Time-domain analysis of discrete-time LTI systems

- Discrete-time signals
- Difference equation single-input, single-output systems in discrete time
- The zero-input response (ZIR): characteristic values and modes
- The zero (initial) state response (ZSR): the unit-pulse response, convolution
- System stability
- The eigenresponse and (zero state) system transfer function
Discrete-time signal by sampling a continuous-time signal

- Consider a continuous-time signal \( x : \mathbb{R} \to \mathbb{R} \) sampled every \( T > 0 \) seconds
  \[
x(kT + t_0) =: x[k] \quad \text{for } k \in \mathbb{Z},
\]
  where
  - \( t_0 \) is the sampling time of the 0\(^{th} \) sample, and
  - \( T \) is assumed less than the Nyquist sampling period of \( x \), and
  - \( x[k] \) (with square brackets) is the \( k \)\(^{th} \) sample itself.

- Here \( x[\cdot] \) is a discrete-time signal defined on \( \mathbb{Z} \).

Example of sampling with \( t_0 = 0 \) and positive signal \( x \)
Introduction to signals and systems in discrete time

- A discrete-time function (or signal) \( x : A \rightarrow B \) is one with countable (time) domain \( A \).
- We will take the range \( B = \mathbb{R} \) or \( B = \mathbb{C} \).
- Typically, we will herein take domain \( A = \mathbb{Z} \) or \( \mathbb{Z}^{\leq n} \) for some (finite) integer \( n \geq 0 \).
- Some properties of signals are as in continuous time: e.g., periodic, causal, bounded, even or odd.
- Similarly, some signal operations are as in continuous time: e.g., spatial shift/scale, superposition, time reflection, and (integer valued) time shift.

Time scaling: decimation and interpolation

- Time scaling can be implemented in continuous time prior to sampling at a fixed rate, or the sampling rate itself could be varied (again recall the Nyquist sampling rate).
- In discrete time, a signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) can be decimated (subsampled) by an integer factor \( L \neq 0 \) to create the signal \( x_L \) defined by
  \[
  x_L[k] = x[kL], \quad \forall k \in \mathbb{Z},
  \]
  i.e., \( x_L \) is defined only by every \( L^{th} \) sample of \( x \).
- A discrete-time signal \( x \) can also be interpolated by an integer factor \( L > 0 \) to create \( x_L \) satisfying
  
  \[
  x_L[kL] = x[k], \quad \forall k \in \mathbb{Z}.
  \]
- For an interpolated signal \( x_L \), the values of \( x_L[r] \) for \( r \) not a multiple of \( L \) (i.e., \( \forall k \in \mathbb{Z} \) s.t. \( r \neq kL \)) can be set in different ways, e.g., between consecutive samples:
  - (piecewise constant) hold: \( x_L[r] = x_L[r/L] = x[r/L] \)
  - linear interpolation:
    
    \[
    x_L[r] = x[r/L] + \frac{r - L[r/L]}{L}(x[r/L] + 1) - x[r/L])
    \]
Is the functional mapping $x \rightarrow x_L$ causal for linear interpolation?

Is the hold causal?

**Exercise:** Show that if a periodic, continuous-time signal $x(t)$, with period $T_0$, is periodically sampled every $T$ seconds, then the resulting discrete-time signal $x[k]$ is periodic if and only if $T/T_0$ is rational.

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**Unit pulse $\delta$, unit step $u$, unit delay $\Delta$, and convolution**

- Some important signals in discrete time are as those in continuous time, e.g., polynomials, exponentials, unit step.

- In discrete time, rather than the (unit) impulse, there is unit pulse (Kronecker delta):
  \[
  \delta[k] = \begin{cases} 
  1 & \text{if } k = 0 \\
  0 & \text{else} 
  \end{cases}
  \]

- Any discrete-time signal $x$ can thus be written as
  \[
  x[k] = \sum_{r=-\infty}^{\infty} x[r] \delta[k-r] = \sum_{r=-\infty}^{\infty} x[k-r] \delta[r] = (x * \delta)[k]
  \]

- or just $x = x * \delta$, i.e., the unit pulse $\delta$ is the identity of discrete-time convolution.

- Define the operator $\Delta$ as unit delay (time-shift), i.e., $\forall$ signals $y$ and $\forall k, r \in \mathbb{Z}$,
  \[
  (\Delta^r y)[k] := y[k-r].
  \]

- The discrete-time unit step $u$ satisfies $\delta = u - \Delta u$, equivalently: $\forall k \in \mathbb{Z}$,
  \[
  \delta[k] = u[k] - u[k-1] \quad \text{and} \quad u[k] = \sum_{r=0}^{\infty} (\Delta^r \delta)[k] = \sum_{r=0}^{\infty} \delta[k-r].
  \]
Unit pulse and unit step functions

- **Exercise:** For any signal causal \( f \{ \{ f[k], k \geq 0 \} \), show that

\[
\forall k \geq 0, (f * u)[k] = \sum_{r=0}^{k} f[r].
\]

Exponential signals in discrete time

- Real-valued exponential (geometric) signals have the form \( x[k] = A \gamma^k, k \in \mathbb{Z} \), where \( A, \gamma \in \mathbb{R} \).

- Consider the scalar \( z = \gamma e^{j \Omega} \in \mathbb{C} \) with \( \gamma > 0, \Omega \in \mathbb{R} \), where again \( j := \sqrt{-1} \).

- Generally, complex-valued exponential signals have the (polar) form

\[
x[k] = A e^{j \phi} z^k = A \gamma^k e^{j(\Omega k + \phi)}, k \in \mathbb{Z},
\]

where w.l.o.g. we can take

\[
-\pi < \Omega, \phi \leq \pi \quad \text{and real} \quad A > 0.
\]

- **Exercise:** Show this complex-valued exponential is periodic if and only if \( \Omega/\pi \) is rational.

- By the Euler-De Moivre identity,

\[
x[k] = A \gamma^k e^{j(\Omega k + \phi)} = A \gamma^k \cos(\Omega k + \phi) + j A \gamma^k \sin(\Omega k + \phi), k \in \mathbb{Z}.
\]
Systems - single input, single output (SISO)

In the figure, \( f \) is an input signal that is being transformed into an output signal, \( y \), by the depicted system (box).

To emphasize this functional transformation, and clarify system properties, we will write the output signal (i.e., system “response” to the input \( f \)) as

\[
y = Sf,
\]

where, again, we are making a statement about functional equivalence:

\[
\forall k \in \mathbb{Z}, \quad y[k] = (Sf)[k].
\]

Again, \( Sf \) is not \( S \) “multiplied by” \( f \), rather a functional transformation of \( f \).

SISO systems (cont)

The \( n \) signals \( \{x_1, x_2, ..., x_n\} \) are the internal states of the system.

The states can be taken as outputs of unit-delay operators, \( \Delta \), i.e.,

\[
\forall k \in \mathbb{Z}, \quad (\Delta y)[k] = y[k-1].
\]

Some properties of systems are as in continuous time: e.g., linear, time invariant, causal, memoryless, stable (with different conditions for stability as we shall see).
Difference equation for an discrete time, LTI, SISO system

• For linear and time-invariant systems in discrete time, relate output $y$ to input $f$ via difference equation in standard (time-advance operator) form:

$$\forall k \geq -n, \quad y[k + n] + a_{n-1}y[k + n - 1] + \ldots + a_1y[k + 1] + a_0y[k] = b_m f[k + m] + b_{m-1}f[k + m - 1] + \ldots + b_1 f[k + 1] + b_0 f[k],$$

given

– scalars $a_k$ for $0 \leq k \leq n$, with $a_n := 1$, and scalars $b_k$ for $0 \leq k \leq m$,
– $a_0 \neq 0$ or $b_0 \neq 0$ (so that $P, Q$ are of minimal degree), and
– initial conditions $y[-n], y[-n + 1], \ldots, y[-2], y[-1]$.

• Compact representation of the above difference equation:

$$Q(\Delta^{-1})y = P(\Delta^{-1})f,$$

where polynomials

$$Q(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad P(z) = \sum_{k=0}^{m} b_k z^k,$$

and $\Delta^{-1}$ is the unit time-advance operator: $(\Delta^{-1}y)[k] \equiv y[k+1], (\Delta^{-r}y)[k] \equiv y[k+r]$

Discussion: conditions for causality and difference equation in $\Delta$

• Exercise: Show that the difference equation $Q(\Delta^{-1})y = P(\Delta^{-1})f$ is not causal if $\deg(P) = m > n = \deg(Q)$, i.e., the system is not proper.

• An anti-causal difference equation can be implemented simply using memory to store a sliding window of prior values of the input $f$ and delaying the output.

• Example: Decoding B (bidirectional) frames of MPEG video.
Numerical solution to difference equation by recursive substitution

- Given the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \), the input \( f[k] \) for \( k \geq 0 \), and initial conditions \( y[-n], ..., y[-1] \),

- one can recursively solve for \( y \) (\( y[k] \) for \( k \geq 0 \)) by rewriting the system equation as

\[
y[k + n] = \sum_{r=0}^{n-1} \alpha_r y[k + r] + \sum_{r=0}^{m} b_r f[k + r] \quad \text{for} \quad k \geq -n
\]

\[
\Rightarrow y[k] = \sum_{r=0}^{n-1} \alpha_r y[k + r - n] + \sum_{r=0}^{m} b_r f[k + r - n] \quad \text{for} \quad k \geq 0.
\]

- For example, the difference equation in standard form,

\[
y[k + 1] + 3y[k] = 7f[k + 1] \quad \text{for} \quad k \geq -1,
\]

can be rewritten as

\[
y[k] = -3y[k - 1] + 7f[k] \quad \text{for} \quad k \geq 0.
\]

- So, given \( f \) and \( y[-1] \) we can recursively compute

\[
y[0] = -3y[-1] + 7f[0], \quad y[1] = -3y[0] + 7f[1], \quad y[2] = -3y[1] + 7f[2], \quad \text{etc.}
\]

- Exercise: If \( f = u \) and \( y[-1] = 7 \) then find \( y[3] \) for this example.

Approach to closed-form solution: ZIR and ZSR

- The total response \( y \) of \( P(\Delta^{-1})f = Q(\Delta^{-1})y \) to the given initial conditions and input \( f \) is a sum of two parts:

  - the ZSR, \( y_{ZS} \), which solves

  \[
P(\Delta^{-1})f = Q(\Delta^{-1})y_{ZS} \quad \text{with zero i.c.'s, i.e., with} \quad 0 = y[-n] = ... = y[-1];
\]

  - the ZIR, \( y_{ZI} \), which solves

  \[
  0 = Q(\Delta^{-1})y_{ZI} \quad \text{with the given initial conditions.}
\]

- The total response \( y \) of the system to \( f \) and the given initial conditions is, by linearity,

\[
y = y_{ZI} + y_{ZS}.
\]

- We will determine the ZIR by finding the characteristic modes of the system.

- We will determine the ZSR by convolution of the input with the (zero state) unit-pulse response, the latter also in terms of characteristic modes.
• Consider again the difference equation:
  \[ \forall k \geq -1, \quad y[k + 1] + 3y[k] = 7f[k + 1], \]

• i.e., \( Q(z) = z + 3 \) with degree \( n = 1 \), and \( P(z) = 7z \) with degree \( m = 1 \),

• Exercise: Show that the following system corresponds to this difference equation.

\[
\begin{array}{cccc}
& f & \rightarrow & 7 \\
& \Delta & -3\Delta y & \rightarrow & y \\
\end{array}
\]

• By recursive substitution, the total response is, \( \forall k \geq -1 \):

\[
y[k] = -3y[k - 1] + 7f[k] = -3(-3y[k - 2] + 7f[k - 1]) + 7f[k] = (-3)^2y[k - 2] - 3 \cdot 7f[k - 1] + 7f[k] = ...
\]

\[
= (-3)^{k+1}y[-1] + \sum_{r=0}^{k} 7(-3)^{k-r}f[r]
\]

\[
= (-3)^{k+1}y[-1] + \sum_{r=0}^{\infty} h[k - r] f[r] =: (-3)^{k+1}y[-1] + (h \ast f)[k],
\]

• where \( h[k] := 7(-3)^k u[k] \) is the (zero state) unit-pulse response,

• \( y[-1] \) is the given \( (n = 1) \) initial condition, and

• we have defined the discrete-time convolution operator with \( \sum_{r=0}^{\infty} (...) := 0 \).
• Exercise: Prove by induction this expression for $y[k]$ for all $k \geq -1$.

• Exercise: Prove convolution is commutative: $h \ast f = f \ast h$.

• So, we can write the total response $y = y_{ZI} + y_{ZS}$ starting from the time of oldest initial condition:

$\forall k \geq -1, \quad y_{ZI}[k] = (-3)^{k+1}y[-1]$ 
$\forall k \geq -1, \quad y_{ZS}[k] = u[k] \sum_{r=0}^{k} 7(-3)^{k-r}f[r] = u[k](h \ast f)[k]$ 

where $y_{ZS}[k] = 0$ when $k < 0$.

• Obviously, this example involves a linear, time-invariant and causal system as described by the difference equation above.

Total response - discussion

• Note that in CMPSC 360, we don’t restrict our attention to linear and time-invariant difference equations.

• We use recursive substitution to guess at the form of the solution and then verify our guess by an inductive proof.

• In this course, we will describe a systematic approach to solve any LTIC difference equation,

• i.e., to solve for the output of a DT-LTIC system given the input and initial conditions.

• And again as in continuous time, we will see important insights about discrete-time signals and LTIC systems through frequency-domain representations and analysis.
ZIR - the characteristic values

- Note that \( \forall k, \Delta^{-r} z^k = z^{k+r} = z^r z^k \), i.e., the \( r \)-units time-advance operator, \( \Delta^{-r} \), is replaced by the scalar \( z^r \) for all \( r \in \mathbb{Z} \).

- Our objective is to solve for the ZIR, i.e., solve
  \[
  Q(\Delta^{-1}) y \equiv 0 \quad \text{given } y[-n], y[-n+1], ..., y[-2], y[-1].
  \]

- Note that exponential (or “geometric”) functions, \( \{ z^k \mid k \in \mathbb{Z} \} \) for \( z \in \mathbb{C} \), are eigenfunctions of time-shift operators of the form \( Q(\Delta^{-1}) \) for a polynomial \( Q \).

- That is, for any non-zero scalar \( z \in \mathbb{C} \), if we substitute \( y[k] = z^k \forall k \in \mathbb{Z} \) we get:
  \[
  \forall k \in \mathbb{Z}, \quad (Q(\Delta^{-1}) y)[k] = Q(\Delta^{-1}) z^k = Q(z) z^k.
  \]

- So, to solve \( Q(z) z^k \equiv 0 \) for all time \( k \geq 0 \), when \( z \neq 0 \) we require
  \[
  Q(z) = 0, \quad \text{the characteristic equation of the system.}
  \]

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ZIR - the characteristic values (cont)

- If \( z \) is a root of the characteristic polynomial \( Q \) of the system, then
  - \( z \) would be a characteristic value of the system, and
  - the signal \( \{ z^k \}_{k \geq 0} \) is a characteristic mode of the system when \( z \neq 0 \), i.e.,
    \[
    Q(\Delta^{-1}) z^k = 0, \quad \forall k \geq 0.
    \]

- Since \( Q \) has degree \( n \), there are \( n \) roots of \( Q \) in \( \mathbb{C} \), each a system characteristic value.
• Let \( n' \leq n \) be the number of non-zero roots of \( Q \), i.e., \( \tilde{Q}(z) = Q(z)/z^{n-n'} \) is a polynomial satisfying \( \tilde{Q}(0) \neq 0 \).

