Signals and Linear and Time-Invariant Systems in Discrete Time

- Properties of signals and systems (difference equations)
- Time-domain analysis
  - ZIR, system characteristic values and modes
  - ZSR, unit-pulse response and convolution
  - stability, eigenresponse and transfer function
- Frequency-domain analysis

Time-domain analysis of discrete-time LTI systems

- Discrete-time signals
- Difference equation single-input, single-output systems in discrete time
- The zero-input response (ZIR): characteristic values and modes
- The zero (initial) state response (ZSR): the unit-pulse response, convolution
- System stability
- The eigenresponse and (zero state) system transfer function
Consider a continuous-time signal $x : \mathbb{R} \to \mathbb{R}$ sampled every $T > 0$ seconds

$$x(kT + t_0) =: x[k] \text{ for } k \in \mathbb{Z},$$

where

- $t_0$ is the sampling time of the 0th sample, and
- $T$ is assumed less than the Nyquist sampling period of $x$, and
- $x[k]$ (with square brackets) is the $k$th sample itself.

Here $x[\cdot]$ is a discrete-time signal defined on $\mathbb{Z}$.

Example of sampling with $t_0 = 0$ and positive signal $x$
Introduction to signals and systems in discrete time

- A discrete-time function (or signal) $x: A \rightarrow B$ is one with countable (time) domain $A$.
- We will take the range $B = \mathbb{R}$ or $B = \mathbb{C}$.
- Typically, we will herein take domain $A = \mathbb{Z}$ or $\mathbb{Z}^{2-n}$ for some (finite) integer $n \geq 0$.
- Some properties of signals are as in continuous time: e.g., periodic, causal, bounded, even or odd.
- Similarly, some signal operations are as in continuous time: e.g., spatial shift/scale, superposition, time reflection, and (integer valued) time shift.

Time scaling: decimation and interpolation

- Time scaling can be implemented in continuous time prior to sampling at a fixed rate, or the sampling rate itself could be varied (again recall the Nyquist sampling rate).
- In discrete time, a signal $x = \{x[k] \mid k \in \mathbb{Z}\}$ can be decimated (subsampled) by an integer factor $L \neq 0$ to create the signal $x_L$ defined by
  $$x_L[k] = x[kL], \ \forall k \in \mathbb{Z},$$
i.e., $x_L$ is defined only by every $L^{th}$ sample of $x$.
- A discrete-time signal $x$ can also be interpolated by an integer factor $L > 0$ to create $x_L$ satisfying
  $$x_L[kL] = x[k], \ \forall k \in \mathbb{Z}.$$  
- For an interpolated signal $x_L$, the values of $x_L[r]$ for $r$ not a multiple of $L$ (i.e., $\forall k \in \mathbb{Z}$ s.t. $r \neq kL$) can be set in different ways, e.g., between consecutive samples:
  - (piecewise constant) hold: $x_L[r] = x_L[L \lfloor r/L \rfloor] = x(L \lfloor r/L \rfloor)$
  - linear interpolation:
    $$x_L[r] = x(L \lfloor r/L \rfloor) + \frac{r - L \lfloor r/L \rfloor}{L} (x(L \lfloor r/L \rfloor + 1) - x(L \lfloor r/L \rfloor))$$
Time scaling: decimation and interpolation - Questions

- Is the functional mapping $x \rightarrow x_L$ causal for linear interpolation?
- Is the hold causal?
- **Exercise**: Show that if a periodic, continuous-time signal $x(t)$, with period $T_0$, is periodically sampled every $T$ seconds, then the resulting discrete-time signal $x[k]$ is periodic if and only if $T/T_0$ is rational.

