• Properties of signals and systems (difference equations)

• Time-domain analysis
  – ZIR, system characteristic values and modes
  – ZSR, unit-pulse response and convolution
  – stability, eigenresponse and transfer function

• Frequency-domain analysis

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Time-domain analysis of discrete-time LTI systems

• Discrete-time signals

• Difference equation single-input, single-output systems in discrete time

• The zero-input response (ZIR): characteristic values and modes

• The zero (initial) state response (ZSR): the unit-pulse response, convolution

• System stability

• The eigenresponse and (zero state) system transfer function
Discrete-time signal by sampling a continuous-time signal

- Consider a continuous-time signal \( x : \mathbb{R} \to \mathbb{R} \) sampled every \( T > 0 \) seconds

\[
x(kT + t_0) = x[k] \quad \text{for} \ k \in \mathbb{Z},
\]

where
- \( t_0 \) is the sampling time of the 0\(^{th}\) sample, and
- \( T \) is assumed less than the Nyquist sampling period of \( x \), and
- \( x[k] \) (with square brackets) is the \( k\(^{th}\) sample itself.

- Here \( x[\cdot] \) is a discrete-time signal defined on \( \mathbb{Z} \).

Example of sampling with \( t_0 = 0 \) and positive signal \( x \)

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Introduction to signals and systems in discrete time

- A discrete-time function (or signal) \( x : A \to B \) is one with countable (time) domain \( A \).  

- We will take the range \( B = \mathbb{R} \) or \( B = \mathbb{C} \).  

- Typically, we will herein take domain \( A = \mathbb{Z} \) or \( \mathbb{Z}^{\geq n} \) for some (finite) integer \( n \geq 0 \).  

- Some properties of signals are as in continuous time: e.g., periodic, causal, bounded, even or odd.  

- Similarly, some signal operations are as in continuous time: e.g., spatial shift/scale, superposition, time reflection, and (integer valued) time shift.  

Time scaling: decimation and interpolation

- Time scaling can be implemented in continuous time prior to sampling at a fixed rate,  

- or the sampling rate itself could be varied (again recall the Nyquist sampling rate).  

- In discrete time, a signal \( x = \{x[k] \mid k \in \mathbb{Z}\} \) can be *decimated* (subsamped) by an integer factor \( L \neq 0 \) to create the signal \( x_L \) defined by  

\[
x_L[k] = x[kL], \quad \forall k \in \mathbb{Z},
\]

i.e., \( x_L \) is defined only by every \( L^{th} \) sample of \( x \).  

- A discrete-time signal \( x \) can also be *interpolated* by an integer factor \( L > 0 \) to create \( x_L \) satisfying  

\[
x_L[kL] = x[k], \quad \forall k \in \mathbb{Z}.
\]

- For an interpolated signal \( x_L \), the values of \( x_L[r] \) for \( r \) not a multiple of \( L \) (i.e., \( \forall k \in \mathbb{Z} \) s.t. \( r \neq kL \)) can be set in different ways, e.g., between consecutive samples:  

  - (piecewise constant) hold: \( x_L[r] = x_L[L \lfloor r/L \rfloor] = x[\lfloor r/L \rfloor] \)  

  - linear interpolation:  

\[
x_L[r] = x[\lfloor r/L \rfloor] + \frac{r - L \lfloor r/L \rfloor}{L}(x[\lfloor r/L \rfloor + 1] - x[\lfloor r/L \rfloor])
\]
Time scaling: decimation and interpolation - Questions

- Is the functional mapping $x \rightarrow x_L$ causal for linear interpolation?

- Is the hold causal?

- **Exercise**: Show that if a periodic, continuous-time signal $x(t)$, with period $T_0$, is periodically sampled every $T$ seconds, then the resulting discrete-time signal $x[k]$ is periodic if and only if $T/T_0$ is rational.

Unit pulse $\delta$, unit step $u$, unit delay $\Delta$, and convolution *

- Some important signals in discrete time are as those in continuous time, e.g., polynomials, exponentials, unit step.

- In discrete time, rather than the (unit) impulse, there is unit pulse (Kronecker delta):
  
  $\delta[k] = \begin{cases} 
  1 & \text{if } k = 0 \\
  0 & \text{else}
  \end{cases}$

- Any discrete-time signal $x$ can thus be written as
  
  $x[k] = \sum_{r=-\infty}^{\infty} x[r] \delta[k-r] = \sum_{r=-\infty}^{\infty} x[k-r] \delta[r] = (x * \delta)[k]$ 

- or just $x = x * \delta$, i.e., the unit pulse $\delta$ is the identity of discrete-time convolution.

- Define the operator $\Delta$ as unit delay (time-shift), i.e., $\forall$ signals $y$ and $\forall k, r \in \mathbb{Z}$,
  
  $(\Delta^r y)[k] := y[k-r]$. 

- The discrete-time unit step $u$ satisfies $\delta = u - \Delta u$, equivalently: $\forall k \in \mathbb{Z},$
  
  $\delta[k] = u[k] - u[k-1]$ and $u[k] = \sum_{r=0}^{\infty} (\Delta^r \delta)[k] = \sum_{r=0}^{\infty} \delta[k-r]$. 

Exponential signals in discrete time

- Real-valued exponential (geometric) signals have the form $x[k] = A\gamma^k$, $k \in \mathbb{Z}$, where $A, \gamma \in \mathbb{R}$.

- Consider the scalar $z = \gamma e^{j\Omega} \in \mathbb{C}$ with $\gamma > 0$, $\Omega \in \mathbb{R}$, where again $j := \sqrt{-1}$.

- Generally, complex-valued exponential signals have the (polar) form
  
  $x[k] = Ae^{j\phi}z^k = A\gamma^k e^{j(\Omega k + \phi)}$, $k \in \mathbb{Z}$,

  where w.l.o.g. we can take

  $-\pi < \Omega, \phi \leq \pi$ and real $A > 0$.

- **Exercise:** Show this complex-valued exponential is periodic if and only if $\Omega/\pi$ is rational.

- By the Euler-De Moivre identity,
  
  $x[k] = A\gamma^k e^{j(\Omega k + \phi)} = A\gamma^k \cos(\Omega k + \phi) + jA\gamma^k \sin(\Omega k + \phi)$, $k \in \mathbb{Z}$.
Systems - single input, single output (SISO)

- In the figure, \( f \) is an input signal that is being transformed into an output signal, \( y \), by the depicted system (box).

- To emphasize this functional transformation, and clarify system properties, we will write the output signal (i.e., system “response” to the input \( f \)) as
  \[ y = Sf, \]
  where, again, we are making a statement about functional equivalence:
  \[ \forall k \in \mathbb{Z}, \quad y[k] = (Sf)[k]. \]

- Again, \( Sf \) is not \( S \) “multiplied by” \( f \), rather a functional transformation of \( f \).

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SISO systems (cont)

- The \( n \) signals \( \{x_1, x_2, ..., x_n\} \) are the internal states of the system.

- The states can be taken as outputs of unit-delay operators, \( \Delta \), i.e.,
  \[ \forall k \in \mathbb{Z}, \quad (\Delta y)[k] = y[k-1]. \]

- Some properties of systems are as in continuous time: e.g., linear, time invariant, causal, memoryless, stable (with different conditions for stability as we shall see).
For linear and time-invariant systems in discrete time, relate output \( y \) to input \( f \) via difference equation in standard (time-advance operator) form:

\[
\forall k \geq -n, \quad y[k+n] + a_{n-1}y[k+n-1] + \ldots + a_1y[k+1] + a_0y[k] = b_m f[k+m] + b_{m-1}f[k+m-1] + \ldots + b_1f[k+1] + b_0f[k],
\]
given

- scalars \( a_k \) for \( 0 \leq k \leq n \), with \( a_n := 1 \), and scalars \( b_k \) for \( 0 \leq k \leq m \),
- \( a_0 \neq 0 \) or \( b_0 \neq 0 \) (so that \( P, Q \) are of minimal degree), and
- initial conditions \( y[-n], y[-n+1], \ldots, y[-2], y[-1] \).

Compact representation of the above difference equation:

\[
Q(\Delta^{-1})y = P(\Delta^{-1})f,
\]
where polynomials

\[
Q(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad P(z) = \sum_{k=0}^{m} b_k z^k,
\]
\( \Delta^{-1} \) is the unit time-advance operator: \( (\Delta^{-1}y[k]) \equiv y[k+1], (\Delta^{-r}y)[k] \equiv y[k+r] \)

Discussion: conditions for causality and difference equation in \( \Delta \)

- **Exercise:** Show that the difference equation \( Q(\Delta^{-1})y = P(\Delta^{-1})f \) is not causal if \( \deg(P) = m > n = \deg(Q) \), i.e., the system is not proper.

- A not anti-causal difference equation can be implemented simply using memory to store a sliding window of prior values of the input \( f \) and delaying the output.

- **Example:** Decoding B (bidirectional) frames of MPEG video.
Numerical solution to difference equation by recursive substitution

- Given the system $Q(\Delta^{-1}) y = P(\Delta^{-1}) f$, the input $f[k]$ for $k \geq 0$, and initial conditions $y[-n], ..., y[-1]$,

- one can recursively solve for $y$ ($y[k]$ for $k \geq 0$) by rewriting the system equation as
  \[
y[k + n] = -\sum_{r=0}^{n-1} a_r y[k + r] + \sum_{r=0}^{m} b_r f[k + r] \quad \text{for } k \geq -n
  \]
  \[
  \Rightarrow y[k] = -\sum_{r=0}^{n-1} a_r y[k + r - n] + \sum_{r=0}^{m} b_r f[k + r - n] \quad \text{for } k \geq 0.
  \]

- For example, the difference equation in standard form,
  \[
y[k + 1] + 3y[k] = 7f[k + 1] \quad \text{for } k \geq -1,
  \]
can be rewritten as
  \[
y[k] = -3y[k - 1] + 7f[k] \quad \text{for } k \geq 0.
  \]

- So, given $f$ and $y[-1]$ we can recursively compute

- Exercise: If $f = u$ and $y[-1] = 7$ then find $y[3]$ for this example.

Approach to closed-form solution: ZIR and ZSR

- The total response $y$ of $P(\Delta^{-1}) f = Q(\Delta^{-1}) y$ to the given initial conditions and input $f$ is a sum of two parts:
  - the ZSR, $y_{ZS}$, which solves
    \[
P(\Delta^{-1}) f = Q(\Delta^{-1}) y_{ZS} \quad \text{with zero i.c.'s, i.e., with } 0 = y[-n] = ... = y[-1];
    \]
  - the ZIR, $y_{ZI}$, which solves
    \[
    0 = Q(\Delta^{-1}) y_{ZI} \quad \text{with the given initial conditions}.
    \]

- The total response $y$ of the system to $f$ and the given initial conditions is, by linearity,
  \[
y = y_{ZI} + y_{ZS}.
  \]

- We will determine the ZIR by finding the characteristic modes of the system.

- We will determine the ZSR by convolution of the input with the (zero state) unit-pulse response, the latter also in terms of characteristic modes.
Consider again the difference equation:
\[
\forall k \geq -1, \quad y[k + 1] + 3y[k] = 7f[k + 1],
\]
i.e., \( Q(z) = z + 3 \) with degree \( n = 1 \), and \( P(z) = 7z \) with degree \( m = 1 \).

**Exercise:** Show that the following system corresponds to this difference equation.

```
\[ f \rightarrow 7 \rightarrow y \]
```

By recursive substitution, the total response is, \( \forall k \geq -1 \):

\[
y[k] = -3y[k - 1] + 7f[k] \\
= -3(-3y[k - 2] + 7f[k - 1]) + 7f[k] \\
= (-3)^2y[k - 2] - 3 \cdot 7f[k - 1] + 7f[k] \\
= \ldots \\
= (-3)^{k+1}y[-1] + \sum_{r=0}^{k} (-3)^{k-r}f[r] \\
= (-3)^{k+1}y[-1] + \sum_{r=0}^{\infty} h[k - r]f[r] =: (-3)^{k+1}y[-1] + (h * f)[k],
\]

where \( h[k] := 7(-3)^{k}u[k] \) is the (zero state) unit-pulse response,
\( y[-1] \) is the given (\( n = 1 \)) initial condition, and
we have defined the discrete-time convolution operator with \( \sum_{r=0}^{1}(...) := 0 \).
• **Exercise:** Prove by induction this expression for $y[k]$ for all $k \geq -1$.

• **Exercise:** Prove convolution is commutative: $h * f = f * h$.

• So, we can write the total response $y = y_{ZI} + y_{ZS}$ starting from the time of oldest initial condition:
  \[
  \forall k \geq -1, \quad y_{ZI}[k] = (-3)^{k+1}y[-1]
  \]
  \[
  \forall k \geq -1, \quad y_{ZS}[k] = u[k] \sum_{r=0}^{k} 7(-3)^{k-r}f[r] = u[k](h * f)[k]
  \]
  where $y_{ZS}[k] = 0$ when $k < 0$.

• Obviously, this example involves a linear, time-invariant and causal system as described by the difference equation above.

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**Total response - discussion**

• Note that in CMPSC 360, we don’t restrict our attention to linear and time-invariant difference equations.

• We use recursive substitution to guess at the form of the solution and then verify our guess by an inductive proof.

• In this course, we will describe a systematic approach to solve any LTIC difference equation,
  \[i.e.,\] to solve for the output of a DT-LTIC system given the input and initial conditions.

• And again as in continuous time, we will see important insights about discrete-time signals and LTIC systems through frequency-domain representations and analysis.
ZIR - the characteristic values

- Note that $\forall k$, $\Delta^{-r} z^k = z^{k+r} = r_z z^k$, i.e., the $r$-units time-advance operator, $\Delta^{-r}$, is replaced by the scalar $z^r$ for all $r \in \mathbb{Z}$.

- Our objective is to solve for the ZIR, i.e., solve $Q(\Delta^{-1})y \equiv 0$ given $y[-n], y[-n+1], ..., y[-2], y[-1]$.

- Note that exponential (or "geometric") functions, $\{z^k \mid k \in \mathbb{Z}\}$ for $z \in \mathbb{C}$, are eigenfunctions of time-shift operators of the form $Q(\Delta^{-1})$ for a polynomial $Q$.

- That is, for any non-zero scalar $z \in \mathbb{C}$, if we substitute $y[k] = z^k \forall k \in \mathbb{Z}$ we get: $\forall k \in \mathbb{Z}$, $(Q(\Delta^{-1})y)[k] = Q(\Delta^{-1})z^k = Q(z)z^k$.

- So, to solve $Q(z)z^k \equiv 0$ for all time $k \geq 0$, when $z \neq 0$ we require $Q(z) = 0$, the characteristic equation of the system.

ZIR - the characteristic values (cont)

- If $z$ is a root of the characteristic polynomial $Q$ of the system, then
  - $z$ would be a characteristic value of the system, and
  - the signal $\{z^k\}_{k \geq 0}$ is a characteristic mode of the system when $z \neq 0$, i.e., $Q(\Delta^{-1})z^k = 0 \forall k \geq 0$.

- Since $Q$ has degree $n$, there are $n$ roots of $Q$ in $\mathbb{C}$, each a system characteristic value.
• Let \( n' \leq n \) be the number of non-zero roots of \( Q \), i.e., \( \tilde{Q}(z) = Q(z)/z^{n-n'} \) is a polynomial satisfying \( \tilde{Q}(0) \neq 0 \).

• Though there may be some repeated roots of the characteristic polynomial \( Q \), there will always be \( n' \) different, linearly independent characteristic modes, \( \mu_k \), i.e.,

\[
\forall k \geq -n, \sum_{r=1}^{n'} c_r \mu_r[k] = 0 \iff \forall r, \text{ scalars } c_r = 0.
\]

• When \( n = n' \), by system linearity, we will be able to write

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{r=1}^{n} c_r \mu_r[k],
\]

for scalars \( c_r \in \mathbb{C} \) that are found by considering the given initial conditions

\[
y[k] = \sum_{r=1}^{n} c_r \mu_r[k] \quad \text{for } k \in \{-n, \ldots, -2, -1\},
\]

i.e., \( n \) equations in \( n \) unknowns \( (c_r) \).