• Though there may be some repeated roots of the characteristic polynomial \( Q \), there will always be \( n' \) different, linearly independent characteristic modes, \( \mu_k \), i.e.,

\[
\forall k \geq -n, \quad \sum_{r=1}^{n'} c_r \mu_r[k] = 0 \iff \forall r, \text{ scalars } c_r = 0.
\]

• When \( n = n' \), by system linearity, we will be able to write

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{r=1}^{n} c_r \mu_r[k],
\]

for scalars \( c_r \in \mathbb{C} \) that are found by considering the given initial conditions

\[
y[k] = \sum_{r=1}^{n} c_r \mu_r[k] \quad \text{for } k \in \{-n, \ldots, -2, -1\},
\]

i.e., \( n \) equations in \( n \) unknowns (\( c_r \)).

• The linear independence of the modes implies linear independence of these \( n \) equations in \( c_r \), and so they have a unique solution.

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**ZIR - the case of different, non-zero, real characteristic values**

• If there are \( n \) different non-zero roots of \( Q \) in \( \mathbb{R} \), \( z_1, z_2, \ldots, z_n \), then there are \( n \) characteristic modes: for \( r \in \{1, 2, \ldots, n\} \),

\[
\forall \text{ time } k, \quad \mu_r[k] = z_r^k.
\]

• Therefore,

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{r=1}^{n} c_r z_r^k.
\]

• The \( n \) unknown scalars \( c_k \in \mathbb{R} \) can be solved using the \( n \) equations:

\[
y[k] = \sum_{r=1}^{n} c_r z_r^k, \quad \text{for } k \in \{-n, -n+1, \ldots, -2, -1\}.
\]
ZIR - the case of different, non-zero, real characteristic values

- **Example**: Consider the difference equation:
  \[ \forall k \geq -3, \quad 2y[k+3] - 10y[k+2] + 12y[k+1] = 3f[k+2], \]
  with \( y[-2] = 1 \) and \( y[-1] = 3 \).

- That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, P(z) = (3/2)z \) and \( m = 1 \).

- So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and the ZIR is
  \[ \forall k \geq -n = -2, \quad y_{ZI}[k] = c_1 3^k + c_2 2^k \]

- Using the initial conditions to find the scalars \( c_1, c_2 \):
  \[
  
  1 = y[-2] = c_1 3^{-2} + c_2 2^{-2} \quad \text{and} \quad 3 = y[-1] = c_1 3^{-1} + c_2 2^{-1}.
  
  \]

- **Exercise**: Now solve for \( c_1 \) and \( c_2 \).

- **Note**: When a coefficient \( c \) is worked out to be zero, it may not be exactly zero in practice, and the corresponding characteristic mode \( z^k \) will increasingly contribute to ZIR \( y_{ZI} \) over time if \( |z| > 1 \) (i.e., an “unstable” mode in discrete time).

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ZIR - the case of not-real characteristic values

- The characteristic polynomial \( Q \) may have non-real roots, but such roots come in complex-conjugate pairs because \( Q \)'s coefficients \( a_k \) are all real.

- For example, if the characteristic polynomial is
  \[ Q(z) = (z - 1)(z^2 - 2z - 2) \]
  then the characteristic values (\( Q \)'s roots) are
  \[ -1, \quad 1 \pm j\sqrt{3} \quad \text{again recalling } j = \sqrt{-1}. \]

- Because we have three different characteristic values \( \in \mathbb{C} \), we can specify three corresponding characteristic modes,
  \[ (-1)^k, (1 + j\sqrt{3})^k, (1 - j\sqrt{3})^k, \forall k \geq 0, \]
  and construct the ZIR as
  \[ \forall k \geq -n = -3, \quad y_{ZI}[k] = c_1 (-1)^k + c_2 (1 + j\sqrt{3})^k + c_3 (1 - j\sqrt{3})^k \]
  \[ = c_1 (-1)^k + c_2 2^k e^{k j\pi/3} + c_3 2^k e^{-k j\pi/3} \]
  where
  - \( c_1 \in \mathbb{R} \) and \( c_2 = c_3 \in \mathbb{C} \) so that \( y_{ZI} \) is real-valued, and again,
  - these scalars are determined by the \( n = 3 \) given (real) initial conditions: \( y[-3], y[-2], y[-1] \).
ZIR - not-real characteristic values with real characteristic modes

• By the Euler-De Moivre identity for the previous example,

\[ y_{ZI}[k] = c_1(-1)^k + (c_2 + c_3)2^k \cos(k\pi/3) + j(c_2 - c_3)2^k \sin(k\pi/3) \]

\[ = c_1(-1)^k + 2\text{Re}\{c_2\}2^k \cos(k\pi/3) - 2\text{Im}\{c_2\}2^k \sin(k\pi/3) \]

• Again, because all initial conditions are real and \( Q \) has real coefficients, \( y_{ZI} \) is real valued and so \( c_3 = \overline{c_3} \Rightarrow c_2 + c_3, j(c_2 - c_3) \in \mathbb{R} \).

• In general, consider two complex conjugate characteristic values \( \nu \pm j\eta \) corresponding to two complex-valued characteristic modes \( |z|^k e^{\pm jk\angle z} \), where \( |z| = \sqrt{\nu^2 + \eta^2} \) and \( \angle z = \arctan(\eta/\nu) \).

• One can use Euler’s identity to show that the corresponding real-valued characteristic modes are

\[ |z|^k \cos(k\angle z), |z|^k \sin(k\angle z) \]

ZIR - the case of repeated characteristic values

• Consider the case where at least one characteristic value is of order > 1, i.e., there are repeated roots of the characteristic polynomial, \( Q \).

• For example, \( Q(z) = (z + 0.75)^3(z - 0.5) \) has a triple (twice repeated) root at \(-0.75\) and a single root at \(0.5\).

• Again, \( \{-0.75\}^k \) is a characteristic mode because \( Q(\Delta^{-1})(-.75)^k \equiv 0 \) follows from

\[ (\Delta^{-1} + .75)(-.75)^k = \Delta^{-1}(-.75)^k + .75(-.75)^k \]

\[ = (-.75)^{k+1} + .75(-.75)^k \]

\[ = 0. \]

• Similarly, \( \{0.5\}^k \) is a characteristic mode since \( (\Delta^{-1} - 0.5)(0.5)^k \equiv 0 \).

• Also, \( \{k(-.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})k(-.75)^k \equiv 0 \) follows from

\[ (\Delta^{-1} + .75)^2k(-.75)^k \]

\[ = (\Delta^{-2} + 1.5\Delta^{-1} + (.75)^2)k(-.75)^k \]

\[ = \Delta^{-2}k(-.75)^k + 1.5\Delta^{-1}k(-.75)^k + (.75)^2k(-.75)^k \]

\[ = (k + 2)(-.75)^{k+2} + 1.5(k + 1)(-.75)^{k+1} + (.75)^2k(-.75)^k \]

\[ = (-.75)^{k+2}((k + 2) - 2(k + 1) + k) \]

\[ = 0. \]
ZIR - the case of repeated characteristic values (cont)

• Similarly, \(k^2(-0.75)^k\) is also a characteristic mode because 
  \((\Delta^{-1} + 0.75)^3k^2(-0.75)^k = 0\).

• Note that without three such linearly independent characteristic modes 
  \(\{(-0.75)^k, k(-0.75)^k, k^2(-0.75)^k \mid k \geq 0\}\)
  for the twice-repeated (triple) characteristic value -.75, the initial conditions will create an 
  "overspecified" set of \(n\) equations involving fewer than \(n\) "unknown" coefficients \(c_k\) of 
  the linear combination of modes forming the ZIR.

• For this example,
  
  \[y_{ZI}[k] = c_0(-0.75)^k + c_1 k(-0.75)^k + c_2 k^2(-0.75)^k + c_3(0.5)^k, \quad k \geq -4.\]

• If the given initial conditions are, say, 
  
  \[y[-4] = 12, \quad y[-3] = 6, \quad y[-2] = -5, \quad y[-1] = 10,\]
  the four equations to solve for the four unknown coefficients \(c_k\) are:

  \[y_{ZI}[-4] = (-0.75)^{-4}c_0 + (-4)(-0.75)^{-4}c_1 + (-4)^2(-0.75)^{-4}c_2 + (.5)^{-4}c_3 = 12\]
  \[y_{ZI}[-3] = (-0.75)^{-3}c_0 + (-3)(-0.75)^{-3}c_1 + (-3)^2(-0.75)^{-3}c_2 + (.5)^{-3}c_3 = 6\]
  \[y_{ZI}[-2] = (-0.75)^{-2}c_0 + (-2)(-0.75)^{-2}c_1 + (-2)^2(-0.75)^{-2}c_2 + (.5)^{-2}c_3 = -5\]
  \[y_{ZI}[-1] = (-0.75)^{-1}c_0 + (-1)(-0.75)^{-1}c_1 + (-1)^2(-0.75)^{-1}c_2 + (.5)^{-1}c_3 = 10\]

ZIR - general case of repeated, non-zero characteristic values

• In general, a set of \(r\) linearly independent modes corresponding to a non-zero characteristic 
  value \(z \in \mathbb{C}\) repeated \(r - 1\) times are 
  
  \[k^{r-1}z^k, k^{r-2}z^k, ..., kz^k, z^k, \quad \text{for } k \geq 0.\]

• Also, if \(v \pm jq\) are characteristic values repeated \(r - 1\) times, with \(v, q \in \mathbb{R}\) and \(q \neq 0\), 
  we can use the \(2k\) real-valued modes 
  \[k^a|z|^k \cos(\angle z), k^a|z|^k \sin(\angle z), \quad \text{for } a \in \{0, 1, 2, ..., r - 1\},\]
  where \(|z| = \sqrt{v^2 + q^2}\) and \(\angle z = \arctan(q/v)\).
ZIR - when some characteristic values are zero

- Again let $n' \leq n$ be the number of non-zero roots of $Q$ (characteristic values),
- i.e., $r := n - n' \geq 0$ is the order (1+repetition) of the characteristic value 0, and
- $r \geq 0$ is the smallest index such that (the coefficient of $Q$) $a_r \neq 0$.
- So, there is a polynomial $\tilde{Q}$ such that $Q(z) = z^r \tilde{Q}(z)$ and $\tilde{Q}(0) \neq 0$.
- Because the constant signal zero cannot be a characteristic mode, we add $r = n - n'$
  time-advanced unit-pulses:
  \[
  \forall k \geq -n, \ y_{ZI}[k] = \sum_{i=1}^{r} C_i \delta[k + i] + y_N[k] \\
  = C_r \delta[k + r] + C_{r-1} \delta[k + r - 1] + \ldots + C_1 \delta[k + 1] + y_N[k]
  \]
  where $y_N$ is a “natural response” (linear combination of $n'$ characteristic modes).
- The $n$ initial conditions are then met by the $r$ coefficients $C_i$ of the advanced unit pulses
  together with the $n' = n - r$ coefficients of the characteristic modes in $y_N$.

ZIR - when some characteristic values are zero - example

- Consider a fourth-order system with characteristic polynomial
  $Q(z) = z^2(z + 1)^2$.
- Thus the poles are 0, $-1$ each repeated and the (non-zero) characteristic modes are
  $(-1)^k, k(-1)^k$.
- So, the ZIR is, for $k \geq -4$:
  \[
  y_{ZI}[k] = C_2 \delta[k + 2] + C_1 \delta[k + 1] + c_1(-1)^k + c_2 k(-1)^k
  \]
- That is, the ZIR has four unknown coefficients $C_2, C_1, c_1, c_2$ to account for the four (given)
  initial conditions $y[-4], y[-3], y[-2], y[-1]$. 
Zero State Response - the unit-pulse response

- Recall the LTIC system
  \[ \sum_{r=0}^{n} a_r \Delta^{-r} y = Q(\Delta^{-1})y = P(\Delta^{-1})f = \sum_{r=0}^{m} b_r \Delta^{-r} f \]
  with \( a_n = 1 \), \( a_0 \neq 0 \) or \( b_0 \neq 0 \), \( m \leq n \).

- We can express any input signal
  \[ f[k] = \sum_{r=0}^{\infty} f[r] \delta[k-r] \quad \forall k \geq 0, \quad \text{i.e., } f = f * \delta. \]

- So the unit pulse \( \delta \) is the identity of the convolution operator in discrete time.

- Thus, by LTI, the ZSR \( y_{ZS} \) is the convolution of input \( f \) and ZSR \( h \) to unit pulse \( \delta \),
  \[ y_{ZS}[k] = \sum_{r=0}^{\infty} f[r] h[k-r] = (f * h)[k], \quad \forall k \geq 0, \]

- \( h \) is called the unit-pulse response of the LTIC system, i.e.,
  \[ Q(\Delta^{-1})h = P(\Delta^{-1})\delta \quad \text{s.t. } h[k] = 0 \quad \forall k < 0. \]

Computing an LTIC system’s unit-pulse response, \( h \)

- For the LTIC system in standard form, if \( a_0 \neq 0 \) then
  \[ h = (b_0/a_0) \delta + y_N u \]
  where \( y_N \) is a natural response of the system (linear combination of characteristic modes).

- Note that \( h[k] = 0 \) for all \( k < 0 \) owing to the unit step \( u \).

- The \( n \) scalars of the natural response \( y_N \) component of \( h \) are solved using
  \[ (Q(\Delta^{-1})h)[k] = (P(\Delta^{-1})\delta)[k] \quad \text{for } k \in \{-n,-n+1,...,-2,-1\} \]
Unit-pulse response when zero is a characteristic value

- If \( r \geq 0 \) is the smallest index such that \( a_r \neq 0 \) (0 is a char. mode of order \( r \)), then may need to add \( r \) delayed unit-pulse terms to \( h \):

\[
h = \sum_{i=0}^{r-1} A_i \Delta^i \delta + (b_0 / a_r) \Delta^r \delta + y_n u,
\]

where
- by definition of the standard form of the difference equation, if \( r > 0 \), \( a_0 = 0 \) so \( b_0 \neq 0 \), and
- \( r \leq n \) since \( 0 \neq a_n \) := 1.

- So if \( r = 0 \) (i.e., \( a_0 \neq 0 \)), then \( A_0 = b_0 / a_0 \) as above, where \( \sum_{i=0}^{-1}(...) := 0 \).

- Exercise: Prove \( A_r = b_0 / a_r \) for \( 0 \leq r \leq n \).

- Thus, zero is a characteristic value of degree \( r \geq 0 \), and

- there are \( r \) characteristic modes that will all be zero.

- The additional unit-pulse terms introduce \( r \) degrees of freedom in the form of the coefficients \( A_0, A_1, ..., A_{r-1} \) to accommodate the \( n = r + n' \) initial conditions of the unit-pulse response: \( h[-n] = h[-n+1] = ... = h[-2] = h[-1] = 0 \).

