unit-pulse response \( h \) to obtain a new row of pixels, and thus a new image \( y \):

\[
    \forall k, \ f[k, \cdot] \rightarrow h \rightarrow y[k, \cdot]
\]

- Alternatively, each of \( f \)’s columns of pixels can be processed by a system with unit-pulse response \( h \) to obtain a new column of pixels, and thus a new image \( y \):

\[
    \forall i, \ f[\cdot, i] \rightarrow h \rightarrow y[\cdot, i]
\]

Image Processing: High-Pass and Low-Pass Filtering

- The system \( h \) may have a specific signal processing objective.

- The output pixels \( y[k, i] \) may be quantized to fewer bpp than those of the input, thus achieving image compression.

- The simple low-pass filter (L)

\[
    h[k] = \frac{1}{2}(\delta[k] + \delta[k - 1]) \quad (\Rightarrow y[k] = \frac{1}{2}(f[k] + f[k - 1]))
\]

can capture shading and texture in the image.

- The simple high-pass filter (H)

\[
    h[k] = \frac{1}{2}(\delta[k] - \delta[k - 1]) \quad (\Rightarrow y[k] = \frac{1}{2}(f[k] - f[k - 1]))
\]

can capture edges in the image.

- Typically more compression possible in higher-frequency bands (H).
Image Processing: Tandem Row and Column Filtering

\[ f \rightarrow \text{row filtering} \rightarrow \text{column filtering} \rightarrow y \]

- Define \( y_{LH} \) as the output of
  \[ f \rightarrow [L] \rightarrow [H] \rightarrow y_{LH} \]
- Similarly define \( y_{LL} \), \( y_{HH} \) and \( y_{HL} \).
- The \( y \) images are downsampled by a factor of four (two in each direction).
- The \( y_{LL} \) image will have a lot of energy while \( y_{HH} \) will have the least energy.
- This motivates non-uniform quantization (bit allotment per pixel) of these images.
- Together with a coding strategy for the quantized images (particularly for the regions of zero pixel-values), this is the basic approach used in JPEG leading to very good compression, e.g., from 8 bpp to 0.2-0.5 bpp.

Sampling Continuous-Time Signals (A/D)

- Consider continuous-time signal \( x \) with \( X = \mathcal{F}x \).
- Recall that by sampling at period \( T \) with impulses in continuous time \( t \in \mathbb{R} \), we get
  \[ x_T(t) := \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} x(kT)e^{-j\omega k} =: X_T(w),\]
  equivalently, \( X_T(w) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X\left(w - v\frac{2\pi}{T}\right) \).
- Now define the sampled process in discrete-time \( k \in \mathbb{Z} \) and its DTFT,
  \[ \hat{x}[k] := x(kT) \xrightarrow{\mathcal{F}} \hat{X}(\omega) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{-j\omega k}. \]
- Substituting \( w = \Omega/T \) we get
  \[ \hat{X}(\Omega) = X_T\left(\frac{\Omega}{T}\right) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X\left(\frac{\Omega - v2\pi}{T}\right). \]
- Exercise: Read decimation (downsampling) and interpolation (upsampling) of Lathi Figs. 8.17 & 10.9.
We are particularly interested in the case where
- the continuous-time signal \( x \) is band-limited, i.e., \( \exists w' > 0 \) s.t. \( X(w) = 0 \) for \( |w| > w' \), and
- the sampling frequency is greater than Nyquist’s, i.e., \( 2\pi/T > 2w' \Rightarrow w'T < \pi \).

**Example:** For fixed \( w' > 0 \), consider the cts-time signal \( x(t) = A\text{sinc}(w't) \) with FT
\[
X(w) = \frac{A\pi}{w'}(u(w + w') - u(w - w')).
\]

Sampling \( x \) at period \( T < \pi/w' \) we get the discrete-time signal \( x[k] = A\text{sinc}(w'kT) \).

Using inverse DTFT, recall that we can easily check that the DTFT of \( x \) is,
\[
\hat{X}(\Omega) = \sum_{v=-\infty}^{\infty} \frac{A\pi}{w'T}(u(\Omega + w'T - 2\pi v) - u(\Omega - w'T - 2\pi v))
\]
\[
= \sum_{v=-\infty}^{\infty} \frac{1}{T}X \left( \frac{\Omega - 2\pi v}{T} \right),
\]
noting \( \forall T > 0, u(\frac{\Omega}{T} \pm w') = u(\frac{1}{T}(\Omega \pm w'T)) = u(\Omega \pm w'T), \Omega := \Omega - 2\pi v \).

---

Sampling Continuous-Time Signals - example (cont), \( w'T < \pi \)
Suppose the signal \( f \) is sampled every \( T_s \) seconds, i.e., at sampling frequency \( w_s := 2\pi / T_s \).

Recall Poisson’s identity (the Fourier series of the picket-fence function)
\[
p_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}
\]

Let’s rederive the relationship between the spectrum of a sampled continuous-time signal and its discrete-time counterpart by first defining the discrete-time signal
\[\forall k \in \mathbb{Z}, \quad \tilde{f}[k] = f(kT_s).\]

We want to relate the (continuous-time) Fourier transform of \( f \) to the (discrete-time) Fourier transform of \( \tilde{f} \),
\[
\hat{F}(\Omega) := \sum_{k=-\infty}^{\infty} \tilde{f}[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f(kT_s)e^{-j\Omega k}.
\]

To this end, recall
\[
f(t)p_T(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} f(t)e^{jkw_s t} \xrightarrow{\mathcal{F}} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s), \quad \text{and also}
\]
\[
f(t)p_T(t) = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} f(kT_s)e^{-j\omega kT_s} = \hat{F}(wT_s).
\]

Equating these two expressions for \( \mathcal{F}\{fp_T\} \) we get,
\[
\hat{F}(wT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s).
\]

Substituting \( w = \Omega / T_s \) we get,
\[
\hat{F}(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F\left(\frac{\Omega - k2\pi}{T_s}\right).
\]
Sampled Data Systems: D/A (digital to analog conversion)

- Now consider a discrete time signal $\hat{y}[k]$.

- We implement at D/A with a $T_s$-second hold, i.e., construct the continuous-time signal
  $$y(t) := \sum_{k=-\infty}^{\infty} \hat{y}[k] r_{T_s}(t - kT_s), \text{ where}$$
  $$r_{T_s}(t) := u(t) - u(t - T_s) \overset{T_s}{\rightarrow} T_s \text{sinc}(w T_s/2) e^{-jw T_s/2} =: R_{T_s}(w).$$

- Note that $y$ is in the form of a convolution, so:
  $$Y(w) = \sum_{k=-\infty}^{\infty} \hat{y}[k] R_{T_s}(w) e^{-jwkT_s} = R_{T_s}(w) \hat{Y}(w T_s).$$

Sampled Data Systems: equivalent cts-time transfer function

- Consider a digital system $\hat{H}(\Omega)$ (or $\hat{H}(e^{j\Omega})$ depending on notation), whose (ZS) output is $\hat{y}$ when the input is $\hat{f}$, i.e., $\hat{Y} = \hat{H}\hat{F}$.