Unit pulse $\delta$, unit step $u$, unit delay $\Delta$, and convolution *

- Some important signals in discrete time are as those in continuous time, e.g., polynomials, exponentials, unit step.
- In discrete time, rather than the (unit) impulse, there is unit pulse (Kronecker delta):
  \[ \delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{else} \end{cases} \]
- Any discrete-time signal $x$ can thus be written as
  \[ x[k] = \sum_{r=-\infty}^{\infty} x[r] \delta[k-r] = \sum_{r=-\infty}^{\infty} x[k-r] \delta[r] \]
  \[ = \left( x \ast \delta \right)[k] \]
- or just $x = x \ast \delta$, i.e., the unit pulse $\delta$ is the identity of discrete-time convolution.
- Define the operator $\Delta$ as unit delay (time-shift), i.e., $\forall$ signals $y$ and $\forall k, r \in \mathbb{Z}$,
  \[ (\Delta y)[k] := y[k-r]. \]
- The discrete-time unit step $u$ satisfies $\delta = u - \Delta u$, equivalently: $\forall k \in \mathbb{Z},$
  \[ \delta[k] = u[k] - u[k-1] \quad \text{and} \quad u[k] = \sum_{r=0}^{\infty} (\Delta \delta)[k] = \sum_{r=0}^{\infty} \delta[k-r]. \]
Unit pulse and unit step functions

- Exercise: For any signal causal \( f \left( \{ f[k], k \geq 0 \} \right) \), show that

\[
\forall k \geq 0, \quad (f \star u)[k] = \sum_{r=0}^{k} f[r].
\]

Exponential signals in discrete time

- Real-valued exponential (geometric) signals have the form \( x[k] = A^k, k \in \mathbb{Z} \), where \( A, \gamma \in \mathbb{R} \).

- Consider the scalar \( z = e^{j\Omega} \in \mathbb{C} \) with \( \gamma > 0, \Omega \in \mathbb{R} \), where again \( j := \sqrt{-1} \).

- Generally, complex-valued exponential signals have the (polar) form

\[
x[k] = Ae^{j\phi} z^k = A\gamma^k e^{j(\Omega k + \phi)}, \quad k \in \mathbb{Z},
\]

where w.l.o.g. we can take

\[-\pi < \Omega, \phi \leq \pi \quad \text{and real } \quad A > 0.\]

- Exercise: Show this complex-valued exponential is periodic if and only if \( \Omega/\pi \) is rational.

- By the Euler-De Moivre identity,

\[
x[k] = A\gamma^k e^{j(\Omega k + \phi)} = A\gamma^k \cos(\Omega k + \phi) + jA\gamma^k \sin(\Omega k + \phi), \quad k \in \mathbb{Z}.
\]
Systems - single input, single output (SISO)

\[ f \rightarrow x_1, x_2, \ldots, x_n \rightarrow y \]

- In the figure, \( f \) is an input signal that is being transformed into an output signal, \( y \), by the depicted system (box).

- To emphasize this functional transformation, and clarify system properties, we will write the output signal (i.e., system "response" to the input \( f \)) as

\[ y = Sf, \]

where, again, we are making a statement about functional equivalence:

\[ \forall k \in \mathbb{Z}, \ y[k] = (Sf)[k]. \]

- Again, \( Sf \) is not \( S \) “multiplied by” \( f \), rather a functional transformation of \( f \).

SISO systems (cont)

\[ f \rightarrow x_1, x_2, \ldots, x_n \rightarrow y \]

- The \( n \) signals \( \{x_1, x_2, \ldots, x_n\} \) are the internal states of the system.

- The states can be taken as outputs of unit-delay operators, \( \Delta \), i.e.,

\[ \forall k \in \mathbb{Z}, \ (\Delta y)[k] = y[k - 1]. \]

- Some properties of systems are as in continuous time: e.g., linear, time invariant, causal, memoryless, stable (with different conditions for stability as we shall see).
Difference equation for an discrete time, LTI, SISO system

- For linear and time-invariant systems in discrete time, relate output $y$ to input $f$ via difference equation in standard (time-advance operator) form:

\[
\forall k \geq -n, \quad y[k + n] + a_{n-1}y[k + n - 1] + \ldots + a_1 y[k + 1] + a_0 y[k] = b_m f[k + m] + b_{m-1} f[k + m - 1] + \ldots + b_1 f[k + 1] + b_0 f[k],
\]

given
- scalars $a_k$ for $0 \leq k \leq n$, with $a_n := 1$, and scalars $b_k$ for $0 \leq k \leq m$,
- $a_0 \neq 0$ or $b_0 \neq 0$ (so that $P, Q$ are of minimal degree), and
- initial conditions $y[-n], y[-n + 1], \ldots, y[-2], y[-1]$.