Computing the ZSR - example 1

- Recall that the difference equation \( y = 7f - 3\Delta y \) corresponds to the above system; in standard form:

\[
\forall k \geq -1, \quad y[k+1] + 3y[k] = 7f[k+1].
\]

with \( Q(z) = z + 3 \), \( P(z) = 7z \) and \( n = 1 = m \).

- Since the system characteristic value is \( -3 \) and \( b_0 = 0 \), the (zero state) unit-pulse response has the form \( h[k] = c(-3)^k u[k] \).

- The scalar \( c \) is solved by evaluating the above difference equation at time \( k = -1 \):

\[
(Q(\Delta^{-1})h)[-1] = (P(\Delta^{-1})\delta)[-1]
\]

\[
i.e., \quad h[0] + 3h[-1] = 7\delta[0]
\]

\[
\Rightarrow c + 3 \cdot 0 = 7 \cdot 1, \quad c = 7
\]
Computing the ZSR - example 1 (cont)

• So, \( h[k] = 7(-3)^ku[k] \).
• If the input is \( f[k] = 4(0.5)^ku[k] \), the system ZSR is, for all \( k \geq 0 \),

\[
yzs[k] = \sum_{r=0}^{k} h[r]f[k-r] = \sum_{r=0}^{k} 7(-3)^r 4(0.5)^{k-r}
\]

\[
= 28(0.5)^k \sum_{r=0}^{k} (-6)^r = 28(0.5)^k \frac{(-6)^{k+1} - 1}{-6 - 1} u[k]
\]

\[
= (24(-3)^k + 4(0.5)^k)u[k].
\]

• Note how the ZIR \( y_{ZI} \) has a term that is a characteristic mode (excited by the input \( f \)) and a term that is proportional to the input \( f \) (this forced response is an eigenresponse).

• Exercise: For the difference equation, \( y[k+1] + 3y[k] = 7f[k] \) \( \forall k \geq -1 \); draw the block diagram, show that \( h[k] = 21(-3)^{k-1}u[k] + (7/3)\delta[k] \), and find the ZSR to the above input \( f \).

• Exercise: Read “sliding tape” method to compute convolution in Lathi, p. 595.

Computing the unit pulse response - example 2

• Find the ZSR of the following system to input \( f[k] = 2(-5)^ku[k] \):

\[
f \rightarrow + \rightarrow \Delta \rightarrow y
\]

• Exercise: show the difference equation for this system (in direct canonical form) is:

\[
\forall k \geq 0, \quad y[k+2] - 5y[k+1] + 6y[k] = 1.5f[k+1]
\]

• That is, \( Q(z) = z^2 - 5z + 6 = (z-3)(z-2) \) and \( n = 2, P(z) = 1.5z \) and \( m = 1 \).

• So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and \( b_0 = 0 \) so the unit-pulse response

\[
h[k] = (c_13^k + c_22^k)u[k].
\]
Computing the unit pulse response - example 2 (cont)

- To find the constants, evaluate the difference equation at \( k = -1 \):
  \[
  2h[1] - 10h[0] + 12h[-1] = 3\delta[0] \\
  \Rightarrow 2h[1] - 10h[0] = 3 \\
  \Rightarrow (2 \cdot 3 - 10 \cdot 1)c_1 + (2 \cdot 2 - 10 \cdot 1)c_2 = 3 \\
  \Rightarrow -4c_1 + -6c_2 = 3
  \]
  and at \( k = -2 \):
  \[
  2h[0] - 10h[-1] + 12h[-2] = 3\delta[-1] \Rightarrow 12h[0] = 0 \Rightarrow h[0] = 0
  \]
  \[
  \Rightarrow c_1 + c_2 = 0.
  \]

- Thus, \( c_2 = -1.5 = -c_1 \) so that \( h[k] = (-1.5(3)^k + 1.5(2)^k)u[k] \) and for \( k \geq 0 \)
  \[
  y_{ZS}[k] = (h * f)[k] = \sum_{r=0}^{k} h[r]f[k-r].
  \]

- **Exercise:** Write the ZSR as a sum of system modes \( 2^k \) and \( 3^k \) and a (force) term like the input, here taken as \( f[k] = 4(-5)^k u[k] \).

---

Convolution - other important properties

- Again, for a LTI system with impulse response \( h \) and input \( f \), the ZSR is \( y_{ZS} = f \ast h \), where
  \[
  (f \ast h)[k] = \sum_{r=-\infty}^{\infty} f[r]h[k-r]
  \]

- By simply changing the dummy variable of summation to \( r' = h - r \), can show convolution is commutative: \( f \ast h = h \ast f \).

- One can directly show that convolution \( f \ast h \) is a bi-linear mapping from pairs of signals \( (f, h) \) to signals \( (y_{ZS}) \), consistent with convolution’s commutative property and the (zero state) system with impulse response \( h \) being LTI;

- that is, \( \forall \) signals \( f, g, h \) and scalars \( \alpha, \beta \in \mathbb{C} \),
  \[
  (\alpha f + \beta g) \ast h = \alpha(f \ast h) + \beta(g \ast h)
  \]

- By changing order of summation (Fubini’s theorem), one can easily show that convolution is associative, i.e., \( \forall \) signals \( f, g, h \),
  \[
  (f \ast g) \ast h = f \ast (g \ast h).
  \]
Convolutions - other important properties (cont)

- We’ll use these properties when composing more complex systems from simpler ones.

- By just changing variables of integration, we can show how to exchange time-shift with convolution, i.e., \( \forall \) signals \( f, h : \mathbb{Z} \to \mathbb{C} \) and times \( k \in \mathbb{Z}, \)

\[
(\Delta^k f) * h = \Delta^k (f * h);
\]

recall how convolution represents the ZSR of linear and time-invariant systems.

- By the ideal sampling property, recall that the identity signal for convolution is the unit pulse \( \delta \), i.e., \( \forall \) signals \( f, \)

\[
f * \delta = \delta * f = f
\]

- **Exercise:** Adapt the proofs of these properties in continuous time to this discrete-time case.

- **Exercise:** In particular, show that if \( f \) and \( h \) are causal signals, then \( y = f * h \) is causal; i.e., if the unit-pulse response \( h \) of a system is a causal signal, then the system is causal.

System stability - ZIR - asymptotically stable

- Consider a SISO system with input \( f \) and output \( y \).

- Recall that the ZIR \( y_{ZI} \) is a linear combination of the system’s characteristic modes, where the coefficients depend on the initial conditions, possibly including some initial unit-pulse terms if zero is a characteristic value (system pole).

- A system is said to be asymptotically stable if for all initial conditions,

\[
\lim_{{k \to \infty}} y_{ZI}[k] = 0.
\]

- So, a system is asymptotically stable if and only if all of its characteristic values have magnitude less than 1.
• If the characteristic polynomial \( Q(z) = (z - 0.5)(z^2 + 0.0625) \), then

• the system’s characteristic values (roots of \( Q \)) are 0.5, \( \pm 0.25j \) each with magnitude less than one,

• and the ZIR is of the form,

\[
y_{ZIR}[k] = (c_1(0.5)^k + c_2(0.25j)^k + c_3(-0.25j)^k) u[k] \\
= (c_1(0.5)^k + 2\text{Re}\{c_2\}(0.25)^k \cos(k\pi/2) - 2\text{Im}\{c_2\}(0.25)^k \sin(k\pi/2)) u[k],
\]

• recalling that \( j^k = e^{jk\pi/2} \).

• So, \( y_{ZIR}[k] \to 0 \) as \( k \to \infty \) for all \( c_1, c_2 \) (i.e., for all initial conditions), and

• hence is asymptotically stable.

System stability - bounded signals

• A signal \( y \) is said to be bounded if

\[ \exists M < \infty \text{ s.t. } \forall k \in \mathbb{Z}, |y[k]| \leq M; \]

otherwise \( y \) is said to be unbounded.

• For example, \( y[k] = 0.25(\frac{1+j\sqrt{3}}{2})^k u[k] \) is bounded (can use \( M = 0.25 \)).

• Also, \( 3 \cos(5k) \) is bounded (can use \( M = 3 \)).

• But both \( 2^k \cos(5k) \) and \( 3 \cdot (-2)^k \) are unbounded.
A system is said to be marginally stable if it is not asymptotically stable but \( y_{ZI} \) is always (for all initial conditions) bounded.

A system is marginally stable if and only if

- it has no characteristic values with magnitude strictly greater than 1,
- it has at least one characteristic value with magnitude exactly 1, and
- all magnitude-1 characteristic values are not repeated.

That is, a marginally stable system has

- some characteristic modes of the form \( \cos(\Omega k) \) or \( \sin(\Omega k) \),
- while the rest of the modes are all of the form \( k^r |z|^k \cos(\Omega k) \) or \( k^r |z|^k \sin(\Omega k) \), with \( |z| < 1 \) and integer degree \( r \geq 0 \).

Exercise: Explain why we can take \( \Omega \in (-\pi, \pi] \) without loss of generality.

Note: the dimension of \( \Omega \), \([\Omega]=\text{radians}\).

The characteristic polynomial is \( Q(z) = z(z^2 + 1)(z - 0.25) \) gives characteristic values 0, 0.25, \( \pm j \).

then the system is marginally stable with modes \( (0.25)^k \cos(k\pi/2), \sin(k\pi/2) \),

the last two of which are bounded but do not tend to zero as time \( k \to \infty \).
System stability - ZIR - unstable

- A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.

- For example, the system with $Q(z) = (z^2 - 0.5)(z + 3)$ is unstable owing to the characteristic value $-3$ with unbounded mode $(-3)^k$.

- For another example, if the characteristic polynomial is $Q(z) = (z^2 + 1)^2(z - 0.5)$ then the purely imaginary characteristic values $\pm j$ are repeated, and hence the two additional modes $k \sin(k\pi/2), k \cos(k\pi/2)$ are unbounded, so this system is unstable.

- Similarly, if $Q(z) = (z^2 - 1)^2(z - 0.5)$ then the characteristic values $\pm 1$ are repeated and the modes $k$ and $k(-1)^k$ are unbounded, so this system is unstable too.

ZIR stability - stability of poles

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• A SISO system is said to be *Bounded Input, Bounded Output* (BIBO) stable if \( \forall \) bounded input signals \( f \), the ZSR \( y_{ZS} \) is bounded.

• A sufficient condition for BIBO stability is absolute summability of the unit-pulse response,

\[
\sum_{k=0}^{\infty} |h[k]| < \infty.
\]

• To see why: If the input \( f \) is bounded (by \( M_f \) with \( 0 \leq M_f < \infty \)) then \( \forall k \geq 0 \):

\[
|y_{ZS}[k]| = |(f \ast h)[k]| = \left| \sum_{r=0}^{k} f[k-r]h[r] \right| \leq \sum_{r=0}^{k} |f[k-r]| |h[r]| \leq M_f \sum_{r=0}^{\infty} |h[r]| =: M_y < \infty.
\]

System stability - ZSR - BIBO stable

• The condition of absolute summability of the unit-pulse response,

\[
\sum_{r=0}^{\infty} |h[r]| < \infty,
\]

is also necessary for, and hence equivalent to, BIBO stability.

• If any component characteristic mode of \( h \) is unbounded, then \( h \) will not to be absolutely summable.

• Thus, if the system (ZIR) is asymptotically stable it will be BIBO stable; the converse is also true.
ZSR - the transfer function, \( H \)

- Recall that for any polynomial \( Q \) and \( z \in \mathbb{C} \) (including \( s = jw, \ w \in \mathbb{R} \)),
  \[
  Q(\Delta^{-1})z^k = Q(z)z^k, \ \forall k \geq 0.
  \]

- So, if we guess that a "particular" solution of the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \) with input \( f[k] = Az^ku[k] \) is of the form \( y_0[k] = AH(z)z^k = H(z)f[k], \ k \geq 0 \), then we get by substitution that \( \forall k \geq 0, z \in \mathbb{C} \),
  \[
  (Q(\Delta^{-1})y_0)[k] = (P(\Delta^{-1})f)[k] \Rightarrow AH(z)Q(z)z^k = AP(z)z^k \Rightarrow H(z) = P(z)/Q(z).
  \]

- The "rational polynomial" \( H = P/Q \) is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

- So, the ZSR (forced response + characteristic modes) would be of the form:
  \[
  y_{ZS}[k] = (AH(z)z^k + \text{linear combination of char. modes})u[k].
  \]

- Recall that for the example with \( Q(z) = z + 3 \) and \( P(z) = 7z \), we computed the unit-pulse response \( h[k] = 7(-3)^ku[k] \) and the ZSR to input \( f[k] = 4(0.5)^ku[k] \) as \( y_{ZS}[k] = (24(-3)^k + 4(0.5)^k)u[k] \).

- Here, note that \( H(0.5) = P(0.5)/Q(0.5) = 1 \), i.e., the forced response component of \( y_{ZS} \) is \( H(0.5)f[k] = 1 \cdot 4(0.5)^ku[k] = 4(0.5)^ku[k] \).

---

**ZSR - unit-pulse response \( h \), transfer function \( H \), and eigenresponse**

- \( y_{ZS}[k] = (H(z)Az^k + \text{linear combination of char. modes})u[k] \) is the ZSR to input \( f[k] = Az^ku[k] \), where \( H(z) = P(z)/Q(z) \).

- The eigenresponse is a special case of the forced response for exponential inputs.

- If \( |z| = 1 \), i.e., \( z = e^{j\Omega} \) for some \( \Omega \in (-\pi, \pi) \) (w.l.o.g.), and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:
  \[
  y[k] \rightarrow AH(e^{j\Omega})e^{j\Omega k} \text{ as } k \rightarrow \infty.
  \]

- Since \( y = h * f \), we get that as \( k \rightarrow \infty \) for a LTIC and asymptotically stable system,
  \[
  y_{ZS}[k] = \sum_{r=0}^{k} h[r]Ae^{j\Omega(k-r)} = Ae^{j\Omega k}H(e^{j\Omega})
  \Rightarrow \sum_{r=0}^{\infty} h[r]e^{-j\Omega r} = H(e^{j\Omega}), \ \forall \Omega \in (-\pi, \pi).
ZSR - transfer function \( H \) and eigenresponse (cont)

\[
e^{j\Omega k} \quad \rightarrow \quad H \equiv \frac{P}{Q} \quad \rightarrow \quad H(e^{j\Omega k})e^{j\Omega k}
\]

- The LTI system transfer function \( H \) is the Discrete-Time Fourier Transform (DTFT) of the system unit-pulse response \( h \):

\[
\forall \Omega \in \mathbb{R}, \quad H(e^{j\Omega}) = \sum_{r=0}^{\infty} h[r]e^{-j\Omega r}.
\]

- Note that \( H(e^{j\Omega}) \) is periodic since \( H(e^{j\Omega}) \equiv H(e^{j\Omega+2\pi k}) \) for any integer \( k \).

- For the \( z \)-transform (and DTFS) we will use this notation for \( H \), but for the DTFT we will instead write \( H(\Omega) \).