• The linear independence of the modes implies linear independence of these \( n \) equations in \( c_r \), and so they have a unique solution.

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### ZIR - the case of different, non-zero, real characteristic values

• If there are \( n \) different non-zero roots of \( Q \) in \( \mathbb{R} \), \( z_1, z_2, \ldots, z_n \), then there are \( n \) characteristic modes: for \( r \in \{1, 2, \ldots, n\} \),

\[
\forall \text{ time } k, \quad \mu_r[k] = z_r^k
\]

• Therefore,

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{r=1}^{n} c_r z_r^k
\]

• The \( n \) unknown scalars \( c_r \in \mathbb{R} \) can be solved using the \( n \) equations:

\[
y[k] = \sum_{r=1}^{n} c_r z_r^k \quad \text{for } k \in \{-n, -n+1, \ldots, -2, -1\}.
\]
ZIR - the case of different, non-zero, real characteristic values

- **Example:** Consider the difference equation:
  \[ \forall k \geq -3, \quad 2y[k+3] - 10y[k+2] + 12y[k+1] = 3f[k+2], \]
  with \( y[-2] = 1 \) and \( y[-1] = 3 \).

- That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, P(z) = (3/2)z \) and \( m = 1 \).

- So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and the ZIR is
  \[ \forall k \geq -n = -2, \quad y_{ZI}[k] = c_13^k + c_22^k \]

- Using the initial conditions to find the scalars \( c_1, c_2 \):
  \[
  1 = y[-2] = c_13^{-2} + c_22^{-2} \quad \text{and} \quad 3 = y[-1] = c_13^{-1} + c_22^{-1}.
  \]

- **Exercise:** Now solve for \( c_1 \) and \( c_2 \).

- Note: When a coefficient \( c \) is worked out to be zero, it may not be exactly zero in practice, and the corresponding characteristic mode \( z^k \) will increasingly contribute to ZIR \( y_{ZI} \) over time if \( |z| > 1 \) (i.e., an "unstable" mode in discrete time).

ZIR - the case of not-real characteristic values

- The characteristic polynomial \( Q \) may have non-real roots, but such roots come in complex-conjugate pairs because \( Q \)'s coefficients \( a_k \) are all real.

- For example, if the characteristic polynomial is
  \[ Q(z) = (z - 1)(z^2 - 2z - 2) \]
  then the characteristic values (\( Q \)'s roots) are
  \[ -1, \ 1 \pm j\sqrt{3} \quad \text{again recalling} \quad j = \sqrt{-1}. \]

- Because we have three different characteristic values \( \in \mathbb{C} \), we can specify three corresponding characteristic modes,
  \[ (-1)^k, (1 + j\sqrt{3})^k, (1 - j\sqrt{3})^k, \quad \forall k \geq 0, \]
  and construct the ZIR as
  \[
  \forall k \geq -n = -3, \quad y_{ZI}[k] = c_1(-1)^k + c_2(1 + j\sqrt{3})^k + c_3(1 - j\sqrt{3})^k
  \]
  \[ = c_1(-1)^k + c_22^k e^{k\pi/3} + c_32^k e^{-k\pi/3} \]
  where
  - \( c_1 \in \mathbb{R} \) and \( c_2 = c_3 \in \mathbb{C} \) so that \( y_{ZI} \) is real-valued, and again,
  - these scalars are determined by the \( n = 3 \) given (real) initial conditions: \( y[-3], y[-2], y[-1] \).
ZIR - not-real characteristic values with real characteristic modes

• By the Euler-De Moivre identity for the previous example,
  \[ y_{ZI}[k] = c_1(-1)^k + (c_2 + c_3)2^k \cos(k\pi/3) + j(c_2 - c_3)2^k \sin(k\pi/3) \]
  \[ = c_1(-1)^k + 2\text{Re}(c_2)2^k \cos(k\pi/3) - 2\text{Im}(c_2)2^k \sin(k\pi/3) \]

• Again, because all initial conditions are real and \( Q \) has real coefficients, \( y_{ZI} \) is real valued and so \( c_3 = c_2 \Rightarrow c_2 + c_3, j(c_2 - c_3) \in \mathbb{R} \).

• In general, consider two complex conjugate characteristic values \( v \pm jq \) corresponding to two complex-valued characteristic modes \( |z|^k e^{\pm jkz} \), where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).

• One can use Euler’s identity to show that the corresponding real-valued characteristic modes are
  \[ |z|^k \cos(k\angle z), \ |z|^k \sin(k\angle z) \]

ZIR - the case of repeated characteristic values

• Consider the case where at least one characteristic value is of order > 1, i.e., there are repeated roots of the characteristic polynomial, \( Q \).

• For example, \( Q(z) = (z + 0.75)^3(z - 0.5) \) has a triple (twice repeated) root at \(-0.75\) and a single root at \(0.5\).

• Again, \( \{-0.75\}^k \) is a characteristic mode because \( Q(\Delta^{-1})(-0.75)^k \equiv 0 \) follows from
  \[ (\Delta^{-1} + .75)(-0.75)^k = \Delta^{-1}(-.75)^k + .75(-.75)^k \]
  \[ = (-.75)^{k+1} + .75(-.75)^k \]
  \[ = 0. \]

• Similarly, \( \{0.5\}^k \) is a characteristic mode since \( (\Delta^{-1} - 0.5)(0.5)^k \equiv 0 \).

• Also, \( \{k(-0.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})k(-.75)^k \equiv 0 \) follows from
  \[ (\Delta^{-1} + .75)^2k(-.75)^k \]
  \[ = (\Delta^{-2} + 1.5\Delta^{-1} + (.75)^2)k(-.75)^k \]
  \[ = \Delta^{-2}k(-.75)^k + 1.5\Delta^{-1}k(-.75)^k + (.75)^2k(-0.75)^k \]
  \[ = (k + 2)(-.75)^{k+2} + 1.5(k + 1)(-.75)^{k+1} + (.75)^2k(-0.75)^k \]
  \[ = (-.75)^{k+2}(k + 2) - 2(k + 1) + k \]
  \[ = 0. \]
• Similarly, \( \{ k^2(-.75)^k \} \) is also a characteristic mode because
\[
(\Delta^{-1} + .75)^3 k^2 (-.75)^k = 0.
\]

• Note that without three such linearly independent characteristic modes
\[
\{ (-.75)^k, k(-.75)^k, k^2(-.75)^k ; \ k \geq 0 \}
\]
for the twice-repeated (triple) characteristic value -.75, the initial conditions will create an "overspecified" set of \( n \) equations involving fewer than \( n \) "unknown" coefficients \( (c_k) \) of the linear combination of modes forming the ZIR.

• For this example,
\[
y_{zi}[k] = c_0(-0.75)^k + c_1(-0.75)^k + c_2k^2(-0.75)^k + c_3(0.5)^k, \ k \geq -4.
\]

• If the given initial conditions are, say,
\[
y[-4] = 12, y[-3] = 6, y[-2] = -5, y[-1] = 10,
\]
the four equations to solve for the four unknown coefficients \( c_k \) are:
\[
\begin{align*}
y_{zi}[-4] &= (-.75)^{-4}c_0 + (-4)(-.75)^{-4}c_1 + (-4)^2(-.75)^{-4}c_2 + (.5)^{-4}c_3 = 12 \\
y_{zi}[-3] &= (-.75)^{-3}c_0 + (-3)(-.75)^{-3}c_1 + (-3)^2(-.75)^{-3}c_2 + (.5)^{-3}c_3 = 6 \\
y_{zi}[-2] &= (-.75)^{-2}c_0 + (-2)(-.75)^{-2}c_1 + (-2)^2(-.75)^{-2}c_2 + (.5)^{-2}c_3 = -5 \\
y_{zi}[-1] &= (-.75)^{-1}c_0 + (-1)(-.75)^{-1}c_1 + (-1)^2(-.75)^{-1}c_2 + (.5)^{-1}c_3 = 10
\end{align*}
\]

ZIR - general case of repeated, non-zero characteristic values

• In general, a set of \( r \) linearly independent modes corresponding to a non-zero characteristic value \( z \in \mathbb{C} \) repeated \( r - 1 \) times are
\[
k^{r-1}z^k, k^{r-2}z^k, \ldots, kz^k, z^k, \text{ for } k \geq 0.
\]

• Also, if \( v \pm jq \) are characteristic values repeated \( r - 1 \) times, with \( v, q \in \mathbb{R} \) and \( q \neq 0 \), we can use the \( 2k \) real-valued modes
\[
k^a|z|^k \cos(kz), k^a|z|^k \sin(kz), \text{ for } a \in \{ 0, 1, 2, \ldots, r - 1 \},
\]
where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).
ZIR - when some characteristic values are zero

- Again let \( n' \leq n \) be the number of non-zero roots of \( Q \) (characteristic values),
- \( i.e., r := n - n' \geq 0 \) is the order (1+repetition) of the characteristic value 0, and
- \( r \geq 0 \) is the smallest index such that \( (\text{the coefficient of } Q) a_r \neq 0 \).
- So, there is a polynomial \( \tilde{Q} \) such that \( Q(z) = z^r \tilde{Q}(z) \) and \( \tilde{Q}(0) \neq 0 \).
- Because the constant signal zero cannot be a characteristic mode, we add \( r = n - n' \)
time-advanced unit-pulses:

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{i=1}^{r} C_i \delta[k + i] + y_N[k] \\
= C_r \delta[k + r] + C_{r-1} \delta[k + r - 1] + ... + C_1 \delta[k + 1] + y_N[k]
\]

where \( y_N \) is a “natural response” (linear combination of \( n' \) characteristic modes).

- The \( n \) initial conditions are then met by the \( r \) coefficients \( C_i \) of the advanced unit pulses
together with the \( n' = n - r \) coefficients of the characteristic modes in \( y_N \).

ZIR - when some characteristic values are zero - example

- Consider a fourth-order system with characteristic polynomial
  \( Q(z) = z^2(z + 1)^2 \).
- Thus the poles are 0, -1 each repeated and the (non-zero) characteristic modes are \((-1)^k, k(-1)^k\).
- So, the ZIR is, for \( k \geq -4 \):

\[
y_{ZI}[k] = C_2 \delta[k + 2] + C_1 \delta[k + 1] + c_1(-1)^k + c_2 k(-1)^k
\]

- That is, the ZIR has four unknown coefficients \( C_2, C_1, c_1, c_2 \) to account for the four (given)
initial conditions \( y[-4], y[-3], y[-2], y[-1] \).
Zero State Response - the unit-pulse response

- Recall the LTIC system

\[ \sum_{r=0}^{n} a_r \Delta^{-r} y =: Q(\Delta^{-1}) y = P(\Delta^{-1}) f := \sum_{r=0}^{m} b_r \Delta^{-r} f \]

with \( a_n \equiv 1, a_0 \neq 0 \) or \( b_0 \neq 0, m \leq n \).

- We can express any input signal

\[ f[k] = \sum_{r=0}^{\infty} f[r] \delta[k - r] \quad \forall k \geq 0, \quad i.e., \forall f, f = f * \delta. \]

- So the unit pulse \( \delta \) is the identity of the convolution operator in discrete time.

- Thus, by LTI, the ZSR \( y_{zs} \) is the convolution of input \( f \) and ZSR \( h \) to unit pulse \( \delta \),

\[ y_{zs}[k] = \sum_{r=0}^{\infty} f[r] h[k - r] = (f * h)[k], \quad \forall k \geq 0, \]

- \( h \) is called the unit-pulse response of the LTIC system, i.e.,

\[ Q(\Delta^{-1}) h = P(\Delta^{-1}) \delta \quad s.t. \quad h[k] = 0 \quad \forall k < 0. \]

Computing an LTIC system’s unit-pulse response, \( h \)

- For the LTIC system in standard form, if \( a_0 \neq 0 \) then

\[ h = (b_0/a_0) \delta + y_{N} u \]

where \( y_{N} \) is a natural response of the system (linear combination of characteristic modes).

- Note that \( h[k] = 0 \) for all \( k < 0 \) owing to the unit step \( u \).

- The \( n \) scalars of the natural response \( y_{N} \) component of \( h \) are solved using

\[ (Q(\Delta^{-1})h)[k] = (P(\Delta^{-1})\delta)[k] \quad \text{for} \quad k \in \{-n, -n+1, \ldots, -2, -1\} \]
Unit-pulse response when zero is a characteristic value

- If \( r \geq 0 \) is the smallest index such that \( a_r \neq 0 \) (0 is a char. mode of order \( r \)), then may need to add \( r \) delayed unit-pulse terms to \( h \):

\[
h = \sum_{i=0}^{r-1} A_i \Delta^i \delta + \left( \frac{b_0}{a_r} \right) \Delta^r \delta + y_n u,
\]

where

- by definition of the standard form of the difference equation, if \( r > 0 \), \( a_0 = 0 \) so \( b_0 \neq 0 \), and
- \( r \leq n \) since \( 0 \neq a_n := 1 \).

- So if \( r = 0 \) (i.e., \( a_0 \neq 0 \)), then \( A_0 = \frac{b_0}{a_0} \) as above, where \( \sum_{i=0}^{r-1} \ldots := 0 \).

- Exercise: Prove \( A_r = b_0/a_r \) for \( 0 \leq r \leq n \).

- Thus, zero is a characteristic value of degree \( r \geq 0 \), and

- there are \( r \) characteristic modes that will all be zero.

- The additional unit-pulse terms introduce \( r \) degrees of freedom in the form of the coefficients \( A_0, A_1, \ldots, A_{r-1} \) to accommodate the \( n = r + n' \) initial conditions of the unit-pulse response: \( h[-n] = h[-n + 1] = \ldots = h[-2] = h[-1] = 0 \).

Computing the ZSR - example 1

- Recall that the difference equation \( y = 7f - 3\Delta y \) corresponds to the above system; in standard form:

\[
\forall k \geq -1, \quad y[k + 1] + 3y[k] = 7f[k + 1].
\]

with \( Q(z) = z + 3 \), \( P(z) = 7z \) and \( n = 1 = m \).

- Since the system characteristic value is \(-3\) and \( b_0 = 0 \), the (zero state) unit-pulse response has the form \( h[k] = c(-3)^k u[k] \).

- The scalar \( c \) is solved by evaluating the above difference equation at time \( k = -1 \):

\[
(Q(\Delta^{-1})h)[-1] = (P(\Delta^{-1})\delta)[-1]
\]

\[
i.e., \quad h[0] + 3h[-1] = 7\delta[0]
\]

\[
\Rightarrow c + 3 \cdot 0 = 7 \cdot 1, \quad c = 7
\]
Computing the ZSR - example 1 (cont)

- So, \( h[k] = 7(-3)^k u[k] \).
- If the input is \( f[k] = 4(0.5)^k u[k] \), the system ZSR is, for all \( k \geq 0 \),
  \[
  y_{ZS}[k] = \sum_{r=0}^{k} h[r] f[k - r] = \sum_{r=0}^{k} 7(-3)^r 4(0.5)^{k-r} 
  \]
  \[
  = 28(0.5)^k \sum_{r=0}^{k} (-6)^r = 28(0.5)^k \frac{(-6)^{k+1} - 1}{-6 - 1} u[k] 
  \]
  \[
  = (24(-3)^k + 4(0.5)^k) u[k]. 
  \]
- Note how the ZIR \( y_{ZI} \) has a term that is a characteristic mode (excited by the input \( f \)) and a term that is proportional to the input \( f \) (this forced response is an eigenresponse).
- Exercise: For the difference equation, \( y[k+1] + 3y[k] = 7f[k] \forall k \geq -1 \): draw the block diagram, show that \( h[k] = 21(-3)^{k-1} u[k] + (7/3)\delta[k] \), and find the ZSR to the above input \( f \).
- Exercise: Read “sliding tape” method to compute convolution in Lathi, p. 595.