- The equivalent continuous-time transformation of the tandem system
  $$f \rightarrow \underbrace{A/D (T_s\text{-sample})}_{\hat{A}/D} \overset{\hat{H}(\Omega)}{\rightarrow} \underbrace{D/A (T_s\text{-hold})}_{\hat{D}/A} \rightarrow y$$
  with input $f$ has (ZS) output
  $$Y(w) = R_{T_s}(w) \hat{Y}(w T_s) = R_{T_s}(w) \hat{H}(w T_s) \hat{F}(w T_s)$$
  $$= R_{T_s}(w) \hat{H}(w T_s) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - k w_s).$$

- Exercise: Show that if $f$ is band-limited by $w_s/2$ (i.e., $w_s$ is greater than $f$’s Nyquist frequency) and the previous sampled data system is followed by an ideal low-pass filter with bandwidth $w_s/2$, then the equivalent (continuous-time) transfer function is
  $$H(w) = \hat{H}(w T_s) T_s^{-1} R_{T_s}(w) \left( u(w + w_s/2) - u(w - w_s/2) \right)$$
Note that the term in the transfer function $H_T$,

$$T_s^{-1} R_{T_s}(w)(u(w + w_s/2) - u(w - w_s/2)) = \text{sinc}(\Omega/2)(u(\Omega + \pi) - u(\Omega - \pi))$$

is not a constant function of $\Omega = wT_s$.

This distortion due to the hold function $R$ can be reduced by putting in tandem with $\hat{H}$ an equalizer system with transfer function approximately

$$\hat{R}^{-1}(\Omega) := \sum_{k=-\infty}^{\infty} \frac{u(\Omega + \pi - k2\pi) - u(\Omega - \pi - k2\pi)}{\text{sinc}((\Omega - k2\pi)/2)}$$

i.e.,

$$\hat{H}(\Omega) \rightarrow \hat{R}^{-1}(\Omega)$$

Sampled Data Systems: equalization of hold $\text{sinc}(\Omega/2)$ by $\hat{R}^{-1}(\Omega)$

- the hold (at left, $R$) distorts the signal by attenuating its higher frequency components
- the equalizer (at right, $R^{-1}$) amplifies at the higher frequencies to cancel out this distortion
Discrete Fourier Transform (DFT)

- Consider the \( N \)-point sequence \( x = \{x[0], ..., x[N-1]\} \).
- Let \( X(\Omega) = \sum_{k=0}^{N-1} x[k] e^{-j k \Omega} \).
- Taking \( \Omega = \frac{2\pi}{N} r \) \((=: \Omega_o r)\) for \( r = 0, 1, ..., N - 1 \), the DFT of \( x \) is the \( N \)-point sequence

\[
X[r] := X\left(\frac{2\pi}{N} r\right) = \sum_{k=0}^{N-1} x[k] e^{-j \frac{2\pi}{N} k r},
\]

for \( r \in \{0, 1, ..., N - 1\} \).
- The inverse DFT is, for \( k \in \{0, 1, ..., N - 1\} \),

\[
x[k] = \frac{1}{N} \sum_{r=0}^{N-1} X[r] e^{j \frac{2\pi}{N} k r}.
\]

DFT (cont)

- Define an \( N \)th complex root of unity ("twiddle" factor)

\[
W_N = e^{-j \frac{2\pi}{N}}
\]

so that

\[
X[r] = \sum_{k=0}^{N-1} x[k] W_N^{r k}.
\]

- **Exercise:** Show that \( \forall \ell \in \mathbb{Z} \), \( W_N^{\ell} = W_N^{(\ell \mod N)} \). So, computing all the \( W_N^{rk} \) terms is only on the order of \( N \) multiplies.
- Given the \( N \) unique twiddle factors \( W_N^k \), to directly compute each \( X[r] \) requires \( N \) (complex) multiplications and \( N - 1 \) additions.
- So, to directly compute \( X = \{X[0], ..., X[N-1]\} \) requires \( N^2 \) multiplications and \( N(N - 1) \) additions.
Fast Fourier Transform (FFT) by time decimation

- If \( N \) is even, then we can separately sum even and odd times \( k \) to compute the DFT:

\[
X[r] = \sum_{k=0}^{\frac{N}{2}-1} x[2k]W_N^{2rk} + \sum_{k=0}^{\frac{N}{2}-1} x[2k+1]W_N^{2rk}
\]

where the last equality is simply because \( W_N^2 = W_{N/2} \).

- So, the first term is an \((N/2)\)-point DFT of (the \((N/2)\)-point sequence) \( x[0], x[2], ..., x[N-2] \).

- while the second term is the \((N/2)\)-point DFT of \( x[1], x[3], ..., x[N-1] \).

So, after one time-decimation step, the computational complexity has become

- \( 2\left(\frac{N}{2}\right)^2 + N = N(N/2) + N \) multiplications and

- \( 2\frac{N}{2}\left(\frac{N}{2} - 1\right) + N = N(N/2) \) additions.

- If \( N \) is a power of 2, then we can repeat this time-decimation until there are only 1-point DFTs, whereupon the computational complexity will be

\[
\approx N \log_2 N.
\]

- For large \( N \), this is a substantial savings over the direct approach with computational complexity

\[
\approx N^2.
\]

- Exercise: Show how decimation can be applied to similarly reduce the computational complexity of the inverse DFT.
Transient analysis in discrete time by unilateral \( z \)-transform

- \( z \)-transform definition and region of convergence.
- Basic \( z \)-transform pairs and properties.
- Inverse \( z \)-transform of rational polynomials by Partial Fraction Expansion (PFE).
- Total transient response of SISO DT LTIC systems \( Q(\Delta^{-1})y = P(\Delta^{-1})f \).
- The steady-state eigenresponse revisited.
- System composition and canonical realizations.

The unilateral \( z \)-transform & region of convergence

- The \( z \)-transform of a signal \( x = \{x[k]\}_{k \geq 0} \) is
  \[
  X(z) = (Zx)(z) = \sum_{k=0}^{\infty} x[k]z^{-k} := \lim_{K \to \infty} \sum_{k=0}^{K} x[k]z^{-k},
  \]
  where \( z \in \mathbb{C} \).