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Frequency-domain methods for discrete-time signals

- Discrete-Time Fourier Series (DTFS) of periodic signals
- Discrete-Time Fourier Transform (DTFT)
- Sampled data systems
  - DFT & FFT
- \( z \)-transform for (complete) transient response
- Eigenresponse
- Canonical system realization of a difference equation
Discrete-time Fourier series of periodic signals

- For all \( r, N \in \mathbb{Z} \), note that the signal \( \{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \} \) “repeats itself” every \( N > 0 \) units of (discrete) time \( k \), in particular
  \[
  \forall r \in \mathbb{Z}, \quad \exp\left(jr \frac{2\pi}{N} k\right)_{k=0} = 1 = \exp\left(jr \frac{2\pi}{N} k\right)_{k=N}.
  \]
- Also the signals \( \{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \} \equiv \{ \exp(jr' \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \} \) whenever \( r' = r \mod N \).
- Suppose \( N \) is the period of periodic signal \( x = \{ x[k] \mid k \in \mathbb{Z} \} \) and \( \Omega_0 = 2\pi/N \) be the fundamental “frequency” of \( x \) (recall \( [\Omega_0] = \) radians).
- We can write \( x \) as a Discrete-Time Fourier Series (DTFS):
  \[
  \forall k \in \mathbb{Z}, \quad x[k] = \sum_{r=0}^{N-1} D_r \exp(jr \Omega_0 k).
  \]
  where \( r \) indexes \( x \)'s \( N \) harmonics.
- Note that the DTFS can also be written for any discrete-time signal \( x : A \rightarrow \mathbb{R} \) defined over any finite interval of time, e.g., \( A = \{ 0, 1, 2, \ldots, N - 1 \} \) or \( A = \{-N, -N + 1, \ldots, -1\} \) for integer \( N < \infty \).

Discrete-time Fourier series of periodic signals (cont)

- Consider the \( N \) signals \( \xi_r[k] := \exp(jr \Omega_0 k) \) over any time-interval \( A \) of length \( N \).
- Equivalently consider these \( N \) signals \( \xi_r \) as \( N \)-vectors in \( \mathbb{R}^N \), i.e., the \( k \)th entry of vector \( \xi_r \) is \( \xi_r[k] \).
- If these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are linearly independent, then they will form a basis spanning all other signals \( x : A \rightarrow \mathbb{R} \), equivalently all other vectors \( x \in \mathbb{R}^N \),
  - i.e., any such \( x \) can be written as a linear combination of the \( \{ \xi_r \}_{r=0}^{N-1} \) giving the DTFS of \( x \):
    \[
    x_r = \sum_{r=0}^{N-1} D_r \xi_r.
    \]
- If we show that these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are orthogonal then
  - linear independence follows
  - the \( r \)th coordinate \( D_r \) (DTFS coefficients) is found by simply projecting \( x \) onto the vector \( \xi_r \):
    \[
    D_r = \langle x, \xi_r \rangle / ||\xi_r||^2.
    \]
DTFS - coefficients (cont)

• Consider any period of \( x : \mathbb{Z} \to \mathbb{R} \), say \( \{0, 1, 2, ..., N - 1\} \).

• First note that for any \( v \in \mathbb{Z} \) that is not a multiple of \( N \) (so \( e^{j\nu \Omega_o} = e^{j\nu(2\pi/N)} \neq 1 \)), the geometric series

\[
\sum_{k=0}^{N-1} e^{j\nu \Omega_o k} = \frac{e^{j\nu(2\pi/N)N} - e^{j\nu(2\pi/N)0}}{e^{j\nu(2\pi/N)} - 1} = 0.
\]

• Thus, for any \( r \neq v \in \mathbb{Z} \) such that \( N \nmid (v - r) \), the inner product \( \langle \xi_r, \xi_v \rangle \) is

\[
\langle \{e^{j\nu(2\pi/N)k}\}, \{e^{j\nu(2\pi/N)k}\} \rangle := \sum_{k=0}^{N-1} e^{j(r-v)(2\pi/N)k} = \sum_{k=0}^{N-1} e^{j(r-v)(2\pi/N)k} = 0,
\]

recalling that the inner product is conjugate-linear in the second argument so that \( <x, x> = ||x||^2 \) when \( x \) is \( \mathbb{C} \)-valued.

• So, these signals are orthogonal and the DTFS coefficients of \( N \)-periodic \( x \) are

\[
D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j\nu \Omega_o k} = \frac{\langle x, \{e^{j\nu \Omega_o k}\} \rangle}{||\{e^{j\nu \Omega_o k}\}||^2}, \text{ where } \Omega_o = \frac{2\pi}{N}.
\]

DTFS - checking coefficients

• Let’s now compute the inner product of \( \xi_v \), for any \( v \in \{0, 1, ..., N - 1\} \), with the DTFS of \( N \)-periodic \( x \):

\[
\langle x, \{e^{j\nu \Omega_o k}\} \rangle = \sum_{k=0}^{N-1} x[k] e^{-j\nu \Omega_o k} = \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} D_r e^{j\nu \Omega_o k} e^{-j\nu \Omega_o r} = \sum_{r=0}^{N-1} D_r e^{j(r-v) \Omega_o k} = \sum_{r=0}^{N-1} D_r N \delta(r-v) = D_v N
\]

• Again, we have verified the DTFS coefficients is

\[
D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j\nu \Omega_o k} = \frac{\langle x, \{e^{j\nu \Omega_o k}\} \rangle}{||\{e^{j\nu \Omega_o k}\}||^2}, \text{ where } \Omega_o = \frac{2\pi}{N}
\]

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DTFS - example

Problem:
Identify the DTFS coefficients (if they exist) for
\[ x[k] = 7 \sin(5.7\pi k) + 2 \cos(3.2\pi k), \quad k \in \mathbb{Z}. \]

Solution:

- First note that the two components of \( x \) are periodic, so their sum is periodic. (Why is this so in discrete time?)

- Since \( \sin \) and \( \cos \) have period \( 2\pi \), we can subtract integer multiples of \( 2\pi \) to get
  \[ x[k] = 7 \sin(1.7\pi k) + 2 \cos(1.2\pi k). \]

- \( 1.7\pi k \) is an integer multiple of \( 2\pi \) when (integer) \( k = 20 \), and when \( k = 5 \) for \( 1.2\pi k \), so least common multiple of these periods is \( k = 20 \).

- (Show that one can alternatively find the greatest common divisor of the component frequencies.)

DTFS - example (cont)

- Thus, the period of \( x \) is \( N = 20 \) and the fund. frequ. is \( \Omega_0 = 2\pi/N = 0.1\pi \).

- By Euler’s identity and adding \( 2\pi k \) to the negative exponents,
  \[ x[k] = \frac{7}{2j}e^{1.7\pi k} - \frac{7}{2j}e^{-1.7\pi k} + e^{1.2\pi k} + e^{-1.2\pi k} = -3.5je^{1.7\pi k} + 3.5je^{0.3\pi k} + e^{1.2\pi k} + e^{0.8\pi k}. \]

- So, the DTFS of \( x[k] = \sum_{r=0}^{19} D_r e^{jr.1\pi k} \) with
  \[ D_{17} = -3.5j = 3.5e^{-j\pi/2}, \quad D_3 = 3.5j = 3.5e^{j\pi/2}, \quad D_{12} = 1, \quad \text{and} \quad D_8 = 1; \]
  else \( D_r = 0 \) (incl. the fundamental \( r \in \{1, 19\} \) & DC \( r = 0 \) components).
DTFS - example and exercise

- Example: The DTFS of an even rectangle wave with period $N = 6$ and duty cycle $3$:

$$x[k] = \sum_{\ell = -\infty}^{\infty} \Delta^6\ell (\Delta^{-1}u - \Delta^2u)[k] = \sum_{\ell = -\infty}^{\infty} (u[k + 1 - 6\ell] - u[k - 2 - 6\ell])$$

is

$$= 5 X r = 0 D r e^{j\ell r \Omega_0},$$

where the fund. freq. $\Omega_0 = 2\pi/6$ and, $\forall r \in \mathbb{Z}$,

$$D_r = \frac{1}{6} \sum_{k=-3}^{2} x[k] e^{-j\ell r \Omega_0} = \frac{1}{6} \sum_{k=-1}^{1} 1 \cdot e^{-j(2\pi/6)k} = \frac{1}{6}(1 + 2 \cos(r(2\pi/6)k)).$$

- Exercise: Plot $x[k]$ as a function of time $k$ and plot its (periodic) spectrum:

$$\forall r \in \{0, 1, 2, ..., 5\}, \ell \in \mathbb{Z},$$

$$\hat{X}(r2\pi/6 + 2\pi\ell) = D_r.$$

DTFS - Parseval’s theorem

- The average power of the $N$-periodic discrete-time signal $x$ is

$$P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} x[k]\overline{x[k]};$$

equivalently, the sum could be taken over any interval of length $N \in \mathbb{Z}^{>0}$.

- Substituting the Fourier series of $x$ separately for $x[k]$ and $\overline{x[k]}$ (using a different summation-index variable for each substitution), leads to Parseval’s theorem

$$P_x = \sum_{r=0}^{N-1} |D_r|^2.$$

- Parseval’s theorem can be used to determine the amount of periodic signal $x$’s power resides in a given frequency band $[\Omega_1, \Omega_2] \subset [0, 2\pi]$ radians:

1. determine the harmonics $r\Omega_0$ of $x$ that reside in this band, $i.e.$, integers $r \in [\Omega_1/\Omega_0, \Omega_2/\Omega_0]$ where $x$’s fundamental frequency $\Omega_0 = 2\pi/N$.

2. sum just over these harmonics to get the answer, $\sum_{[\Omega_1/\Omega_0] \leq r \leq [\Omega_2/\Omega_0]} |D_r|^2$. 

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Find the fraction of $x$'s average power in the "frequency" band $[0.4\pi, 1.1\pi]$ radians where
\[
\forall k \in \mathbb{Z}, \ x[k] = \sum_{\nu=-\infty}^{\infty} (3\delta[k - 4\nu] - 4\delta[k - 1 - 4\nu])
\]

**Solution:** $x$ has period $N = 4$ and average power
\[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{4} \sum_{k=0}^{3} |x[k]|^2 = \frac{1}{4}(3^2 + (-4)^2 + 0^2 + 0^2) = \frac{25}{4}
\]

$x$ has fundamental frequency $\Omega_o = 2\pi/N = \pi/2$ radians and discrete-time Fourier coefficients
\[
D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j(r\Omega_o)k} = \frac{1}{4}(3 - 4e^{-j\pi/2}), \ 0 \leq r \leq N - 1 = 3.
\]

The harmonics $r$ of $x$ that reside in $[0.4\pi, 1.1\pi]$ satisfy $0.4\pi \leq r\Omega_o = r\pi/2 \leq 1.1\pi$, i.e., $r \in \{1, 2\}$.

So, by Parseval’s theorem, the answer is $\left( |D_1|^2 + |D_2|^2 \right) / P_x$.

---

**Periodic extensions**

Consider signal $x : \mathbb{Z} \to \mathbb{R}$ having finite support $\{-M, -M + 1, ..., 0, ..., M - 1, M\}$ for $0 < M < \infty$; i.e., $\forall |k| > M$, $x[k] = 0$.

For $N \geq M$, define $2N$-periodic $x^{(N)}$ such that
\[
x^{(N)}[k] = \begin{cases} 
  x[k] & \text{if } |k| \leq M \\
  0 & \text{if } M < |k| \leq N
\end{cases}
\]

$x^{(N)}$ is a periodic extension of the finite-support signal $x$, where again $x^{(N)}$'s period is $2N$ and
\[
\lim_{N \to \infty} x^{(N)} = x.
\]
DTFS of periodic extension leading to DTFT

- For \( r \in \{-N + 1, -N + 2, \ldots, N - 1, N\} \), the DTFS of \( x^{(N)} \) has coefficients

\[
D_r^{(N)} = \frac{1}{2N} \sum_{k=-N+1}^{N} x^{(N)}[k] e^{-jr \frac{2\pi}{2N} k}
\]

\[
= \frac{1}{2N} \sum_{k=-M}^{M} x[k] e^{-jr \frac{2\pi}{2N} k}
\]

\[
= \frac{1}{2N} \sum_{k=-\infty}^{\infty} x[k] e^{-jr \frac{2\pi}{2N} k}
\]

\[
=: \frac{1}{2N} X \left( r \frac{2\pi}{2N} \right),
\]

where the Discrete-Time Fourier Transform (DTFT) of (aperiodic) \( x : \mathbb{Z} \to \mathbb{R} \) is \( X : \mathbb{R} \to \mathbb{C} \):

\[
X(\Omega) := \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} =: (Fx)(\Omega), \quad \Omega \in \mathbb{R}
\]

- Note that Fourier integrals (spectra of discrete-time signals) are periodic, repeating themselves every 2\( \pi \) radians: \( \forall \Omega \in \mathbb{R}, \ell \in \mathbb{Z}, \)

\[
X(\Omega) = X(\Omega + \ell 2\pi).
\]

Inverse DTFT by Fourier Integral

- Thus, \( \forall k \in \mathbb{Z}, \)

\[
x[k] = \lim_{N \to \infty} x^{(N)}[k]
\]

\[
= \lim_{N \to \infty} \sum_{r=-N+1}^{N} D_r^{(N)} e^{jr \frac{2\pi}{2N} k}
\]

\[
= \lim_{N \to \infty} \sum_{r=-N+1}^{N} X \left( r \frac{2\pi}{2N} \right) e^{jr \frac{2\pi}{2N} k} \frac{1}{2N} \frac{2\pi}{2\pi}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega
\]

where the last equality is by Riemann integration with \( 2\pi/(2N) \to d\Omega. \)

- Thus, we have derived the inverse DTFT by Fourier integral of \( X \) giving (aperiodic) \( x \),

\[
\forall k \in \mathbb{Z}, \quad x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega =: (F^{-1}X)[k].
\]

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• If \( x = \delta \) then obviously \( X = 1 \).

• The geometric signal \( x[k] = \gamma^k u[k] \) for scalar \( \gamma \) s.t. \( |\gamma| < 1 \) has DTFT

\[
X(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k = \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{1}{(1 - \gamma) \cos(\Omega) + j\gamma \sin(\Omega)}
\]

• Note that

\[
|X(\Omega)| = \frac{1}{(1 - \gamma \cos(\Omega))^2 + \gamma^2 \sin^2(\Omega)} = \frac{1}{1 + \gamma^2 - 2\gamma \cos(\Omega)} \\
\angle X(\Omega) = -\arctan \left( \frac{\gamma \sin(\Omega)}{1 - \gamma \cos(\Omega)} \right)
\]

DTFT Examples - exponential signal (cont)

• The plots above are for \( \gamma = 0.5 \).

• Note how \( X \) has period \( 2\pi \).

• Exercise: What are the maximum and minimum values of \( |X| \), i.e., how would this plot depend on \( \gamma > 0 \)? Plot \( x \) and \( \angle X \). How do these plots differ when \( -1 < \gamma < 0 \)?

• Exercise: Find the DTFT of anticausal signal \( x[k] = \gamma^k u[-k] \) for scalar \( \gamma \) s.t. \( |\gamma| > 1 \).