- Compact representation of the above difference equation:

\[
Q(\Delta^{-1})y = P(\Delta^{-1})f,
\]

where polynomials

\[
Q(z) = z^n + \sum_{k=0}^{n-1} a_k z^k, \quad P(z) = \sum_{k=0}^{m} b_k z^k,
\]

$\Delta^{-1}$ is the unit time-advance operator: $(\Delta^{-1} y)[k] \equiv y[k + 1], (\Delta^{-r} y)[k] \equiv y[k + r]$

Discussion: conditions for causality and difference equation in $\Delta$

- **Exercise:** Show that the difference equation $Q(\Delta^{-1})y = P(\Delta^{-1})f$ is not causal if $\deg(P) = m > n = \deg(Q)$, i.e., the system is not proper.

- A not anti-causal difference equation can be implemented simply using memory to store a sliding window of prior values of the input $f$ and delaying the output.

- **Example:** Decoding B (bidirectional) frames of MPEG video.
Given the system \( Q(\Delta^{-1})y = P(\Delta^{-1})f \), the input \( f[k] \) for \( k \geq 0 \), and initial conditions \( y[-n], ..., y[-1] \),

one can recursively solve for \( y \) \( (y[k] \text{ for } k \geq 0) \) by rewriting the system equation as

\[
y[k + n] = -\sum_{r=0}^{n-1} a_r y[k + r] + \sum_{r=0}^{m} b_r f[k + r] \quad \text{for } k \geq -n
\]

\[
\Rightarrow y[k] = -\sum_{r=0}^{n-1} a_r y[k + r - n] + \sum_{r=0}^{m} b_r f[k + r - n] \quad \text{for } k \geq 0.
\]

For example, the difference equation in standard form,

\[
y[k + 1] + 3y[k] = 7f[k + 1] \quad \text{for } k \geq -1,
\]

can be rewritten as

\[
y[k] = -3y[k - 1] + 7f[k] \quad \text{for } k \geq 0.
\]

So, given \( f \) and \( y[-1] \) we can recursively compute

\[
y[0] = -3y[-1] + 7f[0], \quad y[1] = -3y[0] + 7f[1], \quad y[2] = -3y[1] + 7f[2], \quad \text{etc.}
\]

Exercise: If \( f = u \) and \( y[-1] = 7 \) then find \( y[3] \) for this example.

Approach to closed-form solution: ZIR and ZSR

- The total response \( y \) of \( P(\Delta^{-1})f = Q(\Delta^{-1})y \) to the given initial conditions and input \( f \) is a sum of two parts:
  - the ZSR, \( y_{ZS} \), which solves
    \[
P(\Delta^{-1})f = Q(\Delta^{-1})y_{ZS} \quad \text{with zero i.c.'s, i.e., with } 0 = y[-n] = ... = y[-1];
    \]
  - the ZIR, \( y_{ZI} \), which solves
    \[
    0 = Q(\Delta^{-1})y_{ZI} \quad \text{with the given initial conditions.}
    \]
- The total response \( y \) of the system to \( f \) and the given initial conditions is, by linearity,

\[
y = y_{ZI} + y_{ZS}.
\]
- We will determine the ZIR by finding the characteristic modes of the system.
- We will determine the ZSR by convolution of the input with the (zero state) unit-pulse response, the latter also in terms of characteristic modes.
Consider again the difference equation:
\[ \forall k \geq -1, \quad y[k+1] + 3y[k] = 7f[k+1], \]

**i.e.,** \( Q(z) = z + 3 \) with degree \( n = 1 \), and \( P(z) = 7z \) with degree \( m = 1 \).

**Exercise:** Show that the following system corresponds to this difference equation.

![Diagram](image)