- If the signal \( x \) is bounded by an exponential (geometric), i.e.,
  \[ \exists M, \gamma \in \mathbb{R}^{>0} \text{ such that } \forall k \in \mathbb{Z}^{\geq 0}, \ |x[k]| \leq M\gamma^k \ (i.e., \ -M\gamma^k \leq x[k] \leq M\gamma^k) \]
  then the series \( X(z) \) converges in the region outside of a disk centered \( 0 \in \mathbb{C} \),
  \[ \{z \in \mathbb{C} \ | \ |z| > \gamma \} \).

- To see why bounded by an exponential suffices, recall absolute convergence \( \Rightarrow \) convergence:
  \[
  \forall k \geq 0, \ |x[k]z^{-k}| = |x[k]| \cdot |z|^{-k} \leq M\gamma^k |z|^{-k} = M(\gamma/|z|)^k
  \]
  \[
  \Rightarrow \sum_{k=0}^{\infty} |x[k]z^{-k}| \leq M \sum_{k=0}^{\infty} (\gamma/|z|)^k \text{ which converges if } \gamma/|z| < 1.
  \]
Basic $z$-transform pairs and RoCs

$$\delta[k] \xrightarrow{z} 1, \quad z \in \mathbb{C}$$

$$u[k] \xrightarrow{z} \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} \quad \frac{z}{z - 1}, \quad |z| > 1$$

$$\beta^k u[k] \xrightarrow{z} \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{1}{1 - \beta z^{-1}} = \frac{z}{z - \beta}, \quad |z| > |\beta|$$

$$\{\beta^{-1} u[k - 1]\}(z) \xrightarrow{z} \sum_{k=1}^{\infty} \beta^{-1} z^{-k} = z^{-1} \sum_{k=0}^{\infty} \beta^k z^{-k'} = z^{-1} \frac{1}{1 - \beta z^{-1}} = \frac{1}{z - \beta}, \quad |z| > |\beta|$$

$$e^{j\Omega} u[k] \xrightarrow{z} \sum_{k=0}^{\infty} e^{j\Omega} z^{-k} = \frac{1}{1 - e^{j\Omega} z^{-1}}, \quad |z| > 1 \quad (\beta = e^{j\Omega})$$

$$k\beta^k u[k] \xrightarrow{z} \sum_{k=0}^{\infty} k\beta^k z^{-k} = \beta \frac{d}{d\beta} \sum_{k=0}^{\infty} \beta^k z^{-k} = \beta \frac{d}{d\beta} \frac{1}{1 - \beta z^{-1}} = \frac{\beta z^{-1}}{\left(1 - \beta z^{-1}\right)^2}, \quad |z| > |\beta|$$

**Exercise:** Find $\mathcal{Z}\{A \cos(\Omega, k + \phi) u[k]\}$ and $\mathcal{Z}\{A \sin(\Omega, k + \phi) u[k]\}$.

---

**Basic $z$-transform properties: linearity**

- The $z$-transform is a linear operator: for all scalars $a_1, a_2 \in \mathbb{C}$ and all signals $x_1, x_2 : \mathbb{Z}^0 \to \mathbb{C}$ with respective ROCs $C_1, C_2 \subset \mathbb{C}$,

  $$(\mathcal{Z}\{a_1 x_1 + a_2 x_2\})(z) = a_1 (\mathcal{Z} x_1)(z) + a_2 (\mathcal{Z} x_2)(z), \quad z \in C_1 \cap C_2.$$  

- Note that

  $$\{z \mid |z| > \gamma_1\} \cap \{z \mid |z| > \gamma_2\} = \{z \mid |z| > \max\{\gamma_1, \gamma_2\}\} \subset \mathbb{C}.$$  

---
Basic $z$-transform properties: advance time shift

- Advance time shift (no change in RoC): Let $X = \mathcal{Z}x$.

$$\Delta^{-1}x \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k+1]z^{-k} = -zx[0] + \sum_{k=-1}^{\infty} x[k+1]z^{-k}$$

$$= -zx[0] + z \sum_{k=-1}^{\infty} x[k+1]z^{-(k+1)}$$

$$= -zx[0] + z \sum_{k=0}^{\infty} x[k]z^{-k}$$

$$= -zx[0] + zX(z)$$

- Exercise: For $v \in \mathbb{Z}^>0$ show by induction that

$$(\mathcal{Z}\{\Delta^{-v}x\})(z) = -\sum_{k=1}^{v} z^k x[v - k] + z^v X(z)$$

Basic $z$-transform properties: delay time shift

- Delay time shift (no change in RoC): For $v \in \mathbb{Z}^>0$,

$$\Delta^v xu \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k-v]u[k-v]z^{-k}$$

$$= \sum_{k=0}^{\infty} x[k-v]z^{-k} = \sum_{k=0}^{\infty} x[k']z^{-k'-v}$$

$$= z^{-v}X(z).$$

- So in the “zero-state” (input-output) context (i.e., $x[k]u[k] = 0$ for $k < 0$), we identify multiplying by $z^{-1}$ in complex-frequency domain with the unit delay $\Delta$ in the time domain.

- Delay $v \in \mathbb{Z}^>0$ of non-causal $x$:

$$\Delta^v x \xrightarrow{\mathcal{Z}} \sum_{k=0}^{\infty} x[k-v]z^{-k} = \sum_{k=-v}^{\infty} x[k']z^{-k'-v}$$

$$= \sum_{k=-v}^{-1} x[k']z^{-k'-v} + z^{-v}X(z).$$
Basic $z$-transform properties: frequency shift & convolution

- Let $X = Zx$ with RoC $C(\gamma) := \{z \in \mathbb{C} \mid |z| > \gamma\}$.
  \[
  \beta^k x[k] \xrightarrow{z} \sum_{k=0}^{\infty} \beta^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/\beta)^{-k} = X(z/\beta), \quad z \in C(\gamma|\beta|).
  \]
  i.e., $\times \beta^k$ in the time-domain is dilation by $\beta$ in the $z$-domain.

- For signals $x_1, x_2 : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ ($x_1[k], x_2[k] = 0$ for $k < 0$), with respective ROCs $C_1, C_2 \subset \mathbb{C}$,
  \[
  x_1 \ast x_2 \xrightarrow{z} \sum_{k=0}^{\infty} (x_1 \ast x_2)[k] z^{-k} = \sum_{k=0}^{\infty} \sum_{v=0}^{k} x_1[v] x_2[k - v] z^{-(k-v)} z^{-v} = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=v}^{\infty} x_2[k - v] z^{-(k-v)} = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=0}^{\infty} x_2[k'] z^{-k'} = X_1(z) X_2(z), \quad z \in C_1 \cap C_2.
  \]

Basic $z$-transform properties: convolution, IVT & FVT

- So convolution in the time-domain is multiplication in the frequency domain.