Computing the unit pulse response - example 2

- Find the ZSR of the following system to input \( f[k] = 2(-5)^k u[k] \):

- Exercise: show the difference equation for this system (in direct canonical form) is:
  \( \forall k \geq 0, \ y[k+2] - 5y[k+1] + 6y[k] = 1.5f[k+1] \)
- That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, P(z) = 1.5z \) and \( m = 1 \).
- So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and \( b_0 = 0 \) so the unit-pulse response \( h[k] = (c_1 3^k + c_2 2^k) u[k] \).
Computing the unit pulse response - example 2 (cont)

- To find the constants, evaluate the difference equation at $k = -1$:
  \[
  2h[1] - 10h[0] + 12h[-1] = 3\delta[0]
  \Rightarrow 2h[1] - 10h[0] = 3
  \Rightarrow (2 \cdot 3 - 10 \cdot 1)c_1 + (2 \cdot 2 - 10 \cdot 1)c_2 = 3
  \Rightarrow -4c_1 - 6c_2 = 3
  \]
  and at $k = -2$:
  \[
  2h[0] - 10h[-1] + 12h[-2] = 3\delta[-1] \Rightarrow 12h[0] = 0 \Rightarrow h[0] = 0
  \]
  \[
  c_1 + c_2 = 0.
  \]
- Thus, $c_2 = -1.5 = -c_1$ so that $h[k] = (-1.5(3)^k + 1.5(2)^k)u[k]$ and for $k \geq 0$
  \[
y_{ZS}[k] = (h * f)[k] = \sum_{r=0}^{k} h[r]f[k-r].
  \]
- **Exercise:** Write the ZSR as a sum of system modes $2^k$ and $3^k$ and a (force) term like the input, here taken as $f[k] = 4(-5)^ku[k]$.

Convolutions - other important properties

- Again, for a LTI system with impulse response $h$ and input $f$, the ZSR is $y_{ZS} = f * h$, where
  \[
  (f * h)[k] = \sum_{r=-\infty}^{\infty} f[r]h[k-r]
  \]
- By simply changing the dummy variable of summation to $r' = h - r$, can show convolution is commutative: $f * h = h * f$.
- One can directly show that convolution $f * h$ is a *bi-linear* mapping from pairs of signals $(f, h)$ to signals $(y_{ZS})$, consistent with convolution’s commutative property and the (zero state) system with impulse response $h$ being LTI;
- that is, $\forall$ signals $f, g, h$ and scalars $\alpha, \beta \in \mathbb{C}$,
  \[
  (\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h)
  \]
- By changing order of summation (Fubini’s theorem), one can easily show that convolution is associative, i.e., $\forall$ signals $f, g, h$,
  \[
  (f * g) * h = f * (g * h).
  \]
Convolution - other important properties (cont)

- We’ll use these properties when composing more complex systems from simpler ones.

- By just changing variables of integration, we can show how to exchange time-shift with convolution, i.e., \( \forall \) signals \( f, h : \mathbb{Z} \rightarrow \mathbb{C} \) and times \( k \in \mathbb{Z} \),
  \[
  (\Delta^k f) * h = \Delta^k (f * h);
  \]
  recall how convolution represents the ZSR of linear and time-invariant systems.

- By the ideal sampling property, recall that the identity signal for convolution is the unit pulse \( \delta \), i.e., \( \forall \) signals \( f \),
  \[
  f * \delta = \delta * f = f
  \]

- Exercise: Adapt the proofs of these properties in continuous time to this discrete-time case.

- Exercise: In particular, show that if \( f \) and \( h \) are causal signals, then \( y = f * h \) is causal; i.e., if the unit-pulse response \( h \) of a system is a causal signal, then the system is causal.

System stability - ZIR - asymptotically stable

- Consider a SISO system with input \( f \) and output \( y \).

- Recall that the ZIR \( y_{ZI} \) is a linear combination of the system’s characteristic modes, where the coefficients depend on the initial conditions, possibly including some initial unit-pulse terms if zero is a characteristic value (system pole).

- A system is said to be asymptotically stable if for all initial conditions,
  \[
  \lim_{k \to \infty} y_{ZI}[k] = 0.
  \]

- So, a system is asymptotically stable if and only if all of its characteristic values have magnitude less than 1.
System stability - ZIR - asymptotically stable: Example

- If the characteristic polynomial $Q(z) = (z - 0.5)(z^2 + 0.0625)$, then
- the system’s characteristic values (roots of $Q$) are $0.5, \pm 0.25j$ each with magnitude less than one,
- and the ZIR is of the form,
  $$y_{ZIR}[k] = (c_1(0.5)^k + c_2(0.25j)^k + c_3(-0.25j)^k) u[k]$$
  $$= (c_1(0.5)^k + 2\text{Re}(c_2)(0.25)^k \cos(k\pi/2) - 2\text{Im}(c_2)(0.25)^k \sin(k\pi/2)) u[k],$$
- recalling that $j^k = e^{jk\pi/2}$.
- So, $y_{ZIR}[k] \to 0$ as $k \to \infty$ for all $c_1, c_2$ (i.e., for all initial conditions), and hence is asymptotically stable.

System stability - bounded signals

- A signal $y$ is said to be bounded if
  $$\exists M < \infty \text{ s.t. } \forall k \in \mathbb{Z}, \; |y[k]| \leq M;$$
  otherwise $y$ is said to be unbounded.
- For example, $y[k] = 0.25(1+j\sqrt{3})^k u[k]$ is bounded (can use $M = 0.25$).
- Also, $3 \cos(5k)$ is bounded (can use $M = 3$).
- But both $2^k \cos(5k)$ and $3 \cdot (-2)^k$ are unbounded.
System stability - ZIR - marginally stable

• A system is said to be marginally stable if it is not asymptotically stable but $y_{ZI}$ is always (for all initial conditions) bounded.

• A system is marginally stable if and only if
  – it has no characteristic values with magnitude strictly greater than 1,
  – it has at least one characteristic value with magnitude exactly 1, and
  – all magnitude-1 characteristic values are not repeated.

• That is, a marginally stable system has
  – some characteristic modes of the form $\cos(\Omega k)$ or $\sin(\Omega k)$,
  – while the rest of the modes are all of the form $k^r |z|^k \cos(\Omega k)$ or $k^r |z|^k \sin(\Omega k)$, with $|z| < 1$ and integer degree $r \geq 0$.
  – Exercise: Explain why we can take $\Omega \in (-\pi, \pi]$ without loss of generality.
  – Note: the dimension of frequency $\Omega$ is $[\Omega] = \text{radians per unit time}$.

System stability - ZIR - marginally stable: Example

• The characteristic polynomial is $Q(z) = z(z^2 + 1)(z - 0.25)$ gives characteristic values 0, 0.25, $\pm j$.

• then the system is marginally stable with modes $(0.25)^k, \cos(k\pi/2), \sin(k\pi/2)$.

• the last two of which are bounded but do not tend to zero as $k \to \infty$. 
A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.

For example, the system with $Q(z) = (z^2 - 0.5)(z + 3)$ is unstable owing to the characteristic value $-3$ with unbounded mode $(-3)^k$.

For another example, if the characteristic polynomial is $Q(z) = (z^2 + 1)^2(z - 0.5)$ then the purely imaginary characteristic values $\pm j$ are repeated, and hence the two additional modes $k \sin(k\pi/2), k \cos(k\pi/2)$ are unbounded, so this system is unstable.

Similarly, if $Q(z) = (z^2 - 1)^2(z - 0.5)$ then the characteristic values $\pm 1$ are repeated and the modes $k$ and $k(-1)^k$ are unbounded, so this system is unstable too.
• A SISO system is said to be *Bounded Input, Bounded Output* (BIBO) stable if ∀ bounded input signals \( f \), the ZSR \( y_{ZS} \) is bounded.

• A sufficient condition for BIBO stability is absolute summability of the unit-pulse response,

\[
\sum_{k=0}^{\infty} |h[k]| < \infty.
\]

• To see why: If the input \( f \) is bounded (by \( M_f \) with \( 0 \leq M_f < \infty \)) then \( \forall k \geq 0 \):

\[
|y_{ZS}[k]| = |(f * h)[k]| = \left| \sum_{r=0}^{k} f[k-r]h[r] \right| \\
\leq \sum_{r=0}^{k} |f[k-r]| |h[r]| \text{ (by the triangle inequality)} \\
\leq \sum_{r=0}^{k} M_f |h[r]| \\
\leq M_f \sum_{r=0}^{\infty} |h[r]| =: M_y < \infty,
\]

• The condition of absolute summability of the unit-pulse response,

\[
\sum_{r=0}^{\infty} |h[r]| < \infty,
\]

is also necessary for, and hence equivalent to, BIBO stability.

• If any component characteristic mode of \( h \) is unbounded, then \( h \) will not to be absolutely summable.

• Thus, if the system (ZIR) is asymptotically stable it will be BIBO stable; the converse is also true.
ZSR - the transfer function, $H$

- Recall that for any polynomial $Q$ and $z \in \mathbb{C}$ (including $s = jw, w \in \mathbb{R}$),
  \[ Q(\Delta^{-1})z^k = Q(z)z^k, \quad \forall k \geq 0. \]

- So, if we guess that a “particular” solution of the system $Q(\Delta^{-1})y = P(\Delta^{-1})f$ with input $f[k] = Az^ku[k]$ is of the form $y_0[k] = AH(z)z^k = H(z)f[k]$, $k \geq 0$, then we get by substitution that $\forall k \geq 0, \ z \in \mathbb{C}$,
  \[ (Q(\Delta^{-1})y_0)[k] = (P(\Delta^{-1})f)[k] \Rightarrow AH(z)Q(z)z^k = AP(z)z^k \Rightarrow H(z) = P(z)/Q(z). \]

- The “rational polynomial” $H = P/Q$ is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

- So, the ZSR (forced response + characteristic modes) would be of the form:
  \[ y_{ZS}[k] = (AH(z)z^k + \text{linear combination of char. modes})u[k]. \]

- Recall that for the example with $Q(z) = z + 3$ and $P(z) = 7z$, we computed the unit-pulse response $h[k] = 7(-3)^k u[k]$ and the ZSR to input $f[k] = 4(0.5)^ku[k]$ as $y_{ZS}[k] = (24(-3)^k + 4(0.5)^k)u[k]$.

- Here, note that $H(0.5) = P(0.5)/Q(0.5) = 1$, i.e., the forced response component of $y_{ZS}$ is $H(0.5)f[k] = 1 \cdot 4(0.5)^ku[k] = 4(0.5)^ku[k]$.

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**ZSR - unit-pulse response $h$, transfer function $H$, and eigenresponse**

- $y_{ZS}[k] = (H(z)Az^k + \text{linear combination of char. modes})u[k]$ is the ZSR to input $f[k] = Az^ku[k]$, where $H(z) = P(z)/Q(z)$.

- The eigenresponse is a special case of the forced response for exponential inputs.

- If $|z| = 1$, i.e., $z = e^{j\Omega}$ for some $\Omega \in (-\pi, \pi]$ (w.l.o.g.), and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:
  \[ y[k] \rightarrow AH(e^{j\Omega})e^{j\Omega k} \quad \text{as} \quad k \rightarrow \infty. \]

- Since $y = h * f$, we get that as $k \rightarrow \infty$ for a LTI and asymptotically stable system,
  \[ y_{ZS}[k] = \sum_{r=0}^{k} h[r]Ae^{j\Omega(k-r)} = Ae^{j\Omega k} \sum_{r=0}^{k} h[r]e^{-j\Omega r} \rightarrow Ae^{j\Omega k}H(e^{j\Omega}), \]
  \[ \Rightarrow \sum_{r=0}^{\infty} h[r]e^{-j\Omega r} = H(e^{j\Omega}), \ \forall \Omega \in (-\pi, \pi]. \]
The LTIC system transfer function $H$ is the $z$-transform of the system unit-pulse response $h$:

$$H(z) = \sum_{k=0}^{\infty} h[k] z^{-k},$$

where $z \in \mathbb{C}$ is in $H$’s “region of convergence”.

Note that $H(e^{j\Omega})$ is periodic since $H(e^{j\Omega}) \equiv H(e^{j\Omega+2\pi k})$ for any integer $k$.

For the $z$-transform (and DTFS) we will use this notation for $H$, but for the DTFT we will write $H(\Omega)$ instead of $H(e^{j\Omega})$.

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**Frequency-domain methods for discrete-time signals**

- Discrete-Time Fourier Series (DTFS) of periodic signals
- Discrete-Time Fourier Transform (DTFT)
- sampled data systems
- DFT & FFT
- $z$-transform for (complete) transient response
- eigenresponse
- canonical system realization of a difference equation
Discrete-time Fourier series of periodic signals

- For all $r, N \in \mathbb{Z}$, note that the signal $\{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \}$ "repeats itself" every $N > 0$ units of (discrete) time $k$, in particular
  \[
  \forall r \in \mathbb{Z}, \ \exp(jr \frac{2\pi}{N} k) \big|_{k=0} = 1 = \exp(jr \frac{2\pi}{N} k) \big|_{k=N}.
  \]
- Also the signals $\{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \}$ $\equiv \{ \exp(jr' \frac{2\pi}{N} k) \mid k \in \mathbb{Z} \}$ whenever $r' = r \mod N$.
- Suppose $N$ is the period of periodic signal $x = \{ x[k] \mid k \in \mathbb{Z} \}$ and $\Omega_o = 2\pi / N$ be the fundamental frequency of $x$ (recall $[\Omega_o] = \text{radians/unit-time}$).
- We can write $x$ as a Discrete-Time Fourier Series (DTFS):
  \[
  \forall k \in \mathbb{Z}, \ x[k] = \sum_{r=0}^{N-1} D_r \exp(jr \Omega_o k).
  \]
  where $r$ indexes $x$’s $N$ harmonics.
- Note that the DTFS can also be written for any discrete-time signal $x : A \to \mathbb{R}$ defined over any finite interval of time, e.g., $A = \{0, 1, 2, \ldots, N - 1\}$ or $A = \{-N, -N + 1, \ldots, -1\}$ for integer $N < \infty$.

Discrete-time Fourier series of periodic signals (cont)

- Consider the $N$ signals $\xi_r[k] := e^{jr \Omega_o k}$ over any time-interval $A$ of length $N$.
- Equivalently consider these $N$ signals $\xi_r$ as $N$-vectors in $\mathbb{R}^N$, i.e., the $k^{th}$ entry of vector $\xi_r$ is $\xi_r[k]$.
- If these signals/vectors $\{ \xi_r \}_{r=0}^{N-1}$ are linearly independent, then they will form a basis spanning all other signals $x : A \to \mathbb{R}$, equivalently all other vectors $x \in \mathbb{R}^N$.
- i.e., any such $x$ can be written as a linear combination of the $\{ \xi_r \}_{r=0}^{N-1}$ giving the DTFS of $x$:
  \[
  x_r = \sum_{r=0}^{N-1} D_r \xi_r.
  \]
- If we show that these signals/vectors $\{ \xi_r \}_{r=0}^{N-1}$ are orthogonal then
  - linear independence follows
  - the $r^{th}$ coordinate $D_r$ (DTFS coefficients) is found by simply projecting $x$ onto the vector $\xi_r$:
    \[
    D_r = \langle x, \xi_r \rangle / ||\xi_r||^2.
    \]
• Consider any period of $x: \mathbb{Z} \to \mathbb{R}$, say $\{0, 1, 2, \ldots, N - 1\}$.