By recursive substitution, the total response is, \( \forall k \geq -1 \):
\[
\begin{align*}
y[k] &= -3y[k-1] + 7f[k] \\
    &= -3(-3y[k-2] + 7f[k-1]) + 7f[k] \\
    &= (-3)^2y[k-2] - 3 \cdot 7f[k-1] + 7f[k] \\
    &= \ldots \\
    &= (-3)^{k+1}y[-1] + \sum_{r=0}^{k} (-3)^{k-r}f[r] \\
    &=: (-3)^{k+1}y[-1] + \sum_{r=0}^{\infty} h[k-r]f[r] =: (-3)^{k+1}u[-1] + (h \ast f)[k],
\end{align*}
\]

- where \( h[k] := 7(-3)^ku[k] \) is the (zero state) unit-pulse response,
- \( y[-1] \) is the given \( (n = 1) \) initial condition, and
- we have defined the discrete-time convolution operator with \( \sum_{r=0}^{-1}(...) := 0 \).
Exercise: Prove by induction this expression for $y[k]$ for all $k \geq -1$.

Exercise: Prove convolution is commutative: $h * f = f * h$.

So, we can write the total response $y = y_{ZI} + y_{ZS}$ starting from the time of oldest initial condition:

$$\forall k \geq -1, \quad y_{ZI}[k] = (-3)^{k+1}y[-1]$$

$$\forall k \geq -1, \quad y_{ZS}[k] = u[k] \sum_{r=0}^{k} 7(-3)^{k-r}f[r] = u[k](h * f)[k]$$

where $y_{ZS}[k] = 0$ when $k < 0$.

Obviously, this example involves a linear, time-invariant and causal system as described by the difference equation above.

Note that in CMPSC 360, we don’t restrict our attention to linear and time-invariant difference equations.

We use recursive substitution to guess at the form of the solution and then verify our guess by an inductive proof.

In this course, we will describe a systematic approach to solve any LTIC difference equation, i.e., to solve for the output of a DT-LTIC system given the input and initial conditions.

And again as in continuous time, we will see important insights about discrete-time signals and LTIC systems through frequency-domain representations and analysis.
ZIR - the characteristic values

- Note that \( \forall k, \Delta^{-r}z^k = z^{k+r} = z^r z^k \), i.e., the \( r \)-units time-advance operator, \( \Delta^{-r} \), is replaced by the scalar \( z^r \) for all \( r \in \mathbb{Z} \).

- Our objective is to solve for the ZIR, i.e., solve \( Q(\Delta^{-1})y \equiv 0 \) given \( y[-n], y[-n+1], ..., y[-2], y[-1] \).

- Note that exponential (or "geometric") functions, \( \{ z^k \mid k \in \mathbb{Z} \} \) for \( z \in \mathbb{C} \), are eigenfunctions of time-shift operators of the form \( Q(\Delta^{-1}) \) for a polynomial \( Q \).

- That is, for any non-zero scalar \( z \in \mathbb{C} \), if we substitute \( y[k] = z^k \forall k \in \mathbb{Z} \) we get:
  \[
  \forall k \in \mathbb{Z}, \quad (Q(\Delta^{-1})y)[k] = Q(\Delta^{-1})z^k = Q(z)z^k.
  \]

- So, to solve \( Q(z)z^k \equiv 0 \) for all time \( k \geq 0 \), when \( z \neq 0 \) we require \( Q(z) = 0 \), the characteristic equation of the system.

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ZIR - the characteristic values (cont)

- If \( z \) is a root of the characteristic polynomial \( Q \) of the system, then
  - \( z \) would be a characteristic value of the system, and
  - the signal \( \{ z^k \}_{k \geq 0} \) is a characteristic mode of the system when \( z \neq 0 \), i.e., \( Q(\Delta^{-1})z^k = 0 \) \( \forall k \geq 0 \).

- Since \( Q \) has degree \( n \), there are \( n \) roots of \( Q \) in \( \mathbb{C} \), each a system characteristic value.
• Let \( n' \leq n \) be the number of non-zero roots of \( Q \), i.e., \( \tilde{Q}(z) = Q(z)/z^{n-n'} \) is a polynomial satisfying \( \tilde{Q}(0) \neq 0 \).