- The converse is also true.

- Directly by definition of $X = Zx$, we get the initial value theorem
  \[
  \lim_{z \to \infty} X(z) = x[0].
  \]

- There is also a “final value” theorem for $\lim_{k \to \infty} x[k]$. 

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We now study transient analysis of LTI difference equations using $z$-transforms.

Recall our system is defined given polynomials $P, Q$, input $f$ and initial conditions:
- $Q(\Delta^{-1})y = P(\Delta^{-1})f$, where $y$ is the total output and
- input $f[k] = 0$ for $k < 0$,
- degree of polynomial $Q = n \geq m = $ degree of polynomial $P$ (causal system),
- $Q(z) = z^n + \sum_{v=0}^{n-1} a_v z^v$ (i.e., $a_n = 1$) and $P(z) = \sum_{v=0}^{m} b_v z^v$,
- $a_n \neq 0$ or $b_n \neq 0$ for pol'ys $Q, P$ of minimum degree,
- $n$ initial conditions $y[-n], y[-n+1], \ldots, y[-2], y[-1]$.

We can restate the difference equation in terms of delays by delaying both sides by $n$ time-units (i.e., applying with $\Delta^n$), to get

$$\Delta^n Q(\Delta^{-1})y = \Delta^n P(\Delta^{-1})f$$

$$\Rightarrow \tilde{Q}(\Delta)y := \sum_{v=0}^{n} a_v \Delta^{n-v}y = \sum_{v=0}^{m} b_v \Delta^{n-v}f =: \tilde{P}(\Delta)f$$

So, taking the $z$-transform of the (delay) difference equation, we get by the (delay) time-shift and linearity properties that

$$\sum_{v=0}^{n} a_v \sum_{k=-v}^{k} y[k] z^{-k-v} + \tilde{Q}(z^{-1})Y(z) = \tilde{P}(z^{-1})F(z)$$

So, solving for the total response $Y$ we get

$$Y(z) = \frac{\tilde{P}(z^{-1})}{\tilde{Q}(z^{-1})} F(z) - \frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{k} y[k] z^{-k-v}}{\tilde{Q}(z^{-1})} = Y_{ZS}(z) + Y_{ZI}(z)$$

where the ZIR and ZSR in the complex-frequency ($z$) domain respectively are

$$Y_{ZI}(z) := -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{k} y[k] z^{-k-v}}{\tilde{Q}(z^{-1})} = -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{k} y[k] z^{n-k-v}}{\tilde{Q}(z)}$$

$$Y_{ZS}(z) := \frac{\tilde{P}(z^{-1})}{\tilde{Q}(z^{-1})} F(z) = \frac{P(z)}{Q(z)} F(z) = H(z)F(z) \ (\text{transfer function } H).$$

Regarding this total transient response, note how
- the $z$-transform's unilateral aspect captures the impact of initial conditions (ZIR), and
- a greater range of inputs $f$ than under DTFT through $\text{RoC} \subset \mathbb{C}$ (not just $|z|=1$).
• Suppose i.c. \( y[-1] = -1 \), input \( f[k] = 2(-3)^k u[k] \) and output \( y \) s.t.
\[
\forall k \geq -1, \quad 2y[k + 1] + 2y[k] = 3f[k + 1] + 2f[k].
\]

• To find the total response \( y \), we take the \( z \)-transform of the equivalent system: \( \forall k \geq 0, \)
\[
2y[k] + 2y[k - 1] = 3f[k] + 2f[k - 1]
\Rightarrow 2Y(z) + 2(z^{-1}Y(z) + y[-1]) = 3F(z) + 2z^{-1}F(z).
\]

• So by the delay property for non-causal signals \( (y) \), the total response
\[
Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} F(z) + \frac{-2y[-1]}{2 + 2z^{-1}}
= H(z) F(z) + \frac{-y[-1]}{1 + z^{-1}}
= \frac{Y_{ZS}(z) + Y_{ZI}(z)}{1 + z^{-1}}
\]
with RoC for \( Y \) being the intersection of those of \( F \) and \( H \).

Total response of SISO LTIC systems - example

• Here \( F(z) = \mathcal{Z}\{2(-3)^k\} = 2/(1 + 3z^{-1}) \), \( y[-1] = -1 \), so
\[
Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} \cdot \frac{2}{1 + 3z^{-1}} + \frac{1}{1 + z^{-1}}
= \frac{3 + 2z^{-1}}{1 + z^{-1}} + \frac{1}{1 + z^{-1}}
= \frac{1}{1 + z^{-1}}
\]
where for the last equality see PFE below (here in \( z^{-1} \)).

• Understanding that the ZIR begins at \( k = -1 \) (initial condition) and the ZSR at time \( k = 0 \), we get:
\[
\forall k \geq -1, \quad y[k] = (3.5(-3)^k - 0.5(-1)^k) u[k] + (-1)^k = y_{ZS}[k] + y_{ZI}[k],
\]
where we minded the ambiguity \( \mathcal{Z}x = \mathcal{Z}xu \).

• Exercise: Verify this solution using time-domain methods, i.e.,
\[
y = y_{ZI} + y_{ZS} = y_{ZI} + h * f, \text{ where } h \text{ and } y_{ZI} \text{ consist of char. modes.}
\]
Inverse \( z \)-transform of proper rational polynomials

- We now describe how to find \( Z^{-1}X \) of causal signal \( X \) that is rational polynomial in \( z \), i.e., \( X(z) = M(z)/N(z) \) where \( M(z) \) and \( N(z) \) are polynomials in \( z \).

- If \( \deg(M) = \deg(N) + 1 \), we perform long division to write \( X = c + \bar{M}/N \) where \( \deg(N) = \deg(\bar{M}) \) and \( Z^{-1}X = c\delta + Z^{-1}\{\bar{M}/N\} \).

- If \( \deg(M) = \deg(N) \) and \( M(0) = 0 \) (so \( z^{-1}M(z) \) is a polynomial), we can factor \( z \) from \( M \) to get
  \[
  X(z) = \frac{z^{-1}M(z)}{N(z)}.
  \]

- We will find \( Z^{-1}X \) using PFE of the strictly proper rational polynomial \( z^{-1}M(z)/N(z) \).

- Alternatively, we could apply PFE on strictly proper rational polynomials in \( z^{-1} \), \( z^{-K}M(z)/(z^{-K}N(z)) \) where \( K := \deg(N) \), as in the previous example.