• Exercise: Find the DTFT of \( x[k] = |k|, k \in \mathbb{Z} \), for scalar \( \gamma \) s.t. \( |\gamma| < 1 \).
• For \( T \in \mathbb{Z}^+ \), the even rectangle pulse with support \( 2T + 1 \),
\[ x = \Delta_T u - \Delta_{T+1} u \quad (\text{i.e., } x[k] = u[k+T] - u[k-(T+1)]) \]
has DTFT
\[ X(\Omega) = \sum_{k=-T}^{T} 1 - e^{-j\Omega k} = 1 + 2 \sum_{k=1}^{T} \cos(k\Omega), \quad \Omega \in \mathbb{R}. \]

• Exercise (even rectangle pulse in frequency domain):
Show that for fixed \( \Omega' \) s.t. \( 0 < \Omega' < \pi \),
\[ \mathcal{F}^{-1}\{\Delta_{-\Omega} u - \Delta_{\Omega} u\}[k] = \frac{\Omega'}{\pi} \text{sinc}(\Omega'k), \quad k \in \mathbb{Z}. \]

• For \( T \in \mathbb{Z}^+ \), the odd triangle pulse with support \( 2T + 1 \),
\[ x[k] \equiv k(\Delta_T u[k] - \Delta_{T+1} u[k]) \]
has DTFT
\[ X(\Omega) = \sum_{k=-T}^{T} k e^{-j\Omega k} = -2j \sum_{k=1}^{T} k \sin(k\Omega), \quad \Omega \in \mathbb{R}. \]

---

# DTFT Examples - exponential sinusoid

• For fixed time \( K_0 \), clearly
\[ \mathcal{F}\{\delta[k-K_0]\}(\Omega) = e^{jK_0\Omega}, \]
where here \( \delta \) is the unit pulse.

• Note that \( e^{jK_0\Omega} \) is a sinusoidal function of \( \Omega \) with period \( 2\pi/K_0 \) (frequency \( K_0 \)).

• Exercise: For fixed frequency \( \Omega_0 \), show that
\[ \mathcal{F}\{e^{-j\Omega_0 k}\}(\Omega) = 2\pi \sum_{v=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi v), \]
where here \( \delta \) is the Dirac impulse (in the frequency domain \( \Omega \in \mathbb{R} \)). Hint: work with \( \mathcal{F}^{-1} \).

• So, the DTFT of a \( N \)-periodic signal with Fourier series
\[ \sum_{r=0}^{N-1} D_re^{j\Omega r} \xrightarrow{\mathcal{F}} 2\pi \sum_{v=-\infty}^{\infty} \sum_{r=0}^{N-1} D_r\delta(\Omega - r\frac{2\pi}{N} + 2\pi v) \]
DTFT - Time shift and frequency shift properties

• If fixed $K_0 \in \mathbb{Z}$ and $X = \mathcal{F}\{x\}$ then

$$\mathcal{F}\{\Delta^{K_0}x\}(\Omega) = \sum_{k=-\infty}^{\infty} (\Delta^{K_0}x)[k]e^{-j\Omega k}$$

$$= \sum_{k=-\infty}^{\infty} x[k - K_0]e^{-j\Omega k}$$

$$= \sum_{k'=-\infty}^{\infty} x[k']e^{-j(k' + K_0)\Omega}$$

$$= e^{-jK_0\Omega}X(\Omega),$$

i.e., shift in time by $K_0$ corresponds to product with sinusoid of period $2\pi/K_0$ (linear phase shift) in frequency domain.

• Exercise: Prove the dual property that if fixed $\Omega_0 \in \mathbb{R}$ and $X = \mathcal{F}\{x\}$ then

$$\mathcal{F}\{x[k]e^{j\Omega_0 k}\}(\Omega) = X(\Omega - \Omega_0),$$

i.e., modulation (multiplication by a sinusoid) in time domain results in frequency shift.

DTFT - convolution properties

• Let $X_r = \mathcal{F}\{x_r\}$ for $r \in \{1, 2\}$.

$$\mathcal{F}\{x_1 \ast x_2\}(\Omega) := \sum_{k=-\infty}^{\infty} (x_1 \ast x_2)[k]e^{-j\Omega k}$$

$$:= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k-l]e^{-j(k-l)\Omega}e^{-j\Omega l}$$

i.e., $x e^{j\Omega}e^{-j\Omega} = 1$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k']e^{-jk'\Omega}e^{-j\Omega l}$$

where $k' = k - l$

$$= \sum_{l=-\infty}^{\infty} x_1[l]e^{-jl\Omega} \sum_{k=-\infty}^{\infty} x_2[k']e^{-jk'\Omega} =: X_1(\Omega)X_2(\Omega)$$

• Exercise: Prove the dual property that

$$\mathcal{F}\{x_1x_2\}(\Omega) = \frac{1}{2\pi} (X_1 \ast X_2)(\Omega) := \frac{1}{2\pi} \int_{2\pi} X_1(v)X_2(\Omega - v)dv.$$

• Exercise: Use the convolution properties to prove the time and frequency shift properties. Hint: $(\Delta^{K,\delta} \ast x) = \Delta^{K,x}$.

• Exercise: Show that DTFT is a linear operator.
DTFT - Parseval’s Theorem

- The energy of a signal DT $x$ is

$$E_x := \sum_{k=\infty}^{\infty} |x[k]|^2 = \sum_{k=\infty}^{\infty} x[k] \bar{x}[k] = \sum_{k=\infty}^{\infty} (\mathcal{F}^{-1}X)[k] (\mathcal{F}^{-1}X)[k]$$

$$= \sum_{k=\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega) e^{j\Omega k} d\Omega' \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega) e^{j\Omega k} d\Omega$$

$$= \sum_{k=\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega') e^{j\Omega k} d\Omega' \frac{1}{2\pi} \int_{2\pi}^{2\pi} \bar{X}(\Omega) e^{-j\Omega k} d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega') \int_{2\pi}^{2\pi} \bar{X}(\Omega) \frac{1}{2\pi} \left( \sum_{k=\infty}^{\infty} e^{j\Omega k} e^{-j\Omega k} \right) d\Omega d\Omega'$$

$$= \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega') \int_{2\pi}^{2\pi} \bar{X}(\Omega) \frac{1}{2\pi} (2\pi \delta(\Omega - \Omega')) d\Omega d\Omega'$$

$$= \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega') |\bar{X}(\Omega')|^2 d\Omega' = \frac{1}{2\pi} \int_{2\pi}^{2\pi} |X(\Omega')|^2 d\Omega'$$

(recalling that for fixed $\Omega'$: $\mathcal{F}^{-1}\{2\pi \delta(\Omega - \Omega')\}[k] = e^{-j\Omega k}$ & $\int_{2\pi}^{2\pi} \bar{X}(\Omega) \delta(\Omega - \Omega') d\Omega = \bar{X}(\Omega')$).

---

DTFT - Parseval’s Theorem - example

- The even rectangle pulse with support $2T+1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ has energy

$$E_x = \sum_{k=\infty}^{\infty} |x[k]|^2 = \sum_{k=\infty}^{\infty} 1^2 = 2T + 1.$$

- Recall its DTFT is $X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k}$, so

$$\frac{1}{2\pi} \int_{2\pi}^{2\pi} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{2\pi}^{2\pi} X(\Omega) \bar{X}(\Omega) d\Omega = \frac{1}{2\pi} \int_{2\pi}^{2\pi} \left( \sum_{k=-T}^{T} e^{-j\Omega k} \right) \bar{X}(\Omega) d\Omega$$

$$= \frac{1}{2\pi} \int_{2\pi}^{2\pi} \left( \sum_{k=-T}^{T} 1 + \sum_{k<k'} e^{j(k-k')\Omega} \right) d\Omega$$

$$= \sum_{k=-T}^{T} \frac{1}{2\pi} \int_{2\pi}^{2\pi} 1 d\Omega + \sum_{k<k'} \frac{1}{2\pi} \int_{2\pi}^{2\pi} e^{j(k-k')\Omega} d\Omega = \sum_{k=-T}^{T} 1 + 0$$

$$= 2T + 1.$$
• **Exercise:** Repeat this calculation using $X(\Omega) = 1 + 2 \sum_{k=1}^{T} \cos(\Omega k)$.

• **Exercise:** Compute amount of energy of $x$ in the frequency band $[-\pi/6, \pi/6]$, i.e.,

$$\frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} |X(\Omega)|^2 d\Omega$$

---

**Analysis of Stable DT LTI Systems in Steady-State**

• Consider a SISO, DT-LTIC system described by the difference equation

$$Q(\Delta^{-1}) y = P(\Delta^{-1}) f,$$

where $f$ is the input and $y$ is the ZSR (output).

• Recall that by the time-shift property,

$$Q(e^{i\Omega}) Y_{ZS}(\Omega) = P(e^{i\Omega}) F(\Omega) \Rightarrow Y_{ZS}(\Omega) = H(\Omega) F(\Omega).$$

• We now re-derive from first principles the eigenresponse by first recalling that the ZSR $y_{ZS} = f * h$ where $h$ is the unit-pulse response.

• Taking DTFTs, $Y_{ZS} = HF$ where $H = F h$ is the transfer function.

• Suppose the system is BIBO/asymptotically stable, i.e., the $n$ roots of $Q$ (system char. modes/poles) $z$ all have modulus $|z| < 1$.

• The ZSR will consist of a forced response plus characteristic modes, where the latter will $\rightarrow 0$ over time (our stability assumption) so that the forced response becomes the steady-state response.
Analysis of Stable DT LTI Systems in Steady-State (cont)

- The forced response to a persistent sinusoidal input
  \[ f[k] = A_f e^{j(\Omega k + \phi_f)} \]
  will be of the form
  \[ y_{ss}[k] = A_y e^{j(\Omega k + \phi_y)} \]
  where (for \( k \geq 0 \)),
  \[ Q(e^{j\Omega}) y_{ss}[k] = (Q(\Delta^{-1}) y_{ss})[k] = (P(\Delta^{-1}) f)[k] = P(e^{j\Omega}) f[k]. \]
  \[ \Rightarrow y_{ss}[k] = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})} f[k] \]

- Also, the ZSR \( y_{ZS} = h * f \), i.e., for all time \( k \geq 0 \):
  \[ y_{ZS}[k] = \sum_{v=0}^{k} h[v] A_f e^{j(\Omega (k-v) + \phi_f)} = f[k] \sum_{v=0}^{k} h[v] e^{-j\Omega v} \]
  \[ \Rightarrow f[k] H(\Omega_o) =: y_{ss}[k] \text{ as } k \to \infty. \]

Transfer Function and Eigenresponse in Discrete Time (cont)

- Equating the forced responses (steady-state response for a stable system), we again get that the system transfer function is
  \[ H(\Omega) = P(e^{j\Omega}) / Q(e^{j\Omega}) = (Fh)(\Omega). \]

- Note that \( \forall k \in \mathbb{Z}, H(\Omega) = H(\Omega + 2\pi k). \)

- Also, we write \( H(\Omega) \) not \( H(e^{j\Omega}) \) for the DTFT.

- So, the eigenresponse of a BIBO/asymptotically stable SISO, DT-LTIC system is the steady-state response to a sinusoid:
  \[ f[k] = A_f e^{j(\Omega_k + \phi_f)} \rightarrow H(\Omega_o) f[k] = A_y e^{j(\Omega k + \phi_y)} =: y_{ss}[k] \]

- The system magnitude response (gain) is \( |H(\Omega)| = |P(e^{j\Omega})| / |Q(e^{j\Omega})| \).
  \[ i.e., A_y = A_f |H(\Omega_0)|. \]

- The system phase response is \( \angle H(\Omega) = \angle P(e^{j\Omega}) - \angle Q(e^{j\Omega}) \).
  \[ i.e., \phi_y = \phi_f + \angle H(\Omega_0). \]
• **Problem:** For the system $2y[k] = 0.6y[k - 1] - 7f[k]$ find the steady-state response (if it exists) to $f[k] = 4 \cos(5k)u[k]$.

• **Solution:** The difference equation in standard form is

$$ (Q(z)y[k]) = y[k + 1] - 0.3y[k] = -3.5f[k + 1] = (P(z)f[k]), $$

where $Q(z) = z - 0.3$ and $P(z) = -3.5z$.

The sole system characteristic value (root of $Q$, system pole) is $0.3$, hence the system is BIBO/asymptotically stable.

By Euler’s identity $f[k] = (2e^{i5k} + 2e^{i(-5)k})u[k]$.

By linearity, the eigenresponse is therefore

$$ 2H(5)e^{i5k} + 2H(-5)e^{i(-5)k}, $$

where $H(\Omega) = P(e^{j\Omega})/Q(e^{j\Omega}) = -3.5e^{j\Omega}/(e^{j\Omega} - 0.3) = H(-\Omega)$, so that

$$ |H(\Omega)| = \frac{3.5}{\sqrt{(\cos(\Omega) -.3)^2 + \sin^2(\Omega)}}, \quad \angle H(\Omega) = \pi + \Omega - \arctan(\frac{-\sin(\Omega)}{\cos(\Omega) - .3}) $$

• **Exercise:** Show that the eigenresponse is also simply $|H(5)|4 \cos(5k + \angle H(5))$.

---

2D Image Processing Example

• Apply 1-dimensional filtering to a 2-dimensional (2D) image by separately performing row and column operations.

• For $256 \times 256$ pixel (2D) image,

$$ f = \begin{bmatrix} f[1, 1] & f[1, 2] & \ldots & f[1, 256] \\ f[2, 1] & f[2, 2] & \ldots & f[2, 256] \\ \vdots & \vdots & \ddots & \vdots \\ f[256, 1] & f[256, 2] & \ldots & f[256, 256] \end{bmatrix} $$

• If $f[k, i]$ represents the 8-bit (grey) intensity of the pixel in row $k$ and column $i$ (i.e., 8 bits per pixel or bpp), then the "raw" image size will be $256^3 \text{bits} = 16\text{Mb} = 2\text{MB}$.

• Each of $f$’s rows of pixels can be processed by a system with unit-pulse response $h$ to obtain a new row of pixels, and thus a new image $y$:

$$ \forall k, \ f[k, \cdot] \rightarrow [h] \rightarrow y[k, \cdot] $$

• Alternatively, each of $f$’s columns of pixels can be processed by a system with unit-pulse response $h$ to obtain a new column of pixels, and thus a new image $y$:

$$ \forall i, \ f[\cdot, i] \rightarrow [h] \rightarrow y[\cdot, i] $$

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The system $h$ may have a specific signal processing objective.

The output pixels $y[k, i]$ may be quantized to fewer bpp than those of the input, thus achieving image compression.

The simple low-pass filter (L)

$$h[k] = \frac{1}{2}(\delta[k] + \delta[k - 1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] + f[k - 1])$$

can capture shading and texture in the image.

The simple high-pass filter (H)

$$h[k] = \frac{1}{2}(\delta[k] - \delta[k - 1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] - f[k - 1])$$

can capture edges in the image.

Typically more compression possible in higher-frequency bands (H).

Define $y_{LH}$ as the output of

$$f \rightarrow \text{row filtering} \rightarrow \text{column filtering} \rightarrow y$$

Similarly define $y_{LL}$, $y_{HH}$ and $y_{HL}$.

The $y$ images are downsampled by a factor of four (two in each direction).

The $y_{LL}$ image will have a lot of energy while $y_{HH}$ will have the least energy.

This motivates non-uniform quantization (bit allotment per pixel) of these images.