• First note that for any $v \in \mathbb{Z}$ that is not a multiple of $N$ (so $e^{jv\Omega} = e^{jv(2\pi/N)} \neq 1$), the geometric series
  \[ \sum_{k=0}^{N-1} e^{jv(2\pi/N)k} = e^{jv(2\pi/N)} \frac{1 - e^{jv(2\pi/N)}}{1 - e^{jv(2\pi/N)}} = 0. \]

• Thus, for any $r \neq v \in \mathbb{Z}$ such that $N \not| (v - r)$, the inner product $\langle \xi_r, \xi_v \rangle = \langle \{ e^{jr(2\pi/N)k} \}, \{ e^{jv(2\pi/N)k} \} \rangle := \sum_{k=0}^{N-1} e^{jr(2\pi/N)k} \overline{e^{jv(2\pi/N)k}} = \sum_{k=0}^{N-1} e^{j(r-v)(2\pi/N)k} = 0,$
  recalling that the inner product is conjugate-linear in the second argument so that $\langle x, x \rangle = ||x||^2$ when $x$ is $\mathbb{C}$-valued.

• So, these signals are orthogonal and the DTFS coefficients of $N$-periodic $x$ are
  \[ D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-jr\Omega_k} = \frac{\langle x, \{ e^{jr\Omega_k} \} \rangle}{||\{ e^{jr\Omega_k} \}||^2}, \text{ where } \Omega_o = \frac{2\pi}{N}. \]

---

**DTFS - checking coefficients**

• Let’s now compute the inner product of $\xi_v$, for any $v \in \{0, 1, \ldots, N - 1\}$, with the DTFS of $N$-periodic $x$:
  \[ \langle x, \{ e^{jv\Omega_k} \} \rangle = \sum_{k=0}^{N-1} x[k] e^{-jv\Omega_k} = \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} D_r e^{jr\Omega_r} e^{-jv\Omega_k} = \sum_{r=0}^{N-1} D_r e^{j(r-v)\Omega_k} \]
  \[ = \sum_{r=0}^{N-1} D_r N \delta(r-v) = D_v N \]

• Again, we have verified the DTFS coefficients is
  \[ D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-jr\Omega_k} = \frac{\langle x, \{ e^{jr\Omega_k} \} \rangle}{||\{ e^{jr\Omega_k} \}||^2}, \text{ where } \Omega_o = \frac{2\pi}{N}. \]
DTFS - example

Problem:
Identify the DTFS coefficients (if they exist) for
\[ x[k] = 7 \sin(5.7\pi k) + 2 \cos(3.2\pi k), \quad k \in \mathbb{Z}. \]

Solution:

- First note that the two components of \( x \) are periodic, so their sum is periodic. 
  *(Why is this so in discrete time?)*

- Since \( \sin \) and \( \cos \) have period \( 2\pi \), we can subtract integer multiples of \( 2\pi \) to get
  \[ x[k] = 7 \sin(1.7\pi k) + 2 \cos(1.2\pi k). \]

- \( 1.7\pi k \) is an integer multiple of \( 2\pi \) when (integer) \( k = 20 \), and when \( k = 5 \) for \( 1.2\pi k \), 
  *so least common multiple* of these periods is \( k = 20 \).

- *(Show that one can alternatively find the greatest common divisor of the component frequencies.)*
• Thus, the period of \( x \) is \( N = 20 \) and the fund. frequ. is \( \Omega_o = \frac{2\pi}{N} = 0.1\pi \).

• By Euler’s identity and adding \( 2\pi k \) to the negative exponents,
\[
x[k] = \frac{7}{2j}e^{j1.7\pi k} - \frac{7}{2j}e^{-j1.7\pi k} + e^{j1.2\pi k} + e^{-j1.2\pi k}
\]
\[
= -3.5je^{j1.7\pi k} + 3.5je^{j0.3\pi k} + e^{j1.2\pi k} + e^{j0.8\pi k}.
\]

• So, the DTFS of \( x[k] = \sum_{r=0}^{19} D_r e^{jr\Omega_o k} \) with
\[
D_{17} = -3.5j = 3.5e^{-j\pi/2}, \quad D_3 = 3.5j = 3.5e^{j\pi/2}, \quad D_{12} = 1, \quad \text{and} \quad D_8 = 1;
\]
else \( D_r = 0 \) (incl. the fundamental \( r \in \{1, 19\} \) & DC \( r = 0 \) components).

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**DTFS - example and exercise**

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**Example:** The DTFS of an even rectangle wave with period \( N = 6 \) and duty cycle 3:
\[
x[k] = \sum_{\ell=-\infty}^{\infty} \Delta^6(\Delta^{-1}u - \Delta^2u)[k] = \sum_{\ell=-\infty}^{\infty} (u[k + 1 - 6\ell] - u[k - 2 - 6\ell])
\]
\[
is = \sum_{r=0}^{5} D_re^{jr\Omega_o k},
\]
where the fund. freq. \( \Omega_o = \frac{2\pi}{6} \) and, \( \forall r \in \mathbb{Z} \),
\[
D_r = \frac{1}{6} \sum_{k=-3}^{2} x[k] e^{-jr\Omega_o k} = \frac{1}{6} \sum_{k=-1}^{1} 1 \cdot e^{-jr(2\pi/6)k} = \frac{1}{6}(1 + 2 \cos(r(2\pi/6)k)).
\]

**Exercise:** Plot \( x[k] \) as a function of time \( k \) and plot its (periodic) spectrum:
\( \forall r \in \{0, 1, 2, \ldots, 5\}, \ell \in \mathbb{Z} \),
\[
\hat{X}(r2\pi/6 + 2\pi\ell) = D_r.
\]
DTFS - Parseval’s theorem

- The average power of the \( N \)-periodic discrete-time signal \( x \) is
  \[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} x[k][x[k]]^*;
\]
equivalently, the sum could be taken over any interval of length \( N \in \mathbb{Z}^+ \).

- Substituting the Fourier series of \( x \) separately for \( x[k] \) and \( x[k] \) (using a different summation-index variable for each substitution), leads to Parseval’s theorem
  \[
P_x = \sum_{r=0}^{N-1} |D_r|^2.
\]

- Parseval’s theorem can be used to determine the amount of periodic signal \( x \)’s power resides in a given frequency band \([\Omega_1, \Omega_2] \subset [0, 2\pi] \) radians/unit-time:
  1. Determine the harmonics \( r \Omega_o \) of \( x \) that reside in this band, \( i.e., \text{ integers } r \in [\Omega_1/\Omega_o, \Omega_2/\Omega_o] \)
     where \( x \)’s fundamental frequency \( \Omega_o = 2\pi/N \).
  2. Sum just over these harmonics to get the answer, \( \sum_{n=r \Omega_o \leq \Omega \leq \Omega/\Omega_o} |D_r|^2 \).

---

DTFS - Parseval’s theorem example

- Find the fraction of \( x \)’s average power in the frequency band \([0.4\pi, 1.1\pi] \) where
  \[
  \forall k \in \mathbb{Z}, \quad x[k] = \sum_{v=-\infty}^{\infty} \left( 3\delta[k - 4v] - 4\delta[k - 1 - 4v] \right)
  \]

- Solution: \( x \) has period \( N = 4 \) and average power
  \[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{4} \sum_{k=0}^{3} |x[k]|^2 = \frac{1}{4} (3^2 + (-4)^2 + 0^2 + 0^2) = \frac{25}{4}
  \]

- \( x \) has fundamental frequency \( \Omega_o = 2\pi/N = \pi/2 \) radians/unit-time and discrete-time Fourier coefficients
  \[
  D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k]e^{-jkr\Omega_o} = \frac{1}{4} \left( 3 - 4e^{-jr\pi/2} \right), \quad 0 \leq r \leq N - 1 = 3.
  \]

- The harmonics \( r \) of \( x \) that reside in \([0.4\pi, 1.1\pi] \) satisfy \( 0.4\pi \leq r\Omega_o = r\pi/2 \leq 1.1\pi \), \( i.e., r \in \{1, 2\} \).

- So, by Parseval’s theorem, the answer is \( (|D_1|^2 + |D_2|^2)/P_x \).
Periodic extensions

- Consider signal $x : \mathbb{Z} \rightarrow \mathbb{R}$ having finite support $\{-M, -M + 1, ..., 0, ..., M - 1, M\}$ for $0 < M < \infty$; i.e., $\forall |k| > M, x[k] = 0$.

- For $N \geq M$, define $2N$-periodic $x^{(N)}$ such that

$$x^{(N)}[k] = \begin{cases} x[k] & \text{if } |k| \leq M \\ 0 & \text{if } M < |k| \leq N \end{cases}$$

- $x^{(N)}$ is a periodic extension of the finite-support signal $x$, where again $x^{(N)}$’s period is $2N$ and

$$\lim_{N \rightarrow \infty} x^{(N)} = x.$$

DTFS of periodic extension leading to DTFT

- For $r \in \{-N + 1, -N + 2, ..., N - 1, N\}$, the DTFS of $x^{(N)}$ has coefficients

$$D_r^{(N)} = \frac{1}{2N} \sum_{k=-N+1}^{N} x^{(N)}[k] e^{-jr \frac{2\pi}{2N} k}$$

$$= \frac{1}{2N} \sum_{k=-M}^{M} x[k] e^{-jr \frac{2\pi}{2N} k}$$

$$= \frac{1}{2N} \sum_{k=-\infty}^{\infty} x[k] e^{-jr \frac{2\pi}{2N} k}$$

$$=: \frac{1}{2N} \mathcal{X}\left( r, \frac{2\pi}{2N} \right),$$

where the Discrete-Time Fourier Transform (DTFT) of (aperiodic) $x : \mathbb{Z} \rightarrow \mathbb{R}$ is $X : \mathbb{R} \rightarrow \mathbb{C}$:

$$X(\Omega) := \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} =: (\mathcal{F}x)(\Omega), \quad \Omega \in \mathbb{R}$$

- Note that Fourier integrals (spectra of discrete-time signals) are periodic, repeating themselves every $2\pi$ radians: $\forall \Omega \in \mathbb{R}, \ell \in \mathbb{Z}$,

$$X(\Omega) = X(\Omega + \ell 2\pi).$$
Inverse DTFT by Fourier Integral

• Thus, \( \forall k \in \mathbb{Z}, \)

\[
x[k] = \lim_{N \to \infty} x^{(N)}[k]
\]

\[
= \lim_{N \to \infty} \sum_{r=-N+1}^{N} D_r^{(N)} e^{j \frac{2\pi}{2N} k r}
\]

\[
= \lim_{N \to \infty} \sum_{r=-N+1}^{N} X \left( r \frac{2\pi}{2N} \right) e^{j \frac{2\pi}{2N} k r} \frac{1}{2N} \frac{2\pi}{2\pi}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega
\]

where the last equality is by Riemann integration with \( 2\pi / (2N) \to d\Omega. \)

• Thus, we have derived the inverse DTFT by Fourier integral of \( X \) giving (aperiodic) \( x, \)

\[
\forall k \in \mathbb{Z}, \quad x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega =: (\mathcal{F}^{-1}X)[k].
\]

DTFT Examples - exponential signal

• If \( x = \delta \) then obviously \( X \equiv 1. \)

• The geometric signal \( x[k] = \gamma^k u[k] \) for scalar \( \gamma \) s.t. \( |\gamma| < 1 \) has DTFT

\[
X(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k
\]

\[
= \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{1}{(1 - \gamma) \cos(\Omega) + j \gamma \sin(\Omega)}
\]

• Note that

\[
|X(\Omega)| = \frac{1}{(1 - \gamma \cos(\Omega))^2 + \gamma^2 \sin^2(\Omega)} = \frac{1}{1 + \gamma^2 - 2\gamma \cos(\Omega)}
\]

\[
\angle X(\Omega) = -\arctan \left( \frac{\gamma \sin(\Omega)}{1 - \gamma \cos(\Omega)} \right)
\]
DTFT Examples - exponential signal (cont)

- The plots above are for $\gamma = 0.5$.
- Note how $X$ has period $2\pi$.
- Exercise: What are the maximum and minimum values of $|X|$, i.e., how would this plot depend on $\gamma > 0$? Plot $x$ and $\angle X$. How do these plots differ when $-1 < \gamma < 0$?
- Exercise: Find the DTFT of anticausal signal $x[k] = \gamma^k u[-k]$ for scalar $\gamma$ s.t. $|\gamma| > 1$.
- Exercise: Find the DTFT of $x[k] = \gamma^{|k|}$, $k \in \mathbb{Z}$, for scalar $\gamma$ s.t. $|\gamma| < 1$.

DTFT Examples - Square and Triangle Pulse

- For $T \in \mathbb{Z}^{>0}$, the even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ (i.e., $x[k] = u[k + T] - u[k - (T + 1)]$), has DTFT

$$X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k} = 1 + 2 \sum_{k=1}^{T} \cos(k \Omega), \ \Omega \in \mathbb{R}.$$  

- Exercise (even rectangle pulse in frequency domain):
  Show that for fixed $\Omega'$ s.t. $0 < \Omega' < \pi$,

$$\mathcal{F}^{-1}\{\Delta_{-\Omega} u - \Delta_{\Omega} u\}[k] = \frac{\Omega'}{\pi} \frac{1}{\text{sinc}(\Omega' k)}, \ k \in \mathbb{Z}.$$  

- For $T \in \mathbb{Z}^{>0}$, the odd triangle pulse with support $2T + 1$, $x[k] \equiv k(\Delta^{-T}u[k] - \Delta^{T+1}u[k])$ has DTFT

$$X(\Omega) = \sum_{k=-T}^{T} ke^{-j\Omega k} = -2j \sum_{k=1}^{T} k \sin(k \Omega), \ \Omega \in \mathbb{R}.$$  

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DTFT Examples - exponential sinusoid

- For fixed time $K_0$, clearly
  \[ \mathcal{F}\{\delta[k - K_0]\}(\Omega) = e^{jK_0\Omega}, \]
  where here $\delta$ is the unit pulse.

- Note that $e^{jK_0\Omega}$ is a sinusoidal function of $\Omega$ with period $2\pi/K_0$ radians.

- Exercise: For fixed frequency $\Omega_0$ radians/unit-time, show that
  \[ \mathcal{F}\{e^{-j\Omega_0 k}\}(\Omega) = 2\pi \sum_{v=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi v), \]
  where here $\delta$ is the Dirac impulse (in the frequency domain $\Omega \in \mathbb{R}$). Hint: work with $\mathcal{F}^{-1}$.

- So, the DTFT of a $N$-periodic signal with Fourier series
  \[ \sum_{r=0}^{N-1} D_r e^{jr\frac{2\pi}{N}} \xrightarrow{\mathcal{F}} 2\pi \sum_{v=-\infty}^{\infty} \sum_{r=0}^{N-1} D_r \delta(\Omega - r\frac{2\pi}{N} + 2\pi v) \]

DTFT - Time shift and frequency shift properties

- If fixed $K_0 \in \mathbb{Z}$ and $X = \mathcal{F}\{x\}$ then
  \[ \mathcal{F}\{\Delta_{K_0} x\}(\Omega) = \sum_{k=-\infty}^{\infty} (\Delta_{K_0} x)[k] e^{-j\Omega k} \]
  \[ = \sum_{k=-\infty}^{\infty} x[k - K_0] e^{-j\Omega k} \]
  \[ = \sum_{k=-\infty}^{\infty} x[k] e^{-j(k+K_0)\Omega} \]
  \[ = e^{-jK_0\Omega} X(\Omega), \]
  i.e., shift in time by $K_0$ corresponds to product with sinusoid of period $2\pi/K_0$ (linear phase shift) in frequency domain.