---

Partial Fraction Expansion (PFE) example in \( z \) (not \( z^{-1} \))

- For example, suppose
  \[
  X(z) := \frac{z(3z + 2)}{z^2 - 0.64} = \frac{z(3z + 2)}{(z + 0.8)(z - 0.8)} = z \left( \frac{0.25}{z + 0.8} + \frac{2.75}{z - 0.8} \right) = 0.25 \frac{z}{z + 0.8} + 2.75 \frac{z}{z - 0.8}
  \]
  where PFE (below) gave the numerators (residues) 0.25 and 2.75.

- So,
  \[
  (Z^{-1}X)[k] = 0.25(-0.8)^k u[k] + 2.75(0.8)^k u[k]
  \]

- Note that the associated RoC of \( X \) is \( \{ z \in \mathbb{C} \mid |z| > 0.8 \} \).
Partial Fraction Expansion (PFE) - preliminaries

- Let $K = \deg(N) = \deg(M)$ so that we can factor
  \[ N(z) = \prod_{k=1}^{K} (z - p_k), \]
  where the $p_k$ are the roots of $N$ (poles of $M/N$).

- We assume $M$ and $N$ have no common roots, i.e., no "pole-zero cancellation" issue to consider, so that the $p_k$ are the poles of $M/N$.

- Again, we assume $M(0) = 0$ ($0$ is a zero of $M/N$) and so $z^{-1}M(z)$ is a polynomial of degree $K - 1$.

- Note that the RoC for $M(z)/N(z)$ is $\{ z \in \mathbb{C} \mid |z| > \max_k |p_k| \}$.

PFE - the case of no repeated poles

- Suppose there are no repeated poles for $M/N$, i.e., $\forall k \neq l, \quad p_k \neq p_l$.

- In this case, we can write the PFE of $z^{-1}M(z)/N(z)$ as
  \[
  \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} \quad \Rightarrow \quad \frac{M(z)}{N(z)} = \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} \frac{1}{1 - p_l z^{-1}}
  \]
  where the scalars (Heaviside coefficients) $c_l \in \mathbb{C}$ are
  \[
  c_l = \frac{z^{-1}M(z)}{\prod_{k \neq l} (z - p_k)} \bigg|_{z = p_l} = \lim_{z \rightarrow p_l} \frac{z^{-1}M(z)}{N(z)} (z - p_l) = \frac{z^{-1}M(z)}{N(z)} (z - p_l) \bigg|_{z = p_l}.
  \]

- That is, to find the Heaviside coefficient $c_k$ over the term $z - p_k$ in the PFE, we have removed (covered up) the term $z - p_k$ from the denominator $N(z)$ and evaluated the remaining rational polynomial at $z = p_k$.

- This approach, called the Heaviside cover-up method, works even when $p$ is $\mathbb{C}$-valued.

- Given the PFE of $z^{-1}M/N$, $(Z^{-1}M/N)[k] = \sum_{l=1}^{K} c_l p_l^h u[k]$. 

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• To prove that the above formula for the Heaviside coefficient $c_l$ is correct, note that the claimed PFE of $z^{-1} M(z) / N(z)$ is

$$\sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) / N(z)$$

• Thus, the PFE equals $z^{-1} M(z) / N(z)$ if and only if the numerator polynomials are equal, i.e., iff

$$z^{-1} M(z) = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) =: \hat{M}(z).$$

• Again, two polynomials are equal if their degrees, $L$, are equal and either:
  - their coefficients are the same, or
  - they agree at $L + 1$ (or more) different points, e.g., two lines ($L = 1$) are equal if they agree at 2 ($= L + 1$) points.

• Since $z^{-1} M(z)$ is a polynomial of degree $< K$, it suffices to check that whether $z^{-1} M(z) = \hat{M}(z)$ for all $z = p_k$, $k \in \{1, 2, ..., K\}$, i.e., this would create $K$ equations in $< K$ unknowns (the coefficients of $M$).

PFE - proof of Heaviside cover-up method (cont)

• But note that any pole $p_r$ of $z^{-1} M(z) / N(z)$ is a root of all but the $r^{th}$ term in $\hat{M}$, so that

$$\hat{M}(p_r) = c_r \prod_{k \neq r} (p_r - p_k)$$

$$= \left( \left. \frac{z^{-1} M(z)}{\prod_{k \neq r} (z - p_k)} \right|_{z = p_r} \right) \prod_{k \neq r} (p_r - p_k)$$

$$= \frac{p_r^{-1} M(p_r)}{\prod_{k \neq r} (p_r - p_k)} \prod_{k \neq r} (p_r - p_k)$$

$$= p_r^{-1} M(p_r).$$

• Q.E.D.
To find the inverse $z$-transform of a proper rational polynomial $X = M/N$ with $M(0) = 0$, first factor its denominator $N$ and factor $z$ from $M$, e.g.,

$$X(z) = \frac{z^3 + 5z^2}{z^3 + 9z^2 + 26z + 24} = \frac{z^2 + 5z}{(z + 4)(z + 3)(z + 2)} \quad \text{for } |z| > 4.$$

So, by PFE

$$X(z) = z^{-1}M(z) = z^{-1}\left(\frac{c_4}{z + 4} + \frac{c_3}{z + 3} + \frac{c_2}{z + 2}\right) = z^{-1}M(z) \Rightarrow z^{-1}M(z) = 1z^2 + 5z + 0 = c_4(z + 3)(z + 2) + c_3(z + 4)(z + 2) + c_2(z + 4)(z + 3) =: \tilde{M}(z).$$

We can solve for the 3 constants $c_k$ by comparing the 3 coefficients of quadratic $M$ and $\tilde{M}$.

The Heaviside cover-up method suggests we try $z = -2, -3, -4$ to solve for $c_2, c_3, c_4$:

$$c_4 = \frac{z^2 + 5z}{(z + 3)(z + 2)} \bigg|_{z = -4} = -2, \quad c_3 = \frac{z^2 + 5z}{(z + 4)(z + 2)} \bigg|_{z = -3} = 6, \quad c_2 = \frac{z^2 + 5z}{(z + 4)(z + 3)} \bigg|_{z = -2} = -3.$$

Thus, $x[k] = (Z^{-1}X)[k] = (-2(-4)^k + 6(-3)^k - 3(-2)^k)u[k]$.

---

The case of a non-repeated, complex-conjugate pair of poles

Again, recall that for polynomials with all coefficients $\in \mathbb{R}$, all complex poles will come in complex-conjugate pairs, $p_1 = \overline{p_2}$.

The case of non-repeated poles $p_1, p_2 = \alpha \pm j\beta (\alpha, \beta \in \mathbb{R}, j := \sqrt{-1})$ that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients $c_1, c_2$, i.e., $c_1 = \overline{c_2}$.