Together with a coding strategy for the quantized images (particularly for the regions of zero pixel-values), this is the basic approach used in JPEG leading to very good compression, e.g., from 8 bpp to 0.2-0.5 bpp.
Sampling Continuous-Time Signals (A/D)

• Consider continuous-time signal $x$ with $X = \mathcal{F}x$.

• Recall that by sampling at period $T$ with impulses in continuous time $t \in \mathbb{R}$, we get
  $$x_T(t) := \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} x(kT)e^{-jkw} =: X_T(w),$$
equivalently, $X_T(w) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( w - \frac{2\pi}{T} v \right)$.

• Now define the sampled process in discrete-time $k \in \mathbb{Z}$ and its DTFT,
  $$\hat{x}[k] := x(kT) \xrightarrow{\mathcal{F}} \hat{X}(\Omega) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{-j\Omega k}.$$

• Substituting $w = \Omega/T$ we get
  $$\hat{X}(\Omega) = X_T \left( \frac{\Omega}{T} \right) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( \frac{\Omega - v2\pi}{T} \right).$$

• Exercise: Read decimation (downsampling) and interpolation (upsampling) of Lathi Figs. 8.17 & 10.9.

Sampling Continuous-Time Signals - example

• We are particularly interested in the case where
  – the continuous-time signal $x$ is band-limited, i.e., $\exists w' > 0$ s.t. $X(w) = 0$ for $|w| > w'$, and
  – the sampling frequency is greater than Nyquist’s, i.e., $2\pi/T > 2w' \Rightarrow w'T < \pi$.

• Example: For fixed $w' > 0$, consider the cts-time signal $x(t) = A\text{sinc}(w't)$ with FT
  $$X(w) = \frac{A\pi}{w'}(u(w + w') - u(w - w')).$$

• Sampling $x$ at period $T < \pi/w'$ we get the discrete-time signal $x[k] = A\text{sinc}(w'kT)$.

• Using inverse DTFT, recall that we can easily check that the DTFT of $x$ is,
  $$\hat{X}(\Omega) = \sum_{v=-\infty}^{\infty} \frac{A\pi}{w'}(u(\Omega + w'T - 2\pi v) - u(\Omega - w'T - 2\pi v))$$
  $$= \sum_{v=-\infty}^{\infty} \frac{1}{T} X \left( \frac{\Omega - 2\pi v}{T} \right),$$
noting $\forall T > 0$, $u(\frac{\Omega}{T} \pm w') = u(\frac{1}{T}(\Omega \pm w'T)) = u(\hat{\Omega} \pm w'T)$, $\hat{\Omega} := \Omega - 2\pi v$. 
Sampled Data Systems: A/D (analog-to-digital conversion)

- Suppose the signal $f$ is sampled every $T_s$ seconds, i.e., at sampling frequency $w_s := 2\pi / T_s$.

- Recall Poisson’s identity (the Fourier series of the picket-fence function)

$$p_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}$$

- Let’s rederive the relationship between the spectrum of a sampled continuous-time signal and its discrete-time counterpart by first defining the discrete-time signal

$$\forall k \in \mathbb{Z}, \quad \hat{f}[k] = f(kT_s).$$

- We want to relate the (continuous-time) Fourier transform of $f$ to the (discrete-time) Fourier transform of $\hat{f}$.

$$\hat{F}(\Omega) := \sum_{k=-\infty}^{\infty} \hat{f}[k] e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f(kT_s) e^{-j\Omega k}.$$
To this end, recall
\[ f(t) \xrightarrow{T_s} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} f(kT_s) \delta(t - kT_s) \xrightarrow{F} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s) , \]
and also
\[ f(t) \xrightarrow{T_s} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} f(kT_s) e^{-jkwT_s} = \hat{F}(wT_s). \]

Equating these two expressions for \( F\{f_{T_s}\} \) we get,
\[ \hat{F}(wT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s). \]

Substituting \( w = \Omega/T_s \) we get,
\[ \hat{F}(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F \left( \frac{\Omega - k2\pi}{T_s} \right). \]

Now consider a discrete time signal \( \hat{y}[k] \).

We implement at D/A with a \( T_s \)-second hold, i.e., construct the continuous-time signal
\[ y(t) : = \sum_{k=-\infty}^{\infty} \hat{y}[k] r_{T_s}(t - kT_s) , \] where
\[ r_{T_s}(t) : = u(t) - u(t - T_s) \xrightarrow{F} T_s \text{sinc}(wT_s/2)e^{-jwT_s}/2 =: R_{T_s}(w). \]

Note that \( y \) is in the form of a convolution, so:
\[ Y(w) = \sum_{k=-\infty}^{\infty} \hat{y}[k] R_{T_s}(w)e^{-jkwT_s} \]
\[ = R_{T_s}(w)\hat{Y}(wT_s) \]
Consider a digital system $\hat{H}(\Omega)$ (or $\hat{H}(e^{j\Omega})$ depending on notation), whose (ZS) output is $\hat{y}$ when the input is $\hat{f}$, i.e., $\hat{Y} = \hat{H}\hat{F}$.

The equivalent continuous-time transformation of the tandem system

\[ f \rightarrow \text{A/D (}T_s\text{sample)} \rightarrow \hat{H}(\Omega) \rightarrow \text{D/A (}T_s\text{hold)} \rightarrow y \]

with input $f$ has (ZS) output

\[ Y(w) = R_{T_s}(w)\hat{Y}(wT_s) = R_{T_s}(w)\hat{H}(wT_s)\hat{F}(wT_s) = R_{T_s}(w)\hat{H}(wT_s)\frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s). \]

**Exercise**: Show that if $f$ is band-limited by $w_s/2$ (i.e., $w_s$ is greater than $f$’s Nyquist frequency) and the previous sampled data system is followed by an ideal low-pass filter with bandwidth $w_s/2$, then the equivalent (continuous-time) transfer function is

\[ H(w) = \hat{H}(wT_s)T_s^{-1}R_{T_s}(w)(u(w + w_s/2) - u(w - w_s/2)) \]

Note that the term in the transfer function $H$,

\[ T_s^{-1}R_{T_s}(w)(u(w + w_s/2) - u(w - w_s/2)) = \text{sinc}(\Omega/2)(u(\Omega + \pi) - u(\Omega - \pi)) \]

is not a constant function of $\Omega = wT_s$.

This distortion due to the hold function $R$ can be reduced by putting in tandem with $\hat{H}$ an equalizer system with transfer function approximately

\[ \tilde{R}^{-1}(\Omega) := \sum_{k=-\infty}^{\infty} \frac{u(\Omega + \pi - k2\pi) - u(\Omega - \pi - k2\pi)}{\text{sinc}((\Omega - k2\pi)/2)} \]

i.e.,

\[ \hat{H}(\Omega) \rightarrow \tilde{R}^{-1}(\Omega) \]
Sampled Data Systems: equalization of hold sinc(\(\Omega/2\)) by \(\hat{R}^{-1}(\Omega)\)

- the hold (at left, \(R\)) distorts the signal by attenuating its higher frequency components
- the equalizer (at right, \(R^{-1}\)) amplifies at the higher frequencies to cancel out this distortion

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DFT and FFT - Reading Exercise on Computational Issues

- Read Lathi Sec. 5.2 and 5.3 re. continuous-time FS, FT
- Read Lathi Sec. 10.6 re. DTFS, DTFT
Transient analysis in discrete time by unilateral $z$-transform

- $z$-transform definition and region of convergence.
- Basic $z$-transform pairs and properties.
- Inverse $z$-transform of rational polynomials by Partial Fraction Expansion (PFE).
- Total transient response of SISO DT LTIC systems $Q(\Delta^{-1})y = P(\Delta^{-1})f$.
- The steady-state eigenresponse revisited.
- System composition and canonical realizations.

The unilateral $z$-transform & region of convergence

- The $z$-transform of a signal $x = \{x[k]\}_{k \geq 0}$ is
  \[
  X(z) = (Zx)(z) = \sum_{k=0}^{\infty} x[k]z^{-k} := \lim_{K \to \infty} \sum_{k=0}^{K} x[k]z^{-k},
  \]
  where $z \in \mathbb{C}$.
- If the signal $x$ is bounded by an exponential (geometric), i.e.,
  \[
  \exists M, \gamma \in \mathbb{R}_{>0} \text{ such that } \forall k \in \mathbb{Z}_{\geq 0}, |x[k]| \leq M\gamma^k \quad \text{(i.e., } -M\gamma^k \leq x[k] \leq M\gamma^k)\]
  then the series $X(z)$ converges in the region outside of a disk centered $0 \in \mathbb{C}$,
  \[
  \{z \in \mathbb{C} \mid |z| > \gamma\}.
  \]
- To see why bounded by an exponential suffices, recall absolute convergence $\Rightarrow$ convergence:
  \[
  \forall k \geq 0, \quad |x[k]z^{-k}| = |x[k]| \cdot |z|^{-k} \leq M\gamma^k |z|^{-k} = M(\gamma/|z|)^k
  \]
  \[
  \Rightarrow \sum_{k=0}^{\infty} |x[k]z^{-k}| \leq M \sum_{k=0}^{\infty} (\gamma/|z|)^k \quad \text{which converges if } \gamma/|z| < 1.
  \]
\[ \delta[k] \xrightarrow{z} 1, \quad z \in \mathbb{C} \]
\[ u[k] \xrightarrow{z} \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \quad |z| > 1 \]
\[ \beta^k u[k] \xrightarrow{z} \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{1}{1 - \beta z^{-1}} = \frac{z}{z - \beta}, \quad |z| > |\beta| \]
\[ \{ \beta^{k-1} u[k - 1] \}(z) \xrightarrow{z} \sum_{k=1}^{\infty} \beta^{k-1} z^{-k} = z^{-1} \sum_{k=0}^{\infty} \beta^k z^{-k'} = z^{-1} \frac{1}{1 - \beta z^{-1}} = \frac{1}{z - \beta}, \quad |z| > |\beta| \]
\[ e^{j\Omega k} u[k] \xrightarrow{z} \sum_{k=0}^{\infty} e^{j\Omega k} z^{-k} = \frac{1}{1 - e^{j\Omega} z^{-1}}, \quad |z| > 1 \quad (\beta = e^{j\Omega}) \]
\[ k\beta^k u[k] \xrightarrow{z} \sum_{k=0}^{\infty} k\beta^k z^{-k} = \frac{\beta d}{d\beta} \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{\beta}{d\beta} \frac{1}{1 - \beta z^{-1}} = \frac{\beta z^{-1}}{(1 - \beta z^{-1})^2}, \quad |z| > |\beta| \]

**Exercise:** Find \( Z\{ A \cos(\Omega, k + \phi) u[k] \} \) and \( Z\{ A \sin(\Omega, k + \phi) u[k] \} \).
Basic $z$-transform properties: advance time shift

- Advance time shift (no change in RoC): Let $X = Zx$.

$$
\Delta^{-1}x \xrightarrow{Z} \sum_{k=0}^{\infty} x[k + 1]z^{-k} = -zx[0] + \sum_{k=-1}^{\infty} x[k + 1]z^{-k}
$$

$$
= -zx[0] + z \sum_{k=-1}^{\infty} x[k + 1]z^{-(k+1)}
$$

$$
= -zx[0] + z \sum_{k=0}^{\infty} x[k']z^{-k'}
$$

$$
= -zx[0] + zX(z)
$$

- Exercise: For $v \in \mathbb{Z}^>$ show by induction that

$$
(Z\{\Delta^{-v}x\})(z) = -\sum_{k=1}^{v} z^{k}v[k] + z^{v}X(z)
$$

Basic $z$-transform properties: delay time shift

- Delay time shift (no change in RoC): For $v \in \mathbb{Z}^>$,

$$
\Delta^{v}(xu) \xrightarrow{Z} \sum_{k=0}^{\infty} x[k - v]u[k - v]z^{-k}
$$

$$
= \sum_{k=v}^{\infty} x[k - v]z^{-k} = \sum_{k=0}^{\infty} x[k']z^{-k'-v}
$$

$$
= z^{-v}X(z).
$$

- So in the “zero-state” (input-output) context (i.e., $x[k]u[k] = 0$ for $k < 0$), we identify multiplying by $z^{-1}$ in complex-frequency domain with the unit delay $\Delta$ in the time domain.

- Delay $v \in \mathbb{Z}^>$ of non-causal $x$:

$$
\Delta^{v}x \xrightarrow{Z} \sum_{k=0}^{\infty} x[k - v]z^{-k} = \sum_{k=-v}^{\infty} x[k']z^{-k'-v}
$$

$$
= \sum_{k=-v}^{-1} x[k']z^{-k'-v} + z^{-v}X(z).
$$
Basic $z$-transform properties: frequency shift & convolution

- Let $X = \mathcal{Z}x$ with RoC $C(\gamma) := \{z \in \mathbb{C} \mid |z| > \gamma\}$.
  \[ \beta^k x[k] \xrightarrow{z} \sum_{k=0}^{\infty} \beta^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k] (z/\beta)^{-k} = X(z/\beta), \quad z \in C(\gamma/|\beta|). \]
  i.e., $\times \beta^k$ in the time-domain is dilation by $\beta$ in the $z$-domain.

- For signals $x_1, x_2 : \mathbb{Z}^{\geq 0} \to \mathbb{C}$ ($x_1[k], x_2[k] = 0$ for $k < 0$), with respective ROCs $C_1, C_2 \subset \mathbb{C}$,
  \[ x_1 \ast x_2 \xrightarrow{z} \sum_{k=0}^{\infty} (x_1 \ast x_2)[k] z^{-k} = \sum_{k=0}^{\infty} \sum_{v=0}^{k} x_1[v] x_2[k-v] z^{-(k-v)} z^{-v} \]
  \[ = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=v}^{\infty} x_2[k-v] z^{-(k-v)} \]
  \[ = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k'=0}^{\infty} x_2[k'] z^{-k'} \]
  \[ = X_1(z) X_2(z), \quad z \in C_1 \cap C_2. \]

Basic $z$-transform properties: convolution, IVT & FVT

- So convolution in the time-domain is multiplication in the frequency domain.
- The converse is also true.
- Directly by definition of $X = \mathcal{Z}x$, we get the initial value theorem
  \[ \lim_{z \to \infty} X(z) = x[0]. \]
- There is also a “final value” theorem for $\lim_{k \to \infty} x[k]$.

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We now study transient analysis of LTI difference equations using $z$-transforms.

Recall our system is defined given polynomials $P, Q$, input $f$ and initial conditions:
- $Q(\Delta^{-1})y = P(\Delta^{-1})f$, where $y$ is the (total) output and
- input $f[k] = 0$ for $k < 0$,
- degree of polynomial $Q = n \geq m = \text{degree of polynomial } P$ (causal system),
- $Q(z) = z^n + \sum_{v=1}^{n-1} a_v z^v$ (i.e., $a_n = 1$) and $P(z) = \sum_{v=0}^{m} b_v z^v$,
- $a_n \neq 0$ or $b_n \neq 0$ for poly'ls $Q, P$ of minimum degree,
- $n$ initial conditions $y[-n], y[-n+1], \ldots, y[0], y[1]$.