- Exercise: Prove the dual property that if fixed $\Omega_0 \in \mathbb{R}$ and $X = \mathcal{F}\{x\}$ then
  \[ \mathcal{F}\{x[k] e^{j\Omega_0 k}\}(\Omega) = X(\Omega - \Omega_0), \]
  i.e., modulation (multiplication by a sinusoid) in time domain results in frequency shift.
DTFT - convolution properties

• Let \( X_r = \mathcal{F}\{x_r\} \) for \( r \in \{1, 2\}. \)

\[
\mathcal{F}\{x_1 * x_2\}(\Omega) := \sum_{k=-\infty}^{\infty} (x_1 * x_2)[k]e^{-j\Omega k} := \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k-l]e^{-j(k-l)\Omega}e^{-jl\Omega} \ i.e., \ x e^{j\Omega} e^{-j\Omega} = 1
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k']e^{-jk\Omega} e^{-jl\Omega} \ \text{where} \ k' = k - l
\]

\[
= \sum_{l=-\infty}^{\infty} x_1[l]e^{-jl\Omega} \sum_{k=-\infty}^{\infty} x_2[k']e^{-jk\Omega} =: X_1(\Omega)X_2(\Omega)
\]

• Exercise: Prove the dual property that

\[
\mathcal{F}\{x_1x_2\}(\Omega) = \frac{1}{2\pi} (X_1 \ast X_2)(\Omega) := \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(v)X_2(\Omega - v)dv.
\]

• Exercise: Use the convolution properties to prove the time and frequency shift properties. Hint: \((\Delta^{K,\delta} \ast x) = \Delta^{K,x}\).

• Exercise: Show that DTFT is a linear operator.

DTFT - Parseval's Theorem

• The energy of a signal DT \( x \) is

\[
E_x := \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} x[k]x[k^*] = \sum_{k=-\infty}^{\infty} (\mathcal{F}^{-1}X)[k] \overline{(\mathcal{F}^{-1}X)[k]}
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')e^{j\Omega'k}d\Omega' \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega k}d\Omega
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')e^{j\Omega'k}d\Omega' \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{-j\Omega k}d\Omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \int_{-\pi}^{\pi} \overline{X(\Omega)} \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{j\Omega k}e^{-j\Omega k} \right) d\Omega d\Omega'
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \int_{-\pi}^{\pi} \overline{X(\Omega)} \frac{1}{2\pi} (2\pi \delta(\Omega - \Omega')) d\Omega d\Omega'
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')\overline{X(\Omega')} d\Omega' = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega')|^2 d\Omega', \ \text{recalling that for}\ \text{fixed} \ \Omega: \ \mathcal{F}^{-1}(2\pi \delta(\Omega - \Omega'))[k] = e^{-j\Omega k} \ \& \ \int_{-\pi}^{\pi} \overline{X(\Omega)} \delta(\Omega - \Omega') d\Omega = \overline{X(\Omega')}.
\]
The even rectangle pulse with support $2T + 1$, $x = \Delta^{-T}u - \Delta^{T+1}u$ has energy

$$E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-T}^{T} 1^2 = 2T + 1.$$ 

Recall its DTFT is $X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k}$, so

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)X^*(\Omega) d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-T}^{T} e^{-j\Omega k} \sum_{k'=-T}^{T} e^{j\Omega k'} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k=-T}^{T} 1 + \sum_{k<k'} e^{j(k-k')\Omega} \right) d\Omega$$

$$= \sum_{k=-T}^{T} \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\Omega + \sum_{k<k'} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(k-k')\Omega} d\Omega = \sum_{k=-T}^{T} 1 + 0$$

$$= 2T + 1.$$

**Exercise:** Repeat this calculation using $X(\Omega) = 1 + 2 \sum_{k=1}^{T} \cos(\Omega k)$.

**Exercise:** Compute amount of energy of $x$ in the frequency band $[-\pi/6, \pi/6]$ radians/unit-time, i.e.,

$$\frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} |X(\Omega)|^2 d\Omega$$
Consider a SISO, DT-LTIC system described by the difference equation
\[ Q(\Delta^{-1})y = P(\Delta^{-1})f, \]
where \( f \) is the input and \( y \) is the ZSR (output).

Recall that by the time-shift property,
\[ Q(e^{j\Omega})Y_{ZS}(\Omega) = P(e^{j\Omega})F(\Omega) \Rightarrow Y_{ZS}(\Omega) = H(\Omega)F(\Omega). \]

We now re-derive from first principles the eigenresponse by first recalling that the ZSR \( y_{ZS} = f \ast h \) where \( h \) is the unit-pulse response.

Taking DTFTs, \( Y_{ZS} = H F \) where \( H = \mathcal{F} h \) is the transfer function.

Suppose the system is BIBO/asymptotically stable, i.e., the \( n \) roots of \( Q \) (system char. modes/poles) \( z \) all have modulus \( |z| < 1 \).

The ZSR will consist of a forced response plus characteristic modes, where the latter will \( \to 0 \) over time (our stability assumption) so that the forced response becomes the steady-state response.

The forced response to a persistent sinusoidal input
\[ f[k] = A f e^{j(\Omega k + \phi)} \]
will be of the form
\[ y_{SS}[k] = A y e^{j(\Omega k + \phi)} \]
where (for \( k \geq 0 \)),
\[ Q(e^{j\Omega})y_{SS}[k] = (Q(\Delta^{-1})y_{SS})[k] = (P(\Delta^{-1})f)[k] = P(e^{j\Omega})f[k]. \]
\[ \Rightarrow y_{SS}[k] = \frac{P(e^{j\Omega})f[k]}{Q(e^{j\Omega})} \]

Also, the ZSR \( y_{ZS} = h \ast f \), i.e., for all time \( k \geq 0 \):
\[ y_{ZS}[k] = \sum_{v=0}^{k} h[v] A f e^{j(\Omega (k-v) + \phi_v)} = f[k] \sum_{v=0}^{k} h[v] e^{-j\Omega v} \]
\[ \to f[k]H(\Omega_o) =: y_{SS}[k] \text{ as } k \to \infty. \]
Equating the forced responses (steady-state response for a stable system), we again get that the system transfer function is

\[ H(\Omega) = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})} = (Fh)(\Omega). \]

Note that \( \forall k \in \mathbb{Z}, H(\Omega) = H(\Omega + 2\pi k). \)

Also, we write \( H(\Omega) \) not \( H(e^{j\Omega}) \) for the DTFT.

So, the eigenresponse of a BIBO/asymptotically stable SISO, DT-LTIC system is the steady-state response to a sinusoid:

\[ f[k] = A_f e^{j(\Omega k + \phi_f)} \rightarrow H(\Omega_o) f[k] = A_y e^{j(\Omega k + \phi_y)} =: y_{ss}[k] \]

The system magnitude response (gain) is \(|H(\Omega)| = |P(e^{j\Omega})|/|Q(e^{j\Omega})|\), i.e., \( A_y = A_f |H(\Omega_o)| \).

The system phase response is \( \angle H(\Omega) = \angle P(e^{j\Omega}) - \angle Q(e^{j\Omega}) \), i.e., \( \phi_y = \phi_f + \angle H(\Omega_o) \).

---

**Eigenresponse - example**

**Problem:** For the system \( 2y[k] = 0.6y[k - 1] - 7f[k] \) find the steady-state response (if it exists) to \( f[k] = 4 \cos(5k)u[k] \).

**Solution:** The difference equation in standard form is

\( (Q(\Delta^{-1})y)[k] = y[k + 1] - 0.3y[k] = -3.5f[k + 1] = (P(\Delta^{-1})f)[k], \)

where \( Q(z) = z - 0.3 \) and \( P(z) = -3.5z \).

The sole system characteristic value (root of \( Q \), system pole) is 0.3, hence the system is BIBO/asymptotically stable.

By Euler’s identity \( f[k] = (2e^{j5k} + 2e^{j(-5)k})u[k] \)

By linearity, the eigenresponse is therefore

\[ 2H(5)e^{j5k} + 2H(-5)e^{j(-5)k}, \]

where \( H(\Omega) = P(e^{j\Omega})/Q(e^{j\Omega}) = -3.5e^{j\Omega}/(e^{j\Omega} - 0.3) = \frac{H(-\Omega)}{H(\Omega)}, \) so that

\[ |H(\Omega)| = \frac{3.5}{\sqrt{(\cos(\Omega) - .3)^2 + \sin^2(\Omega)}}, \angle H(\Omega) = \pi + \Omega - \arctan \left( \frac{\sin(\Omega)}{\cos(\Omega) - .3} \right) \]

**Exercise:** Show that the eigenresponse is also simply \( |H(5)|4 \cos(5k + \angle H(5)) \).
2D Image Processing Example

- Apply 1-dimensional filtering to a 2-dimensional (2D) image by separately performing row and column operations.

- For $256 \times 256$ pixel (2D) image,

$$
\begin{bmatrix}
  f[1, 1] & f[1, 2] & \ldots & f[1, 256] \\
  \vdots & \vdots & \ddots & \vdots \\
  f[256, 1] & f[256, 2] & \ldots & f[256, 256]
\end{bmatrix}
$$

- If $f[k, i]$ represents the 8-bit (grey) intensity of the pixel in row $k$ and column $i$ (i.e., 8 bits per pixel or bpp), then the “raw” image size will be $256^2 \text{bits} = 16 \text{Mb} = 2 \text{MB}$.

- Each of $f$’s rows of pixels can be processed by a system with unit-pulse response $h$ to obtain a new row of pixels, and thus a new image $y$:

$$
\forall k, \ f[k, \cdot] \rightarrow [h] \rightarrow y[k, \cdot]
$$

- Alternatively, each of $f$’s columns of pixels can be processed by a system with unit-pulse response $h$ to obtain a new column of pixels, and thus a new image $y$:

$$
\forall i, \ f[\cdot, i] \rightarrow [h] \rightarrow y[\cdot, i]
$$

Image Processing: High-Pass and Low-Pass Filtering

- The system $h$ may have a specific signal processing objective.

- The output pixels $y[k, i]$ may be quantized to fewer bpp than those of the input, thus achieving image compression.

- The simple low-pass filter (L)

$$
\begin{align*}
  h[k] &= \frac{1}{2}(\delta[k] + \delta[k - 1]) \\
  \Rightarrow \quad y[k] &= \frac{1}{2}(f[k] + f[k - 1])
\end{align*}
$$

can capture shading and texture in the image.

- The simple high-pass filter (H)

$$
\begin{align*}
  h[k] &= \frac{1}{2}(\delta[k] - \delta[k - 1]) \\
  \Rightarrow \quad y[k] &= \frac{1}{2}(f[k] - f[k - 1])
\end{align*}
$$

can capture edges in the image.

- Typically more compression possible in higher-frequency bands (H).
Image Processing: Tandem Row and Column Filtering

\[
f \rightarrow \begin{array}{c}
\text{row filtering} \\
\text{column filtering}
\end{array} \rightarrow y
\]

- Define \( y_{LH} \) as the output of
  \[
f \rightarrow \begin{array}{c}
L \\
H
\end{array} \rightarrow y_{LH}
\]

- Similarly define \( y_{LL} \), \( y_{HH} \) and \( y_{HL} \).

- The \( y \) images are downsampled by a factor of four (two in each direction).

- The \( y_{LL} \) image will have a lot of energy while \( y_{HH} \) will have the least energy.

- This motivates non-uniform quantization (bit allotment per pixel) of these images.

- Together with a coding strategy for the quantized images (particularly for the regions of zero pixel-values), this is the basic approach used in JPEG leading to very good compression, e.g., from 8 bpp to 0.2-0.5 bpp.

---

Sampling Continuous-Time Signals (A/D)

- Consider continuous-time signal \( x \) with \( X = \mathcal{F}x \).

- Recall that by sampling at period \( T \) with impulses in continuous time \( t \in \mathbb{R} \), we get
  \[
x_T(t) := \sum_{k=-\infty}^{\infty} x(kT)\delta(t - kT) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} x(kT)e^{-jkTw} =: X_T(w),
\]
equivalently,
  \[
X_T(w) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( w - \frac{2\pi v}{T} \right).
\]

- Now define the sampled process in discrete-time \( k \in \mathbb{Z} \) and its DTFT,
  \[
\hat{x}[k] := x(kT) \xrightarrow{\mathcal{F}} \hat{X}(\Omega) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{-j\Omega k}.
\]

- Substituting \( w = \Omega/T \) we get
  \[
\hat{X}(\Omega) = X_T \left( \frac{\Omega}{T} \right) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( \frac{\Omega - v2\pi}{T} \right).
\]

- Exercise: Read decimation (downsampling) and interpolation (upsampling) of Lathi Figs. 8.17 & 10.9.
Sampling Continuous-Time Signals - example

- We are particularly interested in the case where
  - the continuous-time signal $x$ is band-limited, i.e., $\exists w' > 0$ s.t. $X(w) = 0$ for $|w| > w'$, and
  - the sampling frequency is greater than Nyquist’s, i.e., $2\pi/T > 2w' \Rightarrow w'T < \pi$.

- Example: For fixed $w' > 0$, consider the cts-time signal $x(t) = A\text{sinc}(w't)$ with FT
  $$X(w) = \frac{A\pi}{w'} (u(w + w') - u(w - w')).$$

- Sampling $x$ at period $T < \pi/w'$ we get the discrete-time signal $x[k] = A\text{sinc}(w'kT)$.

- Using inverse DTFT, recall that we can easily check that the DTFT of $x$ is,
  $$\hat{X}(\Omega) = \sum_{v=-\infty}^{\infty} \frac{A\pi}{w'T} (u(\Omega + w'T - 2\pi v) - u(\Omega - w'T - 2\pi v))$$
  $$= \sum_{v=-\infty}^{\infty} \frac{1}{T} X \left( \frac{\Omega - 2\pi v}{T} \right),$$
  noting $\forall T > 0$, $u(\tilde{\Omega} \pm w') = u(\frac{1}{T}(\tilde{\Omega} \pm w'T)) = u(\tilde{\Omega} \pm w'T)$, $\tilde{\Omega} := \Omega - 2\pi v$. 

Sampling Continuous-Time Signals - example (cont), $w'T < \pi$
Sampled Data Systems: A/D (analog-to-digital conversion)

• Suppose the signal \( f \) is sampled every \( T_s \) seconds, i.e., at sampling frequency \( w_s := 2\pi/T_s \).