In the PFE, we can alternatively combine the terms

$$\frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} = \frac{r_1z + r_0}{(z - \alpha)^2 + \beta^2}$$

where by equating the two numerator polynomials’ coefficients,

$$r_0 = -c_1p_2 - c_2p_1 = -2\text{Re}\{c_1p_2\} \in \mathbb{R} \quad \text{and} \quad r_1 = c_1 + c_2 = 2\text{Re}\{c_1\} \in \mathbb{R}.$$

Exercise: Show that

$$2|c| \cdot |p|^k \cos(k\angle p + \angle c) \Rightarrow \frac{cz}{z - p} + \frac{\overline{cz}}{z - \overline{p}}$$
To find the inverse $z$-transform of
\[ X(z) = \frac{3z^2 + 2z}{z^3 + 5z^2 + 10z + 12}, \]
first factor the denominator and divide the numerator by $z$ to get
\[ X(z) = \frac{3z + 2}{(z^2 + 2z + 4)(z + 3)}. \]

Note that the poles of $X$ are $-3$ and $-1 \pm j\sqrt{3}$ (so $X$’s RoC is $|z| > 3$).

So, we can expand $X$ to
\[ X(z) = z^{-1}r_0 + z^{-2}r_1 + z^{-3}c_3, \]
where by the Heaviside cover-up method,
\[ c_3 = \frac{3z + 2}{z^2 + 2z + 4} \bigg|_{z=-3} = -1. \]

To find $r_1, r_0$, we will compare coefficients of the numerator polynomials of $X$ (actually $z^{-1}X$) and its PFE, i.e.,
\[ 0z^2 + 3z + 2 = (r_1z + r_0)(z + 3) + c_3(z^2 + 2z + 4) \quad (*) \]
\[ = (r_1 - 1)z^2 + (3r_1 + r_0 - 2)z + 3r_0 - 4. \]

Thus, by comparing coefficients
\[ 0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4 \]
we get
\[ r_0 = 2 \quad \text{and} \quad r_1 = 1. \]

Note how $z = -3$ in $(*)$ gives $c_3 = -1$ as Heaviside cover-up did.
• Thus by substituting, we get

\[ X(z) = \frac{z + 2}{z^2 + 2z + 4} + \frac{-1}{z + 3} \]

• Exercise: Show that

\[ x[k] = (z^{-1}X)[k] = \left( \sqrt{\frac{4}{3}} 2^k \cos(k \frac{2\pi}{3} - \frac{\pi}{6}) - (-3)^k \right) u[k]. \]

PFE - the general case of repeated poles

• If a particular pole \( p \) of \( z^{-1}M(z)/N(z) \) is of order \( r \geq 1 \), i.e., \( N(z) \) has a factor \((z-p)^r\), then the PFE of \( z^{-1}M(z)/N(z) \) has the terms

\[ \frac{c_1}{z-p} + \frac{c_2}{(z-p)^2} + \ldots + \frac{c_r}{(z-p)^r} = \sum_{k=1}^{r} \frac{c_k}{(z-p)^k} = \frac{z^{-1}M(z)}{N(z)} - \Phi(z) \]

with \( c_k \in \mathbb{C} \forall k \in \{1, 2, \ldots, r\} \), where \( \Phi(z) \) represents the other PFE terms of \( z^{-1}M(z)/N(z) \) (i.e., corresponding to poles \( \neq p \)).

• Note that equating \( z^{-1}M(z)/N(z) \) to its PFE and multiplying by \((z-p)^r\) gives

\[ \frac{z^{-1}M(z)}{N(z)}(z-p)^r = c_r + \sum_{k=1}^{r-1} c_k(z-p)^{r-k} + \Phi(z)(z-p)^r \]

\[ \Rightarrow \frac{z^{-1}M(z)}{N(z)}(z-p)^r \bigg|_{z=p} = c_r, \]

i.e., Heaviside cover-up (of the entire term \((z-p)^r\)) works for \( c_r \).
PFE - the general case of repeated poles (cont)

- To find $c_{r-1}$, we differentiate the previous display to get
  \[
  \frac{d}{dz} \left( z^{-1} \frac{M(z)}{N(z)} (z - p)^r \right) = \sum_{k=1}^{r-1} c_k(r - k)(z - p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z - p)^r
  \]
  \[
  = c_{r-1} + \sum_{k=1}^{r-2} c_k(r - k)(z - p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z - p)^r
  \]
  \[
  \Rightarrow c_{r-1} = \left. \left( \frac{d}{dz} z^{-1} M(z) (z - p)^r \right) \right|_{z=p}
  \]

- If we differentiate the original display $k \in \{0, 1, 2, \ldots, r - 1\}$ times and then substitute $z = p$, we get (with $0! := 1$)
  \[
  \left. \left( \frac{d^k}{dz^k} z^{-1} M(z) (z - p)^r \right) \right|_{z=p} = k! c_{r-k}
  \]
  \[
  \Rightarrow c_{r-k} = \frac{1}{k!} \left. \left( \frac{d^k}{dz^k} z^{-1} M(z) (z - p)^r \right) \right|_{z=p}
  \]

---

PFE - the general case of repeated poles - example

- To find the inverse $z$-transform of
  \[
  X(z) = \frac{z(3z + 2)}{(z + 1)(z + 2)^3}
  \]
  write the PFE of $X$ as
  \[
  X(z) = z \left( \frac{c_1}{z + 1} + \frac{c_{2,1}}{z + 2} + \frac{c_{2,2}}{(z + 2)^2} + \frac{c_{2,3}}{(z + 2)^3} \right)
  \]
  so clearly the RoC of causal $X$ is $|z| > 2$.

- By Heaviside cover-up
  \[
  c_1 = \left. \frac{3z + 2}{(z + 2)^3} \right|_{z=-1} = -1 \quad \text{and} \quad c_{2,3} = \left. \frac{3z + 2}{z + 1} \right|_{z=-2} = 4.
  \]
• Also,
\[ c_{2,2} = \left. \frac{1}{1!} \left( \frac{d}{dz} \frac{3z + 2}{z + 1} \right) \right|_{z = -2} = \frac{1}{1!} \frac{1}{(z + 1)^2} \bigg|_{z = -2} = 1 \]
\[ c_{2,1} = \left. \frac{1}{2!} \left( \frac{d^2}{dz^2} \frac{3z + 2}{z + 1} \right) \right|_{z = -2} = \frac{1}{2!} \frac{-2}{(z + 1)^3} \bigg|_{z = -2} = 1 \]

• Thus,
\[ X(z) = \frac{-1}{z + 1} + \frac{1}{z + 2} + \frac{1}{(z + 2)^2} + \frac{4}{(z + 2)^3} \quad \forall |z| > 2 \]
\[ \Rightarrow x[k] = (Z^{-1}X)[k] = (-1)^k + (-2)^k + k(-2)^{k-1} + 4 \frac{k(k-1)}{2} (-2)^{k-2} u[k] \]

• Exercise: Show by induction and integration by parts that: \( \forall m \in \mathbb{Z}^{>0}, \)
\[ \binom{k}{m} \gamma^{k-m} u[k] \xrightarrow{Z} \frac{z}{(z - \gamma)^m} \]

• Exercise: Find the ZSR \( y \) to input \( f[k] = 2^j k u[k] = 2e^{jk\pi/2} u[k] \) of the marginally stable system \( H(z) = 4/(z^2 + 1) \).

\[ \text{PFE of } M/N \text{ when } M(0) \neq 0 \]

• If \( M(0) \neq 0 \) (so cannot factor \( z \) from \( M(z) \)), then just perform long division if \( \deg(M) \geq \deg(N) \) to get a strictly proper rational polynomial, factor \( N \) to find the poles, and find the PFE as before.