We can restate the difference equation in terms of delays by delaying both sides by $n$ time-units (i.e., applying with $\Delta^n$), to get

$$\Delta^n Q(\Delta^{-1})y = \Delta^n P(\Delta^{-1})f$$

$$\Rightarrow \bar{Q}(\Delta)y := \sum_{v=0}^{n-1} a_v \Delta^{n-v} y = \sum_{v=0}^{m} b_v \Delta^{n-v} f =: \bar{P}(\Delta)f$$

So, taking the $z$-transform of the (delay) difference equation, we get by the (delay) time-shift and linearity properties that

$$\sum_{v=0}^{n} a_v \sum_{k=-v}^{1} y[k] z^{-k-v} + \bar{Q}(z^{-1})Y(z) = \bar{P}(z^{-1})F(z)$$

So, solving for the total response $Y$ we get

$$Y(z) = \frac{\bar{P}(z^{-1})F(z)}{\bar{Q}(z^{-1})} - \frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{1} y[k] z^{-k-v}}{\bar{Q}(z^{-1})} = Y_{ZS}(z) + Y_{ZI}(z)$$

where the ZIR and ZSR in the complex-frequency ($z$) domain respectively are

$$Y_{ZI}(z) := -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{1} y[k] z^{-k-v}}{\bar{Q}(z^{-1})} = -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{1} y[k] z^{n-k-v}}{\bar{Q}(z)}$$

$$Y_{ZS}(z) := \frac{\bar{P}(z^{-1})F(z)}{\bar{Q}(z^{-1})} = \frac{P(z)}{Q(z)} F(z) = H(z)F(z) \text{ (transfer function } H).$$

Regarding this total transient response, note how
- the $z$-transform’s unilateral aspect captures the impact of initial conditions (ZIR), and
- a greater range of inputs $f$ than under DTFT through $\text{RoC} \subset \mathbb{C}$ (not just $|z| = 1$).
• Suppose i.c. \( y[-1] = -1 \), input \( f[k] = 2(-3)^k u[k] \) and output \( y \) s.t.
\[
\forall k \geq -1, \quad 2y[k + 1] + 2y[k] = 3f[k + 1] + 2f[k].
\]

• To find the total response \( y \), we take the \( z \)-transform of the equivalent system: \( \forall k \geq 0, \)
\[
2y[k] + 2y[k - 1] = 3f[k] + 2f[k - 1]
\implies 2Y(z) + 2(z^{-1}Y(z) + y[-1]) = 3F(z) + 2z^{-1}F(z).
\]

• So by the delay property for non-causal signals \( (y) \), the total response
\[
Y(z) = \frac{3 + 2z^{-1}}{2} F(z) + \frac{-2y[-1]}{2 + 2z^{-1}}
\]
\[
= H(z) F(z) + \frac{-y[-1]}{1 + z^{-1}}
\]
\[
= Y_{ZS}(z) + Y_{ZI}(z)
\]
with RoC for \( Y \) being the intersection of those of \( F \) and \( H \).

• Understanding that the ZIR begins at \( k = -1 \) (initial condition) and the ZSR at time \( k = 0 \), we get:
\[
\forall k \geq -1, \quad y[k] = (3.5(-3)^k - 0.5(-1)^k)u[k] + (-1)^k = y_{ZS}[k] + y_{ZI}[k],
\]
where we minded the ambiguity \( \mathcal{Z}x = \mathcal{Z}xu \).

• Exercise: Verify this solution using time-domain methods, i.e.,
\[
y = y_{ZI} + y_{ZS} = y_{ZI} + h \ast f, \quad \text{where} \ h \ \text{and} \ y_{ZI} \ \text{consist of char. modes.}
\]
Inverse $z$-transform of proper rational polynomials

- We now describe how to find $\mathcal{Z}^{-1}X$ of causal signal $X$ that is rational polynomial in $z$, i.e., $X(z) = M(z)/N(z)$ where $M(z)$ and $N(z)$ are polynomials in $z$.

- If $\deg(M) = \deg(N) + 1$, we perform long division to write $X = c + \tilde{M}/N$ where $\deg(N) = \deg(\tilde{M})$ and $\mathcal{Z}^{-1}X = c\delta + \mathcal{Z}^{-1}\{\tilde{M}/N\}$.

- If $\deg(M) = \deg(N)$ and $M(0) = 0$ (so $z^{-1}M(z)$ is a polynomial), we can factor $z$ from $M$ to get

$$X(z) = z^{-1}M(z)/N(z).$$

- We will find $\mathcal{Z}^{-1}X$ using PFE of the strictly proper rational polynomial $z^{-1}M(z)/N(z)$.

- Alternatively, we could apply PFE on strictly proper rational polynomials in $z^{1}$, $z^{-K}M(z)/(z^{-K}N(z))$ where $K := \deg(N)$, as in the previous example.

Partial Fraction Expansion (PFE) example in $z$ (not $z^{-1}$)

- For example, suppose

$$X(z) := \frac{z(3z + 2)}{z^2 - 0.64} = \frac{z(3z + 2)}{(z + 0.8)(z - 0.8)} = \frac{z(0.25 \cdot z + 2.75)}{z + 0.8 + \frac{z}{z - 0.8}} = 0.25 \frac{z}{z + 0.8} + 2.75 \frac{z}{z - 0.8}$$

where PFE (below) gave the numerators (residues) 0.25 and 2.75.

- So,

$$\mathcal{Z}^{-1}X[k] = 0.25(-0.8)^k u[k] + 2.75(0.8)^k u[k]$$

- Note that the associated RoC of $X$ is $\{z \in \mathbb{C} \mid |z| > 0.8\}$. 
Partial Fraction Expansion (PFE) - preliminaries

- Let $K = \deg(N) = \deg(M)$ so that we can factor
  \[ N(z) = \prod_{k=1}^{K}(z - p_k), \]
  where the $p_k$ are the roots of $N$ (poles of $M/N$).
- We assume $M$ and $N$ have no common roots, i.e., no “pole-zero cancellation” issue to consider, so that the $p_k$ are the poles of $M/N$.
- Again, we assume $M(0) = 0$ (0 is a zero of $M/N$) and so $z^{-1}M(z)$ is a polynomial of degree $K - 1$.
- Note that the RoC for $M(z)/N(z)$ is $\{z \in \mathbb{C} \mid |z| > \max_{k} |p_k|\}$.

PFE - the case of no repeated poles

- Suppose there are no repeated poles for $M/N$, i.e., $\forall k \neq l, \ p_k \neq p_l$.
- In this case, we can write the PFE of $z^{-1}M(z)/N(z)$ as
  \[ z^{-1}M(z) \begin{array}{c} \begin{array}{cc} N(z) \\ \Rightarrow M(z) \\ \Rightarrow \end{array} \end{array} \begin{array}{c} \begin{array}{c} N(z) \\ \Rightarrow \end{array} \end{array} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} \]
  \[ = \sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} \frac{1}{1 - p_lz^{-1}} \]
  where the scalars (Heaviside coefficients) $c_l \in \mathbb{C}$ are
  \[ c_l = \frac{z^{-1}M(z)}{\prod_{k \neq l}(z - p_k)} \bigg|_{z = p_l} = \lim_{z \to p_l} \frac{z^{-1}M(z)}{N(z)}(z - p_l) = \frac{z^{-1}M(z)}{N(z)}(z - p_l) \bigg|_{z = p_l}. \]
- That is, to find the Heaviside coefficient $c_k$ over the term $z - p_k$ in the PFE, we have removed (covered up) the term $z - p_k$ from the denominator $N(z)$ and evaluated the remaining rational polynomial at $z = p_k$.
- This approach, called the Heaviside cover-up method, works even when $p$ is $\mathbb{C}$-valued.
- Given the PFE of $z^{-1}M/N$, $(Z^{-1}M/N)[k] = \sum_{l=1}^{K} c_l p_l^k u[k]$.
PFE - proof of Heaviside cover-up method

• To prove that the above formula for the Heaviside coefficient $c_l$ is correct, note that the claimed PFE of $z^{-1}M(z)/N(z)$ is

$$
\sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) / N(z)
$$

• Thus, the PFE equals $z^{-1}M(z)/N(z)$ if and only if the numerator polynomials are equal, i.e., iff

$$
z^{-1}M(z) = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) =: \hat{M}(z).
$$

• Again, two polynomials are equal if their degrees, $L$, are equal and either:
  - their coefficients are the same, or
  - they agree at $L + 1$ (or more) different points, e.g., two lines ($L = 1$) are equal if they agree at 2 ($= L + 1$) points.

• Since $z^{-1}M(z)$ is a polynomial of degree $< K$, it suffices to check that whether $z^{-1}M(z) = \hat{M}(z)$ for all $z = p_k$, $k \in \{1, 2, ..., K\}$, i.e., this would create $K$ equations in $< K$ unknowns (the coefficients of $\hat{M}$).

PFE - proof of Heaviside cover-up method (cont)

• But note that any pole $p_r$ of $z^{-1}M(z)/N(z)$ is a root of all but the $r^{th}$ term in $\hat{M}$, so that

$$
\hat{M}(p_r) = c_r \prod_{k \neq r} (p_r - p_k) = \left( \frac{z^{-1}M(z)}{\prod_{k \neq r} (z - p_k)} \right)_{z=p_r} \prod_{k \neq r} (p_r - p_k) = \frac{p_r^{-1}M(p_r)}{\prod_{k \neq r} (p_r - p_k)} \prod_{k \neq r} (p_r - p_k) = p_r^{-1}M(p_r).
$$

• Q.E.D.
PFE - the case of no repeated poles - example

• To find the inverse $z$-transform of a proper rational polynomial $X = M/N$ with $M(0) = 0$, first factor its denominator $N$ and factor $z$ from $M$, e.g.,

$$X(z) = \frac{z^3 + 5z^2}{z^3 + 9z^2 + 26z + 24} = \frac{z^2 + 5z}{(z + 4)(z + 3)(z + 2)} \quad \text{for } |z| > 4.$$

• So, by PFE

$$X(z) = z \left( \frac{c_4}{z + 4} + \frac{c_3}{z + 3} + \frac{c_2}{z + 2} \right) = z \frac{z^{-1} M(z)}{N(z)} \Rightarrow$$

$$z^{-1} M(z) = 1z^2 + 5z + 0 = c_4(z + 3)(z + 2) + c_3(z + 4)(z + 2) + c_2(z + 4)(z + 3) =: \hat{M}(z).$$

• We can solve for the 3 constants $c_k$ by comparing the 3 coefficients of quadratic $M$ and $\hat{M}$.

• The Heaviside cover-up method suggests we try $z = -2, -3, -4$ to solve for $c_2, c_3, c_4$:

$$c_4 = \left. \frac{z^2 + 5z}{(z + 3)(z + 2)} \right|_{z = -4} = -2, \quad c_3 = \left. \frac{z^2 + 5z}{(z + 4)(z + 2)} \right|_{z = -3} = 6, \quad c_2 = \left. \frac{z^2 + 5z}{(z + 4)(z + 3)} \right|_{z = -2} = -3$$

• Thus, $x[k] = (Z^{-1} X)[k] = (-2(-4)^k + 6(-3)^k - 3(-2)^k)u[k]$.

PFE - the case of a non-repeated, complex-conjugate pair of poles

• Again, recall that for polynomials with all coefficients $\in \mathbb{R}$, all complex poles will come in complex-conjugate pairs, $p_1 = \overline{p}_2$.

• The case of non-repeated poles $p_1, p_2 = \alpha \pm j\beta \ (\alpha, \beta \in \mathbb{R}, \ j := \sqrt{-1})$ that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients $c_1, c_2$, i.e., $c_1 = \overline{c}_2$.

• In the PFE, we can alternatively combine the terms

$$\frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} = \frac{r_1 z + r_0}{(z - \alpha)^2 + \beta^2}$$

where by equating the two numerator polynomials' coefficients,

$$r_0 = -c_1 p_2 - c_2 p_1 = -2 \text{Re}\{c_1 p_2\} \in \mathbb{R} \quad \text{and} \quad r_1 = c_1 + c_2 = 2 \text{Re}\{c_1\} \in \mathbb{R}.$$

• Exercise: Show that

$$2|c| \cdot |p|^k \cos(k \angle p + \angle c) \overset{z}{\longrightarrow} \frac{cz}{z - p} + \frac{\overline{c}z}{z - \overline{p}}$$
To find the inverse $z$-transform of

$$X(z) = \frac{3z^2 + 2z}{z^3 + 5z^2 + 10z + 12},$$

first factor the denominator and divide the numerator by $z$ to get

$$X(z) = \frac{3z + 2}{(z^2 + 2z + 4)(z + 3)}.$$

Note that the poles of $X$ are $-3$ and $-1 \pm j\sqrt{3}$ (so $X$'s RoC is $|z| > 3$).

So, we can expand $X$ to

$$X(z) = \frac{r_1 z + r_0}{z^2 + 2z + 4} + \frac{c_3}{z + 3},$$

where by the Heaviside cover-up method,

$$c_3 = \frac{3z + 2}{z^2 + 2z + 4} \bigg|_{z = -3} = -1.$$

Thus, by comparing coefficients

$$0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4$$

we get

$$r_0 = 2 \quad \text{and} \quad r_1 = 1.$$
Thus by substituting, we get

\[ X(z) = \frac{z + 2}{z^2 + 2z + 4} + z \frac{-1}{z + 3} \]

Exercise: Show that

\[ x[k] = (z^{-1}X)[k] = \left( \sqrt{\frac{4}{3}} 2^k \cos(k2\pi/3 - \pi/6) - (-3)^k \right) u[k]. \]

If a particular pole \( p \) of \( z^{-1}M(z)/N(z) \) is of order \( r \geq 1 \), i.e., \( N(z) \) has a factor \((z - p)^r\), then the PFE of \( z^{-1}M(z)/N(z) \) has the terms

\[ \frac{c_1}{z - p} + \frac{c_2}{(z - p)^2} + \cdots + \frac{c_r}{(z - p)^r} = \sum_{k=1}^{r} \frac{c_k}{(z - p)^k} = \frac{z^{-1}M(z)}{N(z)} - \Phi(z) \]

with \( c_k \in \mathbb{C} \forall k \in \{1, 2, ..., r\} \), where \( \Phi(z) \) represents the other PFE terms of \( z^{-1}M(z)/N(z) \) (i.e., corresponding to poles \( \neq p \)).