• Recall Poisson’s identity (the Fourier series of the picket-fence function)
  \[
  p_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}
  \]

• Let’s rederive the relationship between the spectrum of a sampled continuous-time signal and its discrete-time counterpart by first defining the discrete-time signal
  \[\forall k \in \mathbb{Z}, \quad \hat{f}[k] = f(kT_s).\]

• We want to relate the (continuous-time) Fourier transform of \( f \) to the (discrete-time) Fourier transform of \( \hat{f} \),
  \[
  \hat{F}(\Omega) := \sum_{k=-\infty}^{\infty} \hat{f}[k] e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f(kT_s) e^{-j\Omega k}.
  \]

To this end, recall
  \[
  f(t)p_{T_s}(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} f(t) e^{jkw_s t} \xrightarrow{F} \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w-kw_s), \quad \text{and also}
  f(t)p_{T_s}(t) = \sum_{k=-\infty}^{\infty} f(kT_s) \delta(t-kT_s) \xrightarrow{F} \sum_{k=-\infty}^{\infty} f(kT_s) e^{-j\Omega k} = \hat{F}(wT_s).
  \]

• Equating these two expressions for \( F\{fp_{T_s}\} \) we get,
  \[
  \hat{F}(wT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w-kw_s).
  \]

• Substituting \( w = \Omega/T_s \) we get,
  \[
  \hat{F}(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F\left(\frac{\Omega - k2\pi}{T_s}\right).
  \]
Sampled Data Systems: D/A (digital to analog conversion)

- Now consider a discrete time signal \( \hat{y}[k] \).
- We implement at D/A with a \( T_s \)-second hold, i.e., construct the continuous-time signal
  \[
  y(t) := \sum_{k=-\infty}^{\infty} \hat{y}[k] r_{T_s}(t - kT_s),
  \]
  where
  \[
  r_{T_s}(t) := u(t) - u(t - T_s) \overset{T_s}{\rightarrow} T_s \text{sinc}(wT_s/2)e^{-jwT_s/2} =: R_{T_s}(w).
  \]
- Note that \( y \) is in the form of a convolution, so:
  \[
  Y(w) = \sum_{k=-\infty}^{\infty} \hat{y}[k] R_{T_s}(w)e^{-jwkT_s} = R_{T_s}(w) Y(wT_s).
  \]

Sampled Data Systems: equivalent cts-time transfer function

- Consider a digital system \( \hat{H}(\Omega) \) (or \( \hat{H}(e^{j\Omega}) \) depending on notation), whose (ZS) output is \( \hat{y} \) when the input is \( \hat{f} \), i.e., \( \hat{Y} = \hat{H} \hat{F} \).
- The equivalent continuous-time transformation of the tandem system
  \[
  f \xrightarrow{A/D (T_s\text{-sample})} \hat{H}(\Omega) \xrightarrow{D/A (T_s\text{-hold})} y
  \]
  with input \( f \) has (ZS) output
  \[
  Y(w) = R_{T_s}(w) Y(wT_s) = R_{T_s}(w) \hat{H}(wT_s) F(wT_s)
  \]
  \[
  = R_{T_s}(w) \hat{H}(wT_s) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s).
  \]
- Exercise: Show that if \( f \) is band-limited by \( w_s/2 \) (i.e., \( w_s \) is greater than \( f \)'s Nyquist frequency) and the previous sampled data system is followed by an ideal low-pass filter with bandwidth \( w_s/2 \), then the equivalent (continuous-time) transfer function is
  \[
  H(w) = \hat{H}(wT_s) T_s^{-1} R_{T_s}(w)(u(w + w_s/2) - u(w - w_s/2)).
  \]
Sampled Data Systems: equivalent cts-time transfer function - discussion

• Note that the term in the transfer function $H$,

$$T_s^{-1} R_T(w)(u(w + w_s/2) - u(w - w_s/2)) = \text{sinc}(\Omega/2)(u(\Omega + \pi) - u(\Omega - \pi))$$

is not a constant function of $\Omega = wT_s$.

• This distortion due to the hold function $R$ can be reduced by putting in tandem with $\hat{H}$ an equalizer system with transfer function approximately

$$\hat{R}^{-1}(\Omega) := \sum_{k=-\infty}^{\infty} \frac{u(\Omega + \pi - k2\pi) - u(\Omega - \pi - k2\pi)}{\text{sinc}((\Omega - k2\pi)/2)}$$

i.e.,

$$\hat{H}(\Omega) \rightarrow \hat{R}^{-1}(\Omega)$$

Sampled Data Systems: equalization of hold $\text{sinc}(\Omega/2)$ by $\hat{R}^{-1}(\Omega)$

• the hold (at left, $R$) distorts the signal by attenuating its higher frequency components

• the equalizer (at right, $R^{-1}$) amplifies at the higher frequencies to cancel out this distortion
• Consider the \( N \)-point sequence \( x = \{x[0], \ldots, x[N-1]\} \).

• Let \( X(\Omega) = \sum_{k=0}^{N-1} x[k] e^{-jk\Omega} \).

• Taking \( \Omega = \frac{2\pi}{N} r \) \((=: \Omega_{or})\) for \( r = 0, 1, \ldots, N \), the DFT of \( x \) is the \( N \)-point sequence

\[
X[r] := X\left(\frac{2\pi}{N} r\right) = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi}{N} rk},
\]

for \( r \in \{0, 1, \ldots, N-1\} \).

• The inverse DFT is, for \( k \in \{0, 1, \ldots, N-1\} \),

\[
x[k] = N^{-1} \sum_{r=0}^{N-1} X[r] e^{j\frac{2\pi}{N} rk}.
\]

DFT (cont)

• Define an \( N \)th complex root of unity (“twiddle” factor)

\[
W_N = e^{-j\frac{2\pi}{N}}
\]

so that

\[
X[r] = \sum_{k=0}^{N-1} x[k] W_N^r.
\]

• Exercise: Show that \( \forall \ell \in \mathbb{Z}, W_N^\ell = W_N^{(\ell \mod N)} \). So, computing \( \text{all} \) the \( W_N^r \) terms is only on the order of \( N \) multiplies.

• Given the \( N \) unique twiddle factors \( W_N^r \), to directly compute each \( X[r] \) requires \( N \) \((\text{complex})\) multiplications and \( N-1 \) additions.

• So, to directly compute \( X = \{X[0], \ldots, X[N-1]\} \) requires \( N^2 \) multiplications and \( N(N-1) \) additions.
Fast Fourier Transform (FFT) by time decimation

- If \( N \) is even, then we can separately sum even and odd times \( k \) to compute the DFT:

\[
X[r] = \sum_{k=0}^{\frac{N}{2}-1} x[2k]W_N^{2rk} + \sum_{k=0}^{\frac{N}{2}-1} x[2k + 1]W_N^{2r(k+1)}
\]

\[
= \sum_{k=0}^{\frac{N}{2}-1} x[2k]W_N^{2rk} + W_N^r \sum_{k=0}^{\frac{N}{2}-1} x[2k + 1]W_N^{2rk}
\]

\[
= \sum_{k=0}^{\frac{N}{2}-1} x[2k]W_{N/2}^{rk} + W_N^r \sum_{k=0}^{\frac{N}{2}-1} x[2k + 1]W_{N/2}^{rk}
\]

where the last equality is simply because \( W_N^2 = W_{N/2} \).

- So, the first term is an \((N/2)\)-point DFT of (the \((N/2)\)-point sequence)

\( x[0], x[2], …, x[N-2] \).

- while the second term is the \((N/2)\)-point DFT of \( x[1], x[3], …, x[N-1] \).

So, after one time-decimation step, the computational complexity has become

- \( 2(\frac{N}{2})^2 + N = N(N/2) + N \) multiplications and

- \( 2 \frac{N}{2} (\frac{N}{2} - 1) + N = N(N/2) \) additions.

- If \( N \) is a power of 2, then we can repeat this time-decimation until there are only 1-point DFTs, whereupon the computational complexity will be

\[
\approx N \log_2 N.
\]

- For large \( N \), this is a substantial savings over the direct approach with computational complexity

\[
\approx N^2.
\]

- Exercise: Show how decimation can be applied to similarly reduce the computational complexity of the inverse DFT.
Transient analysis in discrete time by unilateral $z$-transform

- $z$-transform definition and region of convergence.
- Basic $z$-transform pairs and properties.
- Inverse $z$-transform of rational polynomials by Partial Fraction Expansion (PFE).
- Total transient response of SISO DT LTIC systems $Q(\Delta^{-1})y = P(\Delta^{-1})f$.
- The steady-state eigenresponse revisited.
- System composition and canonical realizations.

The unilateral $z$-transform & region of convergence

- The $z$-transform of a signal $x = \{x[k]\}_{k \geq 0}$ is
  $$X(z) = (Zx)(z) = \sum_{k=0}^{\infty} x[k]z^{-k} := \lim_{K \to \infty} \sum_{k=0}^{K} x[k]z^{-k},$$
  where $z \in \mathbb{C}$.

- If the signal $x$ is bounded by an exponential (geometric), i.e.,
  $$\exists \ M, \gamma \in \mathbb{R}_{>0} \text{ such that } \forall k \in \mathbb{Z}_{\geq 0}, \ |x[k]| \leq M\gamma^k \ (i.e., \ -M\gamma^k \leq x[k] \leq M\gamma^k)$$
  then the series $X(z)$ converges in the region outside of a disk centered $0 \in \mathbb{C}$,
  $$\{z \in \mathbb{C} \mid |z| > \gamma\}.$$

- To see why bounded by an exponential suffices, recall absolute convergence $\Rightarrow$ convergence:
  $$\forall k \geq 0, \ |x[k]z^{-k}| = |x[k]| \cdot |z|^{-k} \leq M\gamma^k |z|^{-k} = M(\gamma/|z|)^k \Rightarrow \sum_{k=0}^{\infty} |x[k]z^{-k}| \leq M \sum_{k=0}^{\infty} (\gamma/|z|)^k$$
  which converges if $\gamma/|z| < 1$. 

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Basic $z$-transform pairs and RoCs

\[\delta[k] \rightarrow \sum_{k=0}^{\infty} z^{-k} = 1, \quad z \in \mathbb{C}\]

\[u[k] \rightarrow \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \quad |z| > 1\]

\[\beta^k u[k] \rightarrow \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{1}{1-\beta z^{-1}} = \frac{z}{z-\beta}, \quad |z| > |\beta|\]

\[\{\beta^{-1} u[k-1]\}(z) \rightarrow \sum_{k=1}^{\infty} \beta^{-1} \beta^k z^{-k} = z^{-1} \sum_{k=0}^{\infty} \beta^k z^{-k'} = z^{-1} \frac{1}{1-\beta z^{-1}} = \frac{1}{z-\beta}, \quad |z| > |\beta|\]

\[e^{\Omega_k} u[k] \rightarrow \sum_{k=0}^{\infty} e^{\Omega_k} z^{-k} = \frac{1}{1-e^{\Omega} z^{-1}}, \quad |z| > 1 \quad (\beta = e^{i\Omega})\]

\[k\beta^k u[k] \rightarrow \sum_{k=0}^{\infty} k \beta^k z^{-k} = \frac{d}{d\beta} \sum_{k=0}^{\infty} \beta^k z^{-k} = \beta \frac{d}{d\beta} \frac{1}{1-\beta z^{-1}} = \frac{\beta z^{-1}}{(1-\beta z^{-1})^2}, \quad |z| > |\beta|\]

**Exercise:** Find $Z\{A \cos(\Omega_k + \phi) u[k]\}$ and $Z\{A \sin(\Omega_k + \phi) u[k]\}$. 

---

Basic $z$-transform properties: linearity

- The $z$-transform is a linear operator: for all scalars $a_1, a_2 \in \mathbb{C}$ and all signals $x_1, x_2 : \mathbb{Z}^0 \rightarrow \mathbb{C}$ with respective ROCs $C_1, C_2 \subset \mathbb{C}$,

\[(Z\{a_1 x_1 + a_2 x_2\})(z) = a_1 (Z x_1)(z) + a_2 (Z x_2)(z), \quad z \in C_1 \cap C_2.\]

- Note that

\[\{z \mid |z| > \gamma_1\} \cap \{z \mid |z| > \gamma_2\} = \{z \mid |z| > \max\{\gamma_1, \gamma_2\}\} \subset \mathbb{C}.

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Basic $z$-transform properties: advance time shift

- Advance time shift (no change in RoC): Let $X = \mathcal{Z}x$.

\[
\Delta^{-1} x \overset{\mathcal{Z}}{\rightarrow} \sum_{k=0}^{\infty} x[k+1]z^{-k} = -zx[0] + \sum_{k=-1}^{\infty} x[k+1]z^{-k}
\]

\[
= -zx[0] + z \sum_{k=-1}^{\infty} x[k+1]z^{-(k+1)}
\]

\[
= -zx[0] + z \sum_{k=0}^{\infty} x[k]z^{-k}
\]

\[
= -zx[0] + zX(z)
\]

- Exercise: For $v \in \mathbb{Z}^+$ show by induction that

\[
(\mathcal{Z}\{\Delta^{-v}x\})(z) = -\sum_{k=1}^{v} z^{k}x[v-k] + z^{v}X(z)
\]

Basic $z$-transform properties: delay time shift

- Delay time shift (no change in RoC): For $v \in \mathbb{Z}^+$,

\[
\Delta^{v}(xu) \overset{\mathcal{Z}}{\rightarrow} \sum_{k=0}^{\infty} x[k-v]u[k-v]z^{-k}
\]

\[
= \sum_{k=v}^{\infty} x[k-v]z^{-k} = \sum_{k=0}^{\infty} x[k']z^{-k'-v}
\]

\[
= z^{-v}X(z).
\]

- So in the “zero-state” (input-output) context (i.e., $x[k]u[k] = 0$ for $k < 0$), we identify multiplying by $z^{-1}$ in complex-frequency domain with the unit delay $\Delta$ in the time domain.

- Delay $v \in \mathbb{Z}^+$ of non-causal $x$:

\[
\Delta^{v}x \overset{\mathcal{Z}}{\rightarrow} \sum_{k=0}^{\infty} x[k-v]z^{-k} = \sum_{k'=v}^{\infty} x[k']z^{-k'-v}
\]

\[
= \sum_{k'=v}^{-1} x[k']z^{-k'-v} + z^{-v}X(z).
\]
Basic $z$-transform properties: frequency shift & convolution

- Let $X = Zx$ with RoC $C(\gamma) := \{z \in \mathbb{C} \mid |z| > \gamma\}$.

$$
\beta^k x[k] \to \sum_{k=0}^{\infty} \beta^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k](z/\beta)^{-k} = X(z/\beta), \quad z \in C(\gamma|\beta|).
$$

i.e., $\times \beta^k$ in the time-domain is dilation by $\beta$ in the $z$-domain.

- For signals $x_1, x_2 : \mathbb{Z}^+ \to \mathbb{C}$ ($x_1[k], x_2[k] = 0$ for $k < 0$), with respective ROCs $C_1, C_2 \subset \mathbb{C},$

$$
x_1 * x_2 \to \sum_{k=0}^{\infty} (x_1 * x_2)[k] z^{-k} = \sum_{v=0}^{\infty} \sum_{k=v}^{\infty} x_1[v] x_2[k-v] z^{-(k-v)} z^{-v} = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=0}^{\infty} x_2[k-v] z^{-(k-v)} = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=0}^{\infty} x_2[k'] z^{-k'} = X_1(z) X_2(z), \quad z \in C_1 \cap C_2.
$$

Basic $z$-transform properties: convolution, IVT & FVT

- So convolution in the time-domain is multiplication in the frequency domain.

- The converse is also true.

- Directly by definition of $X = Zx$, we get the initial value theorem

$$
\lim_{z \to \infty} X(z) = x[0].
$$

- There is also a “final value” theorem for $\lim_{k \to \infty} x[k]$.
We now study transient analysis of LTI difference equations using $z$-transforms.

Recall our system is defined given polynomials $P, Q$, input $f$ and initial conditions:
- $Q(\triangle^{-1})y = P(\triangle^{-1})f$, where $y$ is the (total) output and
- input $f[k] = 0$ for $k < 0$,
- degree of polynomial $Q = n \geq m = \text{degree of polynomial } P$ (causal system),
- $Q(z) = z^n + \sum_{v=0}^{n-1} a_v z^{-v}$ (i.e., $a_n = 1$) and $P(z) = \sum_{v=0}^{m} b_v z^{v}$,
- $a_n \neq 0$ or $b_n \neq 0$ for poly’ls $Q, P$ of minimum degree,
- $n$ initial conditions $y[-n], y[-n + 1], ..., y[-2], y[-1]$.