• When taking inverse \( z \)-transform, recall the \( z \)-transform pair
\[ \beta^{k-1} u[k - 1] \xrightarrow{Z} \frac{1}{z - \beta}, \quad |z| > |eta| \]
PFE without factoring $z$ from the numerator first

- For example, to find the ZSR to $f[k] = 2(-1)^k u[k]$ of the system
  \[ y[k+1] - 4y[k] = 5f[k], \]
  take the $z$-transform to get
  \[
  Y_{ZS}(z) = H(z)F(z) = \frac{5}{z-4} F(z) = \frac{10z}{(z-4)(z+1)}
  \]
  \[ = \frac{8}{z-4} + \frac{2}{z+1} \text{ (by PFE)} \]
  \[ \Rightarrow y_{ZS}[k] = 8(4)^{k-1} u[k-1] + 2(-1)^{k-1} u[k-1] \]

- Note that the unit-pulse response is
  \[ h[k] = Z^{-1}(H)[k] = 5(4)^{k-1} u[k-1], \]
  and that, by delaying the difference equation to get
  \[ y[k] = -4y[k-1] + 5f[k-1], \]
  we see that (the ZSR) $y_{ZS}[0] = 0$.

- Exercise: First factor $z$ from the numerator of $Y_{ZS}$ before PFE to show that
  \[ y_{ZS}[k] = 2(4)^k u[k] - 2(-1)^k u[k]. \]
  Is this result different? Check for $k = 0$ and $k > 0$.

PFE and eigenresponse for asymptotically stable systems

- The total response of a SISO LTI system to input $f$ is of the form
  \[ Y(z) = H(z)F(z) + \frac{P_1(z)}{Q(z)}F(z) + \frac{P_1(z)}{Q(z)} = Y_{ZS}(z) + Y_{ZI}(z). \]
  where $P_1$ depends on the initial conditions and the RoC is the intersection of that of input $F = Zf$ and the system characteristic modes.

- Unlike for DTFT notation, here write $H(z) = P(z)/Q(z) = (Zh)(z)$.

- Suppose the system is BIBO/asymptotically stable and the input is a sinusoid at frequency $\Omega_\phi$ radians per unit time, $f[k] = A e^{j(\Omega_\phi \omega + \phi)} u[k] = A e^{j\phi}(e^{j\Omega_\phi})^k u[k]$ with $A > 0$
  \[ F(z) = A e^{j\phi z}/(z - e^{j\Omega_\phi}) \text{ with RoC } |z| > 1. \]

- Since $e^{j\Omega_\phi}$ cannot be a system pole (owing to asymptotic stability all poles have modulus strictly less than one), we can use Heaviside cover-up on
  \[ Y_{ZS}(z) = H(z)F(z) = \frac{P(z)}{Q(z)(z - e^{j\Omega_\phi})} A e^{j\phi} \text{ to get} \]
  \[ Y_{ZS}(z) = z^2 H(e^{j\Omega_\phi}) A e^{j\phi} + \text{char. modes} = H(e^{j\Omega_\phi}) F(z) + \text{char. modes.} \]
• Thus, the total response of an asymptotically stable system to a sinusoidal input $f$ at frequency $\Omega_0$ is

$$y[k] = H(e^{j\Omega_0})f[k] + \text{linear combination of characteristic modes.}$$

• So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as $k \to \infty$,

$$y[k] \to H(e^{j\Omega_0})f[k] = H(e^{j\Omega_0})Ae^{j(\Omega_0k + \phi)} = |H(e^{j\Omega_0})|Ae^{j(\Omega_0k + \phi + \angle H(e^{j\Omega_0}))},$$

where again,

• $H = P/Q$ is the system’s transfer function,

• $|H(e^{j\Omega_0})|$ is the system’s magnitude response at frequency $\Omega_0$, radians/unit-time, and

• $\angle H(e^{j\Omega_0})$ is the system’s phase response at $\Omega_0$.

PFE and eigenresponse for asymptotically stable systems (cont)

• Laplace’s approximation: the rate at which the total response converges to the eigenresponse response is according to the characteristic value of largest modulus,

  – which will be $< 1$ owing to the stability assumption,

  – i.e., giving the modes(s) that $\to 0$ slowest.

• In continuous-time systems, it’s the characteristic value of largest real part, which will be negative owing to stability assumption.
Consider the proper \((m \leq n)\) transfer function
\[
H(z) = \frac{P(z)}{Q(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{Y(z)}{F(z)}
\]

The direct-form realization employs the interior system state \(X := F/Q\), i.e., \(F = QX\) and \(Y = PX\) where the former implies (with \(a_n = 1\)),
\[
F(z) = \sum_{r=0}^{n} a_r z^r X(z) \Rightarrow z^n X(z) = F(z) - \sum_{r=0}^{n-1} a_r z^r X(z).
\]

For \(n = 2\), there are two “system states” (outputs of unit delays), \(X\) and \(zX\) (respectively, \(x[k]\) and \((\Delta^{-1}x)[k] = x[k+1]\)).

Now adding \(Y = PX\), we finally get the direct-form canonical system-realization of \(H\):

Again, state variables taken as outputs of unit delays, here: \(x, \Delta^{-1}x, \ldots \Delta^{-(n-1)}x\).

If \(b_n = b_2 \neq 0\), there is direct coupling of input and output, \(H\) is proper but not strictly so, \(h = Z^{-1}H\) has a unit-pulse component \(b_2\delta\).
 Canonical system realizations - direct form (cont)

- Note that this \( n = 2 \) example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for \( n = 2 \), \( H(z) = P(z)/Q(z) \) where

\[
Q(z) = z^2 + a_1 z + a_0 = (z - \alpha)^2 + \beta^2
\]

for \( \alpha, \beta \in \mathbb{R} \), so the poles are \( \alpha \pm j\beta \).