Note that equating \( z^{-1}M(z)/N(z) \) to its PFE and multiplying by \((z - p)^r\) gives

\[ \frac{z^{-1}M(z)}{N(z)}(z - p)^r = c_r + \sum_{k=1}^{r-1} c_k(z - p)^{r-k} + \Phi(z)(z - p)^r \]

\[ \Rightarrow \frac{z^{-1}M(z)}{N(z)}(z - p)^r \bigg|_{z=p} = c_r, \]

i.e., Heaviside cover-up (of the entire term \((z - p)^r\)) works for \( c_r \).
To find $c_{r-1}$, we differentiate the previous display to get

$$
\frac{d}{dz} \left( z^{-1} M(z) (z-p)^r \right) = \sum_{k=1}^{r-1} c_k (r-k)(z-p)^{r-k} \frac{d}{dz} \Phi(z)(z-p)^r \\
= c_{r-1} + \sum_{k=1}^{r-2} c_k (r-k)(z-p)^{r-k} \frac{d}{dz} \Phi(z)(z-p)^r \\
\Rightarrow c_{r-1} = \left. \left( \frac{d}{dz} \left( z^{-1} M(z) \right) (z-p)^r \right) \right|_{z=p} 
$$

If we differentiate the original display $k \in \{0, 1, 2, \ldots, r-1\}$ times and then substitute $z = p$, we get (with $0! := 1$)

$$
\left. \frac{d^k}{dz^k} \left( z^{-1} M(z) \right) (z-p)^r \right|_{z=p} = k! c_{r-k} \\
\Rightarrow c_{r-k} = \frac{1}{k!} \left. \left( \frac{d^k}{dz^k} \left( z^{-1} M(z) \right) (z-p)^r \right) \right|_{z=p} .
$$

**PFE - the general case of repeated poles - example**

To find the inverse $z$-transform of

$$X(z) = \frac{z(3z + 2)}{(z + 1)(z + 2)^3},$$

write the PFE of $X$ as

$$X(z) = z \left( \frac{c_1}{z+1} + \frac{c_{2,1}}{z+2} + \frac{c_{2,2}}{(z+2)^2} + \frac{c_{2,3}}{(z+2)^3} \right),$$

so clearly the RoC of causal $X$ is $|z| > 2$.

By Heaviside cover-up

$$c_1 = \left. \frac{3z + 2}{(z+2)^3} \right|_{z=-1} = -1 \quad \text{and} \quad c_{2,3} = \left. \frac{3z + 2}{z+1} \right|_{z=-2} = 4.$$
• Also,

\[ c_{2,2} = \frac{1}{1!} \left. \left( \frac{d}{dz} \frac{3z + 2}{z + 1} \right) \right|_{z=-2} = \frac{1}{1!} \left. \frac{1}{(z + 1)^2} \right|_{z=-2} = 1 \]

\[ c_{2,1} = \frac{1}{2!} \left. \left( \frac{d^2}{dz^2} \frac{3z + 2}{z + 1} \right) \right|_{z=-2} = \frac{1}{2!} \left. \frac{-2}{(z + 1)^3} \right|_{z=-2} = 1 \]

Thus,

\[ X(z) = \frac{-1}{z + 1} + z \frac{1}{z + 2} + z \frac{1}{(z + 2)^2} + z \frac{4}{(z + 2)^3} \quad \forall |z| > 2 \]

\[ \Rightarrow x[k] = (Z^{-1}X)[k] = \left( -(-1)^k + (-2)^k + k(-2)^{k-1} + \frac{k(k-1)}{2} \right) u[k] \]

• Exercise: Show by induction and integration by parts that: \( \forall m \in \mathbb{Z}^+ \),

\[ \left( \begin{array}{c} k \\ m \end{array} \right) \gamma^{k-m} u[k] \xrightarrow{z} \frac{z}{(z - \gamma)^m} \]

• Exercise: Find the ZSR \( y \) to input \( f[k] = 2^k u[k] = 2e^{jk\pi/2} u[k] \) of the marginally stable system \( H(z) = \frac{4}{(z^2 + 1)} \).

---

**PFE of \( M/N \) when \( M(0) \neq 0 \)**

• If \( M(0) \neq 0 \) (so cannot factor \( z \) from \( M(z) \)), then just perform long division if \( \deg(M) \geq \deg(N) \) to get a strictly proper rational polynomial, factor \( N \) to find the poles, and find the PFE as before.

• When taking inverse \( z \)-transform, recall the \( z \)-transform pair

\[ \beta^{k-1}u[k-1] \xrightarrow{z} \frac{1}{z - \beta}, \quad |z| > |\beta| \]
PFE without factoring $z$ from the numerator first

- For example, to find the ZSR to $f[k] = 2(-1)^ku[k]$ of the system
  
  \[ y[k+1] - 4y[k] = 5f[k], \]
  
  take the $z$-transform to get
  
  \[ \frac{Y_{ZS}(z)}{z-4} = F(z) = \frac{10z}{(z-4)(z+1)} \]

  \[ \Rightarrow y_{ZS}[k] = 8(4)^k u[k-1] + 2(-1)^k u[k-1] \]

- Note that the unit-pulse response is
  
  \[ h[k] = Z^{-1}(H)[k] = 5(4)^k u[k-1], \]
  
  and that, by delaying the difference equation to get
  
  \[ y[k] = -4y[k-1] + 5f[k-1], \]
  
  we see that (the ZSR) $y_{ZS}[0] = 0$.

- Exercise: First factor $z$ from the numerator of $Y_{ZS}$ before PFE to show that
  
  \[ y_{ZS}[k] = 2(4)^k u[k] - 2(-1)^k u[k]. \]
  
  Is this result different? Check for $k = 0$ and $k > 0$.

PFE and eigenresponse for asymptotically stable systems

- The total response of a SISO LTI system to input $f$ is of the form
  
  \[ Y(z) = H(z)F(z) + \frac{P(z)}{Q(z)} \]
  
  where $P_1$ depends on the initial conditions and the RoC is the intersection of that of input $F = Zf$ and the system characteristic modes.

- Unlike for DTFT notation, here write $H(z) = \frac{P(z)}{Q(z)} = (Zh)(z)$.

- Suppose the system is BIBO/asymptotically stable and the input is a sinusoid at frequency (angle) $\Omega_o$, $f[k] = A e^{j(\Omega_o k + \phi)} u[k] = A e^{j\phi} (e^{j\Omega_o})^k u[k]$ with $A > 0$
  
  \[ \Rightarrow F(z) = A e^{j\phi} / (z - e^{j\Omega_o}) \] with RoC $|z| > 1$.

- Since $e^{j\Omega_o}$ cannot be a system pole (owing to asymptotic stability all poles have modulus strictly less than one), we can use Heaviside cover-up on

  \[ \frac{Y_{ZS}(z)}{z-\Omega_o} = H(z)F(z) = z \frac{P(z)}{Q(z)} e^{j\phi} \] to get

  \[ Y_{ZS}(z) = z H(e^{j\Omega_o}) A e^{j\phi} + \text{char. modes} = H(e^{j\Omega_o}) F(z) + \text{char. modes}. \]

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Thus, the total response of an asymptotically stable system to a sinusoidal input $f$ at frequency $\Omega_o$ is

$$y[k] = H(e^{j\Omega_o})f[k] + \text{linear combination of characteristic modes.}$$

So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as $k \to \infty$,

$$y[k] \to H(e^{j\Omega_o})f[k] = H(e^{j\Omega_o})Ae^{j(\Omega_o k + \phi)} = |H(e^{j\Omega_o})|Ae^{j(\Omega_o k + \phi + \angle H(e^{j\Omega_o}))},$$

where again,

- $H = P/Q$ is the system’s transfer function,
- $|H(e^{j\Omega_o})|$ is the system’s magnitude response at frequency (angle) $\Omega_o$, and
- $\angle H(e^{j\Omega_o})$ is the system’s phase response at $\Omega_o$.

Laplace’s approximation: the rate at which the total response converges to the eigenresponse is according to the characteristic value of largest modulus,

- which will be $< 1$ owing to the stability assumption,
- i.e., giving the modes(s) that $\to 0$ slowest.

In continuous-time systems, it’s the characteristic value of largest real part, which will be negative owing to stability assumption.
Canonical (ZS) system-realizations - direct form

• Consider the proper \((m \leq n)\) transfer function

\[
H(z) = \frac{P(z)}{Q(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{Y(z)}{F(z)}
\]

• The direct-form realization employs the interior system state \(X := F/Q\), i.e., \(F = QX\) and \(Y = PX\) where the former implies (with \(a_n = 1\)),

\[
F(z) = \sum_{r=0}^{n} a_r z^r X(z) \Rightarrow z^n X(z) = F(z) - \sum_{r=0}^{n-1} a_r z^r X(z).
\]

• For \(n = 2\), there are two “system states” (outputs of unit delays), \(X\) and \(zX\) (respectively, \(x[k]\) and \((\Delta^{-1}x)[k] = x[k+1]\)):

![Diagram](Image)

Canonical system-realizations - direct form (cont)

• Now adding \(Y = PX\), we finally get the direct-form canonical system-realization of \(H\):

![Diagram](Image)

• Again, state variables taken as outputs of unit delays, here: \(x, \Delta^{-1}x, \ldots, \Delta^{-(n-1)}x\).

• If \(b_n = b_2 \neq 0\), there is direct coupling of input and output, \(H\) is proper but not strictly so, \(h = Z^{-1}H\) has a unit-pulse component \(b_2\delta\).

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Note that this $n = 2$ example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for $n = 2$, $H(z) = P(z)/Q(z)$ where

$$Q(z) = z^2 + a_1z + a_0 = (z - \alpha)^2 + \beta^2$$

for $\alpha, \beta \in \mathbb{R}$, so the poles are $\alpha \pm j\beta$.

---

**Canonical system realizations by PFE**

- In the general case of a proper transfer function, we can use partial-fraction expansion
  - grouping the terms corresponding to a complex-conjugate pair of poles, i.e., a second-order denominator, and
  - using a direct-form realization for these terms.

- Besides the PFE-based and direct-form realizations, there are other (zero-state) system realizations, e.g., "observer" canonical.

- For proper rational-polynomial transfer functions $H = P/Q$, all of these realizations involve $n$ (degree of $Q$) unit delays, the output of each being a different interior state variable of the system.
Canonical system realizations by PFE - example

\[ H(z) = \frac{0.3 z^2 - 0.1}{z^2 - 0.1 z - 0.3} = 0.3 + \frac{0.3 z - 0.01}{(z - 0.6)(z + 0.5)} = 0.3 + \frac{0.17/1.1}{z - 0.6} + \frac{0.16/1.1}{z + 0.5} \]

Note that one cannot factor \( z \) from the numerator of \( H \).

**Exercise:** Find a realization for this transfer function \( H \) by

1. splitting/forking the input signal \( F \),
2. using the direct canonical form for each of these 3 terms of \( H \) found by long division and PFE, and
3. summing three resulting output signals to get the (ZS) output \( Y = H F \).

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**Digital Proportional-Integral (PI) system**

- Consider a continuous-time signal \( x \) sampled every \( T \) seconds,
  \[ \forall k \in \mathbb{Z}^+, \ x[k] = x(kT), \]
  and its integral \( y(t) = \int_0^t x(\tau) d\tau \).
- The sampled integral can be approximated, \( y(kT) \approx y[k] \), by the trapezoid rule,
  \[ y[k] = y[k - 1] + \frac{x[k - 1] + x[k]}{2} T. \]
- In the complex-frequency domain,
  \[ Y(z) = Y(z) z^{-1} + \frac{X(z) z^{-1} + X(z) T}{2} \]
  \[ \Rightarrow \frac{Y(z)}{X(z)} = \frac{T}{2} \frac{1 + z^{-1}}{1 - z^{-1}}. \]
Digital PI system (cont)

- So, a digital PI transfer function would be of the form,
  \[ G(z) = K_p + \frac{K_i T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}. \]
  for constants \( K_p, K_i. \)

- In practice, PID or PI systems \( G \) are commonly used to control a plant \( H \), where \( G \) may be in series with \( H \) or in the feedback branch.

**Exercises:**
- Draw the direct-form canonical realization for \( G \).
- Draw the block diagram for the closed-loop system with negative feedback: \( Y = HX \) and \( X = F - GY \) where \( H \) is the (open-loop) system.
- Find the closed-loop transfer function \( Y/F \) and recall the pole placement problem to stabilize \( H \).

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Recursive Least Squares (RLS) Filter - Introduction

- Consider a LTI system with input \( f \) and output \( y \),
  \[ y[k] = \sum_{r=0}^{K} h[k-r] f[r] + v[k], \quad k \in \mathbb{Z}, \]
  where \( v \) is an additive noise process and \( K \) is the maximum system order.

- The system (unit-pulse response) \( h \) is not known.

- Past values of the output \( y \) are observed (known).

- At time \( k \), the objective is to forecast the next output \( \hat{y}[k+1] \), based on the assumed known/observed quantities:
  - the next input \( f[k+1] \),
  - the past \( R \) input-output pairs \( \{f[r], y[r]\}_{k-R+1 \leq r \leq k} \).
RLS objective and $R^{th}$-order linear tap filter

- The output of an $R^{th}$-order RLS tap-filter at time $k$ is,

$$\hat{y}_k[i] = \sum_{r=i-R+1}^{i} \eta_k[i-r] f[r], \ i \leq k + 1.$$  

- The objective of this filter at time $k$ is to accurately estimate the system output $y[k+1]$ with $\hat{y}_k[k+1]$ by choosing the $R$ filter coefficients $\eta_k[k-R+1], ..., \eta_k[k-1], \eta_k[k]$ that minimize the following sum-of-square-error objective:

$$E_k = \sum_{r=k-R+1}^{k} \lambda^{k-r} |y[r] - \hat{y}_k[r]|^2 = \sum_{r=k-R+1}^{k} \lambda^{k-r} |e_k[r]|^2$$

where
- $\lambda > 0$ is a forgetting factor and
- error $e_k[r] := y[r] - \hat{y}_k[r]$.

Exercise: Prove the last equality.

---

RLS filter

- So, to minimize $E_k$, substitute $\hat{y}_k[r]$ into $E_k$ and solve

$$0 = \frac{\partial E_k}{\partial \eta_k[i]} \text{ for } i \in \{k-R+1, ..., k-1, k\}.$$  

- That is, $R$ equations in $R$ unknowns: for $i \in \{k-R+1, ..., k-1, k\}$,

$$0 = \sum_{r=k-R+1}^{k} 2\lambda^{k-r} e_k[r] \frac{\partial e_k[r]}{\partial \eta_k[i]}$$

$$= \sum_{r=k-R+1}^{k} 2\lambda^{k-r} (y[r] - \hat{y}_k[r]) \left( -\frac{\partial \hat{y}_k[r]}{\partial \eta_k[i]} \right)$$

$$= \sum_{r=k-R+1}^{k} 2\lambda^{k-r} (\hat{y}_k[r] - y[r]) f[r-i]$$

- Exercise: Prove the last equality.
Substituting $\hat{y}_k[r]$, rewrite these equations to get the following $R$ equations in $R$ unknowns $\eta_k[i]$ that are $E_k$-minimizing: for $i \in \{k - R + 1, \ldots, k - 1, k\}$,

$$\sum_{r=k-R+1}^{k} \lambda^{k-r} f[r - i] \sum_{\ell=r-R+1}^{r} f[\ell] \eta_k[r - \ell] = \sum_{r=k-R+1}^{k} \lambda^{k-r} y[r] f[r - i]$$

**Exercise:** Prove the last equality and write it in matrix form.

**Exercise:** Research how the $E_k$-minimizing filter parameters $\eta_k$ can be computed recursively, i.e., using $\eta_{k-1}$.

The filter order $R$ can also be “trial adapted” to discover the system order $K$ so that the error-minimizing filter parameters $\eta_k$ “track” the system unit-pulse response $h$ over time $k$.

Note the required initial “warm-up” period of $R$ time-units where the outputs of system $h$ are simply observed and recorded and no estimates are made.

**Exercise:** If there was no additive noise process $v$ and the system unit-pulse response $h$ had finite support (i.e., a FIR system with $K < \infty$), show how $h$ can be deduced from input-output $(f, y)$ observations.