We can restate the difference equation in terms of delays by delaying both sides by $n$ time-units (i.e., applying with $\Delta^n$), to get
\[
\Delta^n Q(\triangle^{-1})y = \Delta^n P(\triangle^{-1})f \\
\Rightarrow \tilde{Q}(\Delta)y := \sum_{v=0}^{n} a_v \Delta^{n-v}y = \sum_{v=0}^{m} b_v \Delta^{n-v}f =: \tilde{P}(\Delta)f
\]

So, taking the $z$-transform of the (delay) difference equation, we get by the (delay) time-shift and linearity properties that
\[
\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k]z^{-k-v} + \tilde{Q}(z^{-1})Y(z) = \tilde{P}(z^{-1})F(z)
\]

So, solving for the total response $Y$ we get
\[
Y(z) = \frac{\tilde{P}(z^{-1})F(z)}{\tilde{Q}(z^{-1})} - \frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k]z^{-k-v}}{\tilde{Q}(z^{-1})} = Y_{ZS}(z) + Y_{ZI}(z)
\]

where the ZIR and ZSR in the complex-frequency ($z$) domain respectively are
\[
Y_{ZI}(z) := -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k]z^{-k-v}}{\tilde{Q}(z^{-1})} = -\frac{\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k]z^{n-k-v}}{\tilde{Q}(z)}
\]
\[
Y_{ZS}(z) := \frac{\tilde{P}(z^{-1})F(z)}{\tilde{Q}(z^{-1})} = \frac{P(z)}{Q(z)} F(z) = H(z)F(z) \quad \text{(transfer function } H).\]

Regarding this total transient response, note how
- the $z$-transform’s unilateral aspect captures the impact of initial conditions (ZIR), and
- a greater range of inputs $f$ than under DTFT through $\text{RoC} \subset \mathbb{C}$ (not just $|z| = 1$).
• Suppose i.c. $y[-1] = -1$, input $f[k] = 2(-3)^k u[k]$ and output $y$ s.t.

\[
\forall k \geq -1, \quad 2y[k+1] + 2y[k] = 3f[k+1] + 2f[k].
\]

• To find the total response $y$, we take the $z$-transform of the equivalent system: $\forall k \geq 0$,

\[
2y[k] + 2y[k-1] = 3f[k] + 2f[k-1]
\]

\[
\Rightarrow 2Y(z) + 2(z^{-1}Y(z) + y[-1]) = 3F(z) + 2z^{-1}F(z).
\]

• So by the delay property for non-causal signals ($y$), the total response

\[
Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} F(z) + \frac{-2y[-1]}{2 + 2z^{-1}}
\]

\[
= H(z) F(z) + \frac{-y[-1]}{1 + z^{-1}}
\]

\[
= Y_{ZS}(z) + Y_{ZI}(z)
\]

with RoC for $Y$ being the intersection of those of $F$ and $H$.

---

• Here $F(z) = \mathcal{Z}\{2(-3)^k\} = 2/(1 + 3z^{-1})$, $y[-1] = -1$, so

\[
Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} \cdot \frac{2}{1 + 3z^{-1}} + \frac{1}{1 + z^{-1}}
\]

\[
= \frac{3 + 2z^{-1}}{(1 + z^{-1})(1 + 3z^{-1})} + \frac{1}{1 + z^{-1}}
\]

\[
= \left( \frac{3.5}{1 + 3z^{-1}} + \frac{-0.5}{1 + z^{-1}} \right) + \frac{1}{1 + z^{-1}}
\]

where for the last equality see PFE below (here in $z^{-1}$).

• Understanding that the ZIR begins at $k = -1$ (initial condition) and the ZSR at time $k = 0$, we get:

\[
\forall k \geq -1, \quad y[k] = (3.5(-3)^k - 0.5(-1)^k)u[k] + (-1)^k = y_{ZS}[k] + y_{ZI}[k],
\]

where we minded the ambiguity $\mathcal{Z}x = \mathcal{Z}xu$.

• Exercise: Verify this solution using time-domain methods, i.e.,

\[
y = y_{ZI} + y_{ZS} = y_{ZI} + h \ast f, \text{ where } h \text{ and } y_{ZI} \text{ consist of char. modes}.
\]
Inverse \( z \)-transform of proper rational polynomials

- We now describe how to find \( z^{-1}X \) of causal signal \( X \) that is rational polynomial in \( z \), i.e., \( X(z) = M(z)/N(z) \) where \( M(z) \) and \( N(z) \) are polynomials in \( z \).

- If \( \deg(M) = \deg(N) + 1 \), we perform long division to write \( X = c + M/N \) where \( \deg(N) = \deg(M) \) and \( z^{-1}X = c\delta + z^{-1}\{M/N\} \).

- If \( \deg(M) = \deg(N) \) and \( M(0) = 0 \) (so \( z^{-1}M(z) \) is a polynomial), we can factor \( z \) from \( M \) to get
  \[
  X(z) = \frac{z^{-1}M(z)}{N(z)}.
  \]

- We will find \( z^{-1}X \) using PFE of the strictly proper rational polynomial \( z^{-1}M(z)/N(z) \).

- Alternatively, we could apply PFE on strictly proper rational polynomials in \( z^{-1} \), \( z^{-K}M(z)/(z^{-1}N(z)) \) where \( K := \deg(N) \), as in the previous example.

Partial Fraction Expansion (PFE) example in \( z \) (not \( z^{-1} \))

- For example, suppose
  \[
  X(z) := \frac{z(3z + 2)}{z^2 - 0.64} = \frac{3z + 2}{(z + 0.8)(z - 0.8)} = z \left( \frac{0.25}{z + 0.8} + \frac{2.75}{z - 0.8} \right) = 0.25 \frac{z}{z + 0.8} + 2.75 \frac{z}{z - 0.8}
  \]
  where PFE (below) gave the numerators (residues) 0.25 and 2.75.

- So,
  \[
  (z^{-1}X)[k] = 0.25(-0.8)^k u[k] + 2.75(0.8)^k u[k]
  \]

- Note that the associated RoC of \( X \) is \( \{z \in \mathbb{C} \mid |z| > 0.8\} \).
Partial Fraction Expansion (PFE) - preliminaries

- Let $K = \deg(N) = \deg(M)$ so that we can factor
  \[ N(z) = \prod_{k=1}^{K} (z - p_k), \]
  where the $p_k$ are the roots of $N$ (poles of $M/N$).

- We assume $M$ and $N$ have no common roots, i.e., no “pole-zero cancellation” issue to consider, so that the $p_k$ are the poles of $M/N$.

- Again, we assume $M(0) = 0$ (0 is a zero of $M/N$) and so $z^{-1}M(z)$ is a polynomial of degree $K - 1$.

- Note that the RoC for $M(z)/N(z)$ is $\{z \in \mathbb{C} \mid |z| > \max_k |p_k|\}$.

---

PFE - the case of no repeated poles

- Suppose there are no repeated poles for $M/N$, i.e., $\forall k \neq l, \; p_k \neq p_l$.

- In this case, we can write the PFE of $z^{-1}M(z)/N(z)$ as
  \[ \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l}, \]
  \[ \Rightarrow \frac{M(z)}{N(z)} = \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} \frac{1}{1 - p_l z^{-1}} \]
  where the scalars (Heaviside coefficients) $c_l \in \mathbb{C}$ are
  \[ c_l = \left. \frac{z^{-1}M(z)}{\prod_{k \neq l} (z - p_k)} \right|_{z = p_l} = \lim_{z \to p_l} \frac{z^{-1}M(z)}{N(z)} (z - p_l) = \left. \frac{z^{-1}M(z)}{N(z)} (z - p_l) \right|_{z = p_l}. \]

- That is, to find the Heaviside coefficient $c_k$ over the term $z - p_k$ in the PFE, we have removed (covered up) the term $z - p_k$ from the denominator $N(z)$ and evaluated the remaining rational polynomial at $z = p_k$.

- This approach, called the Heaviside cover-up method, works even when $p$ is $\mathbb{C}$-valued.

- Given the PFE of $z^{-1}M/N$, $(Z^{-1}M/N)[k] = \sum_{l=1}^{K} c_l p_l^k u[k]$. 

PFE - proof of Heaviside cover-up method

• To prove that the above formula for the Heaviside coefficient \( c_l \) is correct, note that the claimed PFE of \( z^{-1}M(z)/N(z) \) is

\[
\sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} c_l \prod_{k \neq l} \frac{z - p_k}{N(z)}
\]

• Thus, the PFE equals \( z^{-1}M(z)/N(z) \) if and only if the numerator polynomials are equal, i.e., iff

\[
z^{-1}M(z) = \sum_{l=1}^{K} c_l \prod_{k \neq l} (z - p_k) =: \tilde{M}(z).
\]

• Again, two polynomials are equal if their degrees, \( L \), are equal and either:
  - their coefficients are the same, or
  - they agree at \( L + 1 \) (or more) different points, e.g., two lines \( (L = 1) \) are equal if they agree at 2 \((= L + 1)\) points.

• Since \( z^{-1}M(z) \) is a polynomial of degree \(< K\), it suffices to check that whether \( z^{-1}M(z) = \tilde{M}(z) \) for all \( z = p_k, k \in \{1, 2, ..., K\} \), i.e., this would create \( K \) equations in \(< K \) unknowns (the coefficients of \( M \)).

PFE - proof of Heaviside cover-up method (cont)

• But note that any pole \( p_r \) of \( z^{-1}M(z)/N(z) \) is a root of all but the \( r \)-th term in \( \tilde{M} \), so that

\[
\tilde{M}(p_r) = c_r \prod_{k \neq r} (p_r - p_k)
\]

\[
= \left( \frac{z^{-1}M(z)}{\prod_{k \neq r} (z - p_k)} \right)_{z=p_r} \prod_{k \neq r} (p_r - p_k)
\]

\[
= \frac{p_r^{-1}M(p_r)}{\prod_{k \neq r} (p_r - p_k)} \prod_{k \neq r} (p_r - p_k)
\]

\[
= p_r^{-1}M(p_r).
\]

• Q.E.D.

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PFE - the case of no repeated poles - example

- To find the inverse $z$-transform of a proper rational polynomial $X = M/N$ with $M(0) = 0$, first factor its denominator $N$ and factor $z$ from $M$, e.g.,

$$X(z) = \frac{z^3 + 5z^2}{z^3 + 9z^2 + 26z + 24} = \frac{z^2 + 5z}{(z + 4)(z + 3)(z + 2)}, \text{ for } |z| > 4.$$

- So, by PFE

$$X(z) = z \left( \frac{c_4}{z + 4} + \frac{c_3}{z + 3} + \frac{c_2}{z + 2} \right) = z^{-1}M(z) \Rightarrow z^{-1}M(z) = 1z^2 + 5z + 0 = c_4(z + 3)(z + 2) + c_3(z + 4)(z + 2) + c_2(z + 4)(z + 3) = \hat{M}(z).$$

- We can solve for the 3 constants $c_k$ by comparing the 3 coefficients of quadratic $M$ and $\hat{M}$.

- The Heaviside cover-up method suggests we try $z = -2, -3, -4$ to solve for $c_2, c_3, c_4$:

$$c_4 = \frac{z^2 + 5z}{(z + 3)(z + 2)} \bigg|_{z = -4} = -2, \quad c_3 = \frac{z^2 + 5z}{(z + 4)(z + 2)} \bigg|_{z = -3} = 6, \quad c_2 = \frac{z^2 + 5z}{(z + 4)(z + 3)} \bigg|_{z = -2} = -3$$

- Thus, $x[k] = (Z^{-1}X)[k] = (-2(-4)^k + 6(-3)^k - 3(-2)^k)u[k]$.

PFE - the case of a non-repeated, complex-conjugate pair of poles

- Again, recall that for polynomials with all coefficients $\in \mathbb{R}$, all complex poles will come in complex-conjugate pairs, $p_1 = \overline{p_2}$.

- The case of non-repeated poles $p_1, p_2 = \alpha \pm j\beta$ $(\alpha, \beta \in \mathbb{R}, j := \sqrt{-1})$ that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients $c_1, c_2$, i.e., $c_1 = \overline{c_2}$.

- In the PFE, we can alternatively combine the terms:

$$\frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} = \frac{r_1z + r_0}{(z - \alpha)^2 + \beta^2}$$

where by equating the two numerator polynomials’ coefficients,

$$r_0 = -c_1p_2 - c_2p_1 = -2\text{Re}\{c_1p_2\} \in \mathbb{R} \text{ and } r_1 = c_1 + c_2 = 2\text{Re}\{c_1\} \in \mathbb{R}.$$

- Exercise: Show that

$$2|c| \cdot |p|^k \cos(k\angle p + \angle c) \xrightarrow{z} \frac{cz}{z - p} + \frac{\overline{cz}}{z - \overline{p}}$$
To find the inverse $z$-transform of

$$X(z) = \frac{3z^2 + 2z}{z^3 + 5z^2 + 10z + 12},$$

first factor the denominator and divide the numerator by $z$ to get

$$X(z) = \frac{3z + 2}{(z^2 + 2z + 4)(z + 3)}.$$

Note that the poles of $X$ are $-3$ and $-1 \pm j\sqrt{3}$ (so $X$'s RoC is $|z| > 3$).

So, we can expand $X$ to

$$X(z) = \frac{r_1 z + r_0}{z^2 + 2z + 4} + \frac{c_3}{z + 3},$$

where by the Heaviside cover-up method,

$$c_3 = \frac{3z + 2}{z^2 + 2z + 4} \bigg|_{z = -3} = -1.$$

To find $r_1, r_0$, we will compare coefficients of the numerator polynomials of $X$ (actually $z^{-1}X$) and its PFE, i.e.,

$$0z^2 + 3z + 2 = (r_1 z + r_0)(z + 3) + c_3(z^2 + 2z + 4) \quad (*)$$

$$= (r_1 - 1)z^2 + (3r_1 + r_0 - 2)z + 3r_0 - 4.$$

Thus, by comparing coefficients

$$0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4$$

we get

$$r_0 = 2 \quad \text{and} \quad r_1 = 1.$$

Note how $z = -3$ in $(*)$ gives $c_3 = -1$ as Heaviside cover-up did.
• Thus by substituting, we get

\[
X(z) = \frac{z + 2}{z^2 + 2z + 4} + \frac{-1}{z + 3}
\]

• Exercise: Show that

\[
x[k] = (z^{-1}X)[k] = \left(\frac{\sqrt{4/3}}{2^k} \cos(k2\pi/3 - \pi/6) - (-3)^k\right) u[k].
\]

PFE - the general case of repeated poles

• If a particular pole \(p\) of \(z^{-1}M(z)/N(z)\) is of order \(r \geq 1\), i.e., \(N(z)\) has a factor \((z-p)^r\), then the PFE of \(z^{-1}M(z)/N(z)\) has the terms

\[
\frac{c_1}{z - p} + \frac{c_2}{(z - p)^2} + ... + \frac{c_r}{(z - p)^r} = \sum_{k=1}^{r} \frac{c_k}{(z - p)^k} = \frac{z^{-1}M(z)}{N(z)} - \Phi(z)
\]

with \(c_k \in \mathbb{C} \forall k \in \{1, 2, ..., r\}\), where \(\Phi(z)\) represents the other PFE terms of \(z^{-1}M(z)/N(z)\) (i.e., corresponding to poles \(\neq p\)).

• Note that equating \(z^{-1}M(z)/N(z)\) to its PFE and multiplying by \((z-p)^r\) gives

\[
\frac{z^{-1}M(z)}{N(z)}(z - p)^r = c_r + \sum_{k=1}^{r-1} c_k(z - p)^{r-k} + \Phi(z)(z - p)^r
\]

\[
\Rightarrow \left. \frac{z^{-1}M(z)}{N(z)}(z - p)^r \right|_{z = p} = c_r,
\]

i.e., Heaviside cover-up (of the entire term \((z-p)^r\)) works for \(c_r\).
PFE - the general case of repeated poles (cont)

- To find $c_{r-1}$, we differentiate the previous display to get
  \[
  \frac{d}{dz} z^{-1} M(z) (z-p)^r = \sum_{k=1}^{r-1} c_k (r-k)(z-p)^{r-k} + \frac{d}{dz} \Phi(z)(z-p)^r
  \]
  \[
  = c_{r-1} + \sum_{k=1}^{r-2} c_k (r-k)(z-p)^{r-k} + \frac{d}{dz} \Phi(z)(z-p)^r
  \]
  \[
  \Rightarrow c_{r-1} = \left. \left( \frac{d}{dz} z^{-1} M(z) (z-p)^r \right) \right|_{z=p}
  \]

- If we differentiate the original display $k \in \{0, 1, 2, \ldots, r-1\}$ times and then substitute $z = p$, we get (with $0! := 1$)
  \[
  \left. \left( \frac{d^k}{dz^k} z^{-1} M(z) (z-p)^r \right) \right|_{z=p} = k! c_{r-k}
  \]
  \[
  \Rightarrow c_{r-k} = \frac{1}{k!} \left. \left( \frac{d^k}{dz^k} z^{-1} M(z) (z-p)^r \right) \right|_{z=p}.
  \]

PFE - the general case of repeated poles - example

- To find the inverse $z$-transform of
  \[
  X(z) = \frac{z(3z+2)}{(z+1)(z+2)^3}
  \]
  write the PFE of $X$ as
  \[
  X(z) = z \left( \frac{c_1}{z+1} + \frac{c_{2,1}}{z+2} + \frac{c_{2,2}}{(z+2)^2} + \frac{c_{2,3}}{(z+2)^3} \right),
  \]
  so clearly the RoC of causal $X$ is $|z| > 2$.