Canonical system realizations by PFE

- In the general case of a proper transfer function, we can use partial-fraction expansion
  - grouping the terms corresponding to a complex-conjugate pair of poles, i.e., a second-order denominator, and
  - using a direct-form realization for these terms.

- Besides the PFE-based and direct-form realizations, there are other (zero-state) system realizations, e.g., "observer" canonical.

- For proper rational-polynomial transfer functions \( H = P/Q \), all of these realizations involve \( n \) (= degree of \( Q \)) unit delays, the output of each being a different interior state variable of the system.
Canonical system realizations by PFE - example

\[ H(z) = \frac{.3z^2 - .1}{z^2 - 0.1z - .3} = .3 + \frac{.3z - .01}{(z - 0.6)(z + 0.5)} = .3 + \frac{17/1.1}{z - 0.6} + \frac{16/1.1}{z + 0.5} \]

Note that one cannot factor \( z \) from the numerator of \( H \).

**Exercise:** Find a realization for this transfer function \( H \) by

1. splitting/forking the input signal \( F \),
2. using the direct canonical form for each of these 3 terms of \( H \) found by long division and PFE, and
3. summing three resulting output signals to get the (ZS) output \( Y = HF \).

Digital Proportional-Integral (PI) system

- Consider a continuous-time signal \( x \) sampled every \( T \) seconds,
  \[ \forall k \in \mathbb{Z}^+, \ x[k] = x(kT), \]
  and its integral \( y(t) = \int_0^t x(\tau) d\tau \).
- The sampled integral can be approximated, \( y(kT) \approx y[k] \), by the trapezoid rule,
  \[ y[k] = y[k-1] + \frac{x[k-1] + x[k]}{2} T. \]
- In the complex-frequency domain,
  \[ Y(z) = Y(z)z^{-1} + \frac{X(z)z^{-1} + X(z)}{2} T \]
  \[ \Rightarrow \frac{Y(z)}{X(z)} = \frac{T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}. \]
Digital PI system (cont)

- So, a digital PI transfer function would be of the form,
  \[ G(z) = K_p + \frac{K_i T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}. \]
  for constants \( K_p, K_i \).

- In practice, PID or PI systems \( G \) are commonly used to control a plant \( H \), where \( G \) may be in series with \( H \) or in the feedback branch.

- **Exercises:**
  - Draw the direct-form canonical realization for \( G \).
  - Draw the block diagram for the closed-loop system with negative feedback:
    \[ Y = HX \text{ and } X = F - GY \text{ where } H \text{ is the (open-loop) system.} \]
  - Find the closed-loop transfer function \( Y/F \) and recall the pole placement problem to stabilize \( H \).

Recursive Least Squares (RLS) Filter - Introduction

- Consider a LTI system with input \( f \) and output \( y \),
  \[ y[k] = \sum_{r=0}^{K} h[k - r]f[r] + v[k], \ k \in \mathbb{Z}, \]
  where \( v \) is an additive noise process and \( K \) is the maximum system order.

- The system (unit-pulse response) \( h \) is not known.

- Past values of the output \( y \) are observed (known).

- At time \( k \), the objective is to forecast the next output \( \hat{y}[k+1] \), based on the assumed known/observed quantities:
  - the next input \( f[k+1] \),
  - the past \( R \) input-output pairs \( \{f[r], y[r]\}_{k-R+1 \leq r \leq k} \).
**RLS objective and $R^\text{th}$-order linear tap filter**

- The output of an $R^\text{th}$-order RLS tap-filter at time $k$ is,
  \[ \hat{y}_k[i] = \sum_{r=i-R+1}^{i} \eta_k[i-r]f[r], \quad i \leq k + 1. \]

- The objective of this filter at time $k$ is to accurately estimate the system output $y[k+1]$ with $\hat{y}_k[k+1]$ by choosing the $R$ filter coefficients $\eta_k[k-R+1], \ldots, \eta_k[k-1], \eta_k[k]$ that minimize the following sum-of-square-error objective:
  \[ E_k = \sum_{r=k-R+1}^{k} \lambda^{k-r}|y[r] - \hat{y}_k[r]|^2 = \sum_{r=k-R+1}^{k} \lambda^{k-r}|e_k[r]|^2 \]
  where
  - $\lambda > 0$ is a forgetting factor and
  - error $e_k[r] := y[r] - \hat{y}_k[r]$.

**Exercise:** Prove the last equality.

---

**RLS filter**

- So, to minimize $E_k$, substitute $\hat{y}_k[r]$ into $E_k$ and solve
  \[ 0 = \frac{\partial E_k}{\partial \eta_k[i]} \quad \text{for} \quad i \in \{k - R + 1, \ldots, k - 1, k\}. \]

- That is, $R$ equations in $R$ unknowns: for $i \in \{k - R + 1, \ldots, k - 1, k\}$,
  \[ 0 = \sum_{r=k-R+1}^{k} 2\lambda^{k-r}e_k[r] \frac{\partial e_k[r]}{\partial \eta_k[i]} \]
  \[ = \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(y[r] - \hat{y}_k[r]) \left( -\frac{\partial \hat{y}_k[r]}{\partial \eta_k[i]} \right) \]
  \[ = \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(\hat{y}_k[r] - y[r])f[r-i] \]

- **Exercise:** Prove the last equality.
Substituting \( \hat{y}_k[r] \), rewrite these equations to get the following \( R \) equations in \( R \) unknowns \( \eta_k[i] \) that are \( E_k \)-minimizing: for \( i \in \{ k - R + 1, \ldots, k - 1, k \} \),

\[
\sum_{r=k-R+1}^{k} \lambda^{k-r} f[r-i] \sum_{\ell=r-R+1}^{r} f[\ell] \eta_k[r-\ell] = \sum_{r=k-R+1}^{k} \lambda^{k-r} y[r] f[r-i]
\]

- **Exercise**: Prove the last equality and write it in matrix form.

- **Exercise**: Research how the \( E_k \)-minimizing filter parameters \( \eta_k \) can be computed recursively, i.e., using \( \eta_{k-1} \).

- The filter order \( R \) can also be "trial adapted" to discover the system order \( K \) so that the error-minimizing filter parameters \( \eta_k \) "track" the system unit-pulse response \( h \) over time \( k \).

- Note the required initial "warm-up" period of \( R \) time-units where the outputs of system \( h \) are simply observed and recorded and no estimates are made.

- **Exercise**: If there was no additive noise process \( v \) and the system unit-pulse response \( h \) had finite support (i.e., a FIR system with \( K < \infty \)), show how \( h \) can be deduced from input-output \((f, y)\) observations.