- By Heaviside cover-up
  \[
  c_1 = \left. \frac{3z+2}{(z+2)^3} \right|_{z=-1} = -1 \quad \text{and} \quad c_{2,3} = \left. \frac{3z+2}{z+1} \right|_{z=-2} = 4.
  \]
PFE - the general case of repeated poles - example (cont)

- Also,
  \[
  c_{2,2} = \frac{1}{1!} \left( \frac{d}{dz} \frac{3z + 2}{z + 1} \right) \bigg|_{z=-2} = \frac{1}{1!} \frac{1}{(z + 1)^2} \bigg|_{z=-2} = 1
  \]
  \[
  c_{2,1} = \frac{1}{2!} \left( \frac{d^2}{dz^2} \frac{3z + 2}{z + 1} \right) \bigg|_{z=-2} = \frac{1}{2!} \frac{-2}{(z + 1)^3} \bigg|_{z=-2} = 1
  \]

- Thus,
  \[
  X(z) = \frac{z-1}{z+1} + z \frac{1}{z+2} + z \frac{1}{(z+2)^2} + z \frac{4}{(z+2)^3} \quad \forall |z| > 2
  \]
  \[
  \Rightarrow x[k] = (Z^{-1}X)[k] = \left( (-1)^k + (-2)^k + k(-2)^{k-1} + 4 \frac{k(k-1)}{2} (-2)^{k-2} \right) u[k]
  \]

- Exercise: Show by induction and integration by parts that: \( \forall m \in \mathbb{Z}^>0 \),
  \[
  \binom{k}{m} \gamma^{k-m} u[k] \xrightarrow{Z} \frac{z}{(z-\gamma)^m}
  \]

- Exercise: Find the ZSR \( y \) to input \( f[k] = 2^j k u[k] = 2e^{jk\pi/2} u[k] \) of the marginally stable system \( H(z) = 4/(z^2 + 1) \).

---

PFE of \( M/N \) when \( M(0) \neq 0 \)

- If \( M(0) \neq 0 \) (so cannot factor \( z \) from \( M(z) \)), then just perform long division if \( \deg(M) \geq \deg(N) \) to get a strictly proper rational polynomial, factor \( N \) to find the poles, and find the PFE as before.

- When taking inverse \( z \)-transform, recall the \( z \)-transform pair
  \[
  \beta^{k-1} u[k-1] \xrightarrow{Z} \frac{1}{z-\beta}, \quad |z| > |\beta|
  \]
PFE without factoring $z$ from the numerator first

- For example, to find the ZSR to $f[k] = 2(-1)^k u[k]$ of the system
  
  $$y[k+1] - 4y[k] = 5f[k],$$

  take the $z$-transform to get
  
  $$Y_{ZS}(z) = H(z)F(z) = \frac{5}{z-4} F(z) = \frac{10z}{(z-4)(z+1)}$$
  
  $$= \frac{8}{z-4} + \frac{2}{z+1} \quad \text{(by PFE)}$$
  
  $$\Rightarrow y_{ZS}[k] = 8(4)^{k-1} u[k-1] + 2(-1)^{k-1} u[k-1]$$

- Note that the unit-pulse response is
  
  $$h[k] = Z^{-1}(H)[k] = 5(4)^{k-1} u[k-1],$$

  and that, by delaying the difference equation to get
  
  $$y[k] = -4y[k-1] + 5f[k-1],$$

  we see that (the ZSR) $y_{ZS}[0] = 0$.

- **Exercise:** First factor $z$ from the numerator of $Y_{ZS}$ before PFE to show that
  
  $$y_{ZS}[k] = 2(4)^k u[k] - 2(-1)^k u[k].$$

  Is this result different? Check for $k = 0$ and $k > 0$.

---

PFE and eigenresponse for asymptotically stable systems

- The total response of a SISO LTI system to input $f$ is of the form
  
  $$Y(z) = H(z)F(z) + \frac{P_1(z)}{Q(z)} = \frac{P(z)}{Q(z)} F(z) + \frac{P_1(z)}{Q(z)} = Y_{ZS}(z) + Y_{ZI}(z).$$

  where $P_1$ depends on the initial conditions and the RoC is the intersection of that of input $F = Zf$ and the system characteristic modes.

- **Unlike for DTFT notation, here write** $H(z) = P(z)/Q(z) = (Zh)\{z\}$.

- Suppose the system is BIBO/asymptotically stable and the input is a sinusoid at frequency $\Omega_o$ radians per unit time, $f[k] = A e^{j(\Omega_o k + \phi)} u[k] = A e^{j\phi} (e^{j\Omega_o})^k u[k]$ with $A > 0$

  $$\Rightarrow F(z) = A e^{j\phi} z/(z - e^{j\Omega_o}) \quad \text{with RoC } |z| > 1.$$

- Since $e^{j\Omega_o}$ cannot be a system pole (owing to asymptotic stability all poles have modulus strictly less than one), we can use Heaviside cover-up on

  $$Y_{ZS}(z) = H(z)F(z) = \frac{P(z)}{Q(z)(z-e^{j\Omega_o})} A e^{j\phi}$$

  to get

  $$Y_{ZS}(z) = \frac{H(e^{j\Omega_o})}{z-e^{j\Omega_o}} A e^{j\phi} + \text{char. modes} = H(e^{j\Omega_o}) F(z) + \text{char. modes.}$$

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Thus, the total response of an asymptotically stable system to a sinusoidal input $f$ at frequency $\omega_o$ is
\[ y[k] = H(e^{j\omega_o})f[k] + \text{linear combination of characteristic modes}. \]

So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as $k \to \infty$,
\[ y[k] \to H(e^{j\omega_o})f[k] = H(e^{j\omega_o})Ae^{j(\omega_o \cdot k + \phi)} = |H(e^{j\omega_o})|Ae^{j(\omega_o \cdot k + \phi + \angle H(e^{j\omega_o}))}, \]
where again,
\[ H = \frac{P}{Q} \text{ is the system’s transfer function,} \]
\[ |H(e^{j\omega_o})| \text{ is the system’s magnitude response at frequency } \omega_o \text{ radians/unit-time, and} \]
\[ \angle H(e^{j\omega_o}) \text{ is the system’s phase response at } \omega_o. \]

Laplace’s approximation: the rate at which the total response converges to the eigenresponse response is according to the characteristic value of largest modulus, which will be $< 1$ owing to the stability assumption, i.e., giving the modes(s) that $\to 0$ slowest.

In continuous-time systems, it’s the characteristic value of largest real part, which will be negative owing to stability assumption.
Canonical (ZS) system-realizations - direct form

- Consider the proper \( m \leq n \) transfer function

\[
H(z) = \frac{P(z)}{Q(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{Y(z)}{F(z)}
\]

- The direct-form realization employs the interior system state \( X := F/Q \), i.e., \( F = QX \) and \( Y = PX \) where the former implies (with \( a_n = 1 \)),

\[
F(z) = \sum_{r=0}^{n} a_r z^r X(z) \quad \Rightarrow \quad z^n X(z) = F(z) - \sum_{r=0}^{n-1} a_r z^r X(z).
\]

- For \( n = 2 \), there are two “system states” (outputs of unit delays), \( X \) and \( zX \) (respectively, \( x[k] \) and \( \Delta^{-1}x[k] = x[k+1] \)):

\[
\begin{array}{c}
\text{F} \\
\text{+} \\
z^{-1}X \\
\text{-a_1} \\
\text{z^{-1}}X \\
\text{z^{-1}}X \\
\text{-a_0} \\
X
\end{array}
\]

Canonical system-realizations - direct form (cont)

- Now adding \( Y = PX \), we finally get the direct-form canonical system-realization of \( H \):

\[
\begin{array}{c}
\text{F} \\
\text{+} \\
z^{-1}X \\
\text{-a_1} \\
\text{z^{-1}}X \\
\text{z^{-1}}X \\
\text{z^{-1}}X \\
\text{-a_0} \\
X \\
\text{+} \\
\text{Y}
\end{array}
\]

- Again, state variables taken as outputs of unit delays, here: \( x, \Delta^{-1}x, \ldots, \Delta^{-(n-1)}x \).

- If \( b_0 = b_2 \neq 0 \), there is direct coupling of input and output, \( H \) is proper but not strictly so, \( h = Z^{-1}H \) has a unit-pulse component \( b_2 \delta \).
• Note that this \( n = 2 \) example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for \( n = 2 \), \( H(z) = \frac{P(z)}{Q(z)} \) where

\[
Q(z) = z^2 + a_1 z + a_0 = (z - \alpha)^2 + \beta^2
\]

for \( \alpha, \beta \in \mathbb{R} \), so the poles are \( \alpha \pm j\beta \).
Canonical system realizations by PFE - example

\[ H(z) = \frac{.3z^2 - .1}{z^2 - 0.1z - .3} = .3 + \frac{.3z - .01}{(z - 0.6)(z + 0.5)} = .3 + \frac{17/1.1}{z - 0.6} + \frac{16/1.1}{z + 0.5} \]

Note that one cannot factor \( z \) from the numerator of \( H \).

**Exercise:** Find a realization for this transfer function \( H \) by

1. splitting/forking the input signal \( F \),
2. using the direct canonical form for each of these 3 terms of \( H \) found by long division and PFE, and
3. summing three resulting output signals to get the (ZS) output \( Y = HF \).

Digital Proportional-Integral (PI) system

- Consider a continuous-time signal \( x \) sampled every \( T \) seconds,
  \[ \forall k \in \mathbb{Z}^+, \ x[k] = x(kT), \]
  and its integral \( y(t) = \int_0^t x(\tau)d\tau \).
- The sampled integral can be approximated, \( y(kT) \approx y[k] \), by the trapezoid rule,
  \[ y[k] = y[k-1] + \frac{x[k-1] + x[k]}{2}T. \]
- In the complex-frequency domain,
  \[ Y(z) = Y(z)z^{-1} + \frac{X(z)z^{-1} + X(z)}{2}T \]
  \[ \Rightarrow \frac{Y(z)}{X(z)} = T \cdot \frac{1 + z^{-1}}{2 \cdot 1 - z^{-1}}. \]
Digital PI system (cont)

- So, a digital PI transfer function would be of the form,
  \[ G(z) = K_p + \frac{K_i T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}. \]
  for constants \( K_p, K_i \).

- In practice, PID or PI systems \( G \) are commonly used to control a plant \( H \), where \( G \) may be in series with \( H \) or in the feedback branch.

Exercises:
- Draw the direct-form canonical realization for \( G \).
- Draw the block diagram for the closed-loop system with negative feedback: \( Y = HX \) and \( X = F - GY \) where \( H \) is the (open-loop) system.
- Find the closed-loop transfer function \( Y/F \) and recall the pole placement problem to stabilize \( H \).

Recursive Least Squares (RLS) Filter - Introduction

- Consider a LTI system with input \( f \) and output \( y \),
  \[ y[k] = \sum_{r=0}^{K} h[k - r] f[r] + v[k], \quad k \in \mathbb{Z}, \]
  where \( v \) is an additive noise process and \( K \) is the maximum system order.

- The system (unit-pulse response) \( h \) is not known.

- Past values of the output \( y \) are observed (known).

- At time \( k \), the objective is to forecast the next output \( \hat{y}[k + 1] \), based on the assumed known/observed quantities:
  - the next input \( f[k + 1] \),
  - the past \( R \) input-output pairs \( \{f[r], y[r]\}_{k-R+1 \leq r \leq k} \).
**RLS objective and \( R \)-th-order linear tap filter**

- The output of an \( R \)-th-order RLS tap-filter at time \( k \) is,

\[
\hat{y}[i] = \sum_{r=i-R+1}^{i} \eta_k[i-r]f[r], \quad i \leq k + 1.
\]

- The objective of this filter at time \( k \) is to accurately estimate the system output \( y[k+1] \) with \( \hat{y}[k+1] \) by choosing the \( R \) filter coefficients \( \eta_k[k-R+1], \ldots, \eta_k[k-1], \eta_k[k] \) that minimize the following sum-of-square-error objective:

\[
\mathcal{E}_k = \sum_{r=k-R+1}^{k} \lambda^{k-r}|y[r] - \hat{y}[r]|^2 = \sum_{r=k-R+1}^{k} \lambda^{k-r}|e_k[r]|^2
\]

where
- \( \lambda > 0 \) is a forgetting factor and
- error \( e_k[r] := y[r] - \hat{y}[r] \).

**Exercise:** Prove the last equality.

---

**RLS filter**

- So, to minimize \( \mathcal{E}_k \), substitute \( \hat{y}_k[r] \) into \( \mathcal{E}_k \) and solve

\[
0 = \frac{\partial \mathcal{E}_k}{\partial \eta_k[i]} \quad \text{for} \quad i \in \{k - R + 1, \ldots, k - 1, k\}.
\]

- That is, \( R \) equations in \( R \) unknowns: for \( i \in \{k - R + 1, \ldots, k - 1, k\} \),

\[
0 = \sum_{r=k-R+1}^{k} 2\lambda^{k-r}e_k[r] \frac{\partial e_k[r]}{\partial \eta_k[i]}
\]

\[
= \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(y[r] - \hat{y}_k[r]) \left( -\frac{\partial \hat{y}_k[r]}{\partial \eta_k[i]} \right)
\]

\[
= \sum_{r=k-R+1}^{k} 2\lambda^{k-r}(\hat{y}_k[r] - y[r])f[r - i]
\]

- **Exercise:** Prove the last equality.
Substituting \( \hat{y}_k[r] \), rewrite these equations to get the following \( R \) equations in \( R \) unknowns \( \eta_k[i] \) that are \( E_k \)-minimizing: for \( i \in \{ k - R + 1, \ldots, k - 1, k \} \),

\[
\sum_{r=k-R+1}^{k} \lambda^{k-r} f[r-i] \sum_{\ell=r-R+1}^{r} f[\ell] \eta_k[r-\ell] = \sum_{r=k-R+1}^{k} \lambda^{k-r} y[r] f[r-i]
\]

**Exercise:** Prove the last equality and write it in matrix form.

**Exercise:** Research how the \( E_k \)-minimizing filter parameters \( \eta_k \) can be computed recursively, i.e., using \( \eta_{k-1} \).

The filter order \( R \) can also be "trial adapted" to discover the system order \( K \) so that the error-minimizing filter parameters \( \eta_k \) "track" the system unit-pulse response \( h \) over time \( k \).

Note the required initial “warm-up” period of \( R \) time-units where the outputs of system \( h \) are simply observed and recorded and no estimates are made.

**Exercise:** If there was no additive noise process \( v \) and the system unit-pulse response \( h \) had finite support (i.e., a FIR system with \( K < \infty \)), show how \( h \) can be deduced from input-output \((f, y)\) observations.