• Though there may be some repeated roots of the characteristic polynomial \( Q \), there will always be \( n' \) different, linearly independent characteristic modes, \( \mu_k \), i.e.,

\[
\forall k \geq -n, \quad \sum_{r=1}^{n'} c_r \mu_r[k] = 0 \quad \Leftrightarrow \quad \forall r, \text{ scalars } c_r = 0.
\]

• When \( n = n' \), by system linearity, we will be able to write

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{r=1}^{n} c_r \mu_r[k],
\]

for scalars \( c_r \in \mathbb{C} \) that are found by considering the given initial conditions

\[
y[k] = \sum_{r=1}^{n} c_r \mu_r[k] \quad \text{for } k \in \{-n, ..., -2, -1\},
\]

i.e., \( n \) equations in \( n \) unknowns (\( c_r \)).

• The linear independence of the modes implies linear independence of these \( n \) equations in \( c_r \), and so they have a unique solution.

ZIR - the case of different, non-zero, real characteristic values

• If there are \( n \) different non-zero roots of \( Q \) in \( \mathbb{R} \), \( z_1, z_2, ..., z_n \), then there are \( n \) characteristic modes: for \( r \in \{1, 2, ...n\} \),

\[
\forall \text{ time } k, \quad \mu_r[k] = z_r^k.
\]

• Therefore,

\[
\forall k \geq -n, \quad y_{ZI}[k] = \sum_{r=1}^{n} c_r z_r^k.
\]

• The \( n \) unknown scalars \( c_r \in \mathbb{R} \) can be solved using the \( n \) equations:

\[
y[k] = \sum_{r=1}^{n} c_r z_r^k, \quad \text{for } k \in \{-n, -n+1, ..., -2, -1\}.
\]
Example: Consider the difference equation:
\[
\forall k \geq -3, \quad 2y[k+3] - 10y[k+2] + 12y[k+1] = 3f[k+2],
\]
with \(y[-2] = 1\) and \(y[-1] = 3\).

That is, \(Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2)\) and \(n = 2, P(z) = (3/2)z\) and \(m = 1\).

So, the \(n = 2\) characteristic values are \(z = 3, 2\) and the ZIR is
\[
\forall k \geq -n = -2, \quad y_{ZI}[k] = c_13^k + c_22^k
\]

Using the initial conditions to find the scalars \(c_1, c_2\):
\[
1 = y[-2] = c_13^{-2} + c_22^{-2} \quad \text{and} \quad 3 = y[-1] = c_13^{-1} + c_22^{-1}.
\]

Exercise: Now solve for \(c_1\) and \(c_2\).

Note: When a coefficient \(c\) is worked out to be zero, it may not be exactly zero in practice, and the corresponding characteristic mode \(z^k\) will increasingly contribute to ZIR \(y_{ZI}\) over time if \(|z| > 1\) (i.e., an “unstable” mode in discrete time).

ZIR - the case of not-real characteristic values

The characteristic polynomial \(Q\) may have non-real roots, but such roots come in complex-conjugate pairs because \(Q\)'s coefficients \(a_k\) are all real.

For example, if the characteristic polynomial is
\[
Q(z) = (z - 1)(z^2 - 2z - 2)
\]
then the characteristic values \(Q\)'s roots) are
\[-1, \ 1 \pm j\sqrt{3}\] again recalling \(j = \sqrt{-1}\).

Because we have three different characteristic values \(\in \mathbb{C}\), we can specify three corresponding characteristic modes,
\[
(-1)^k, \ (1 + j\sqrt{3})^k, \ (1 - j\sqrt{3})^k, \ \forall k \geq 0,
\]
and construct the ZIR as
\[
\forall k \geq -n = -3, \quad y_{ZI}[k] = c_1(-1)^k + c_2(1 + j\sqrt{3})^k + c_3(1 - j\sqrt{3})^k
\]
where
- \(c_1 \in \mathbb{R}\) and \(c_2 = \overline{c_2} \in \mathbb{C}\) so that \(y_{ZI}\) is real-valued, and again,
- these scalars are determined by the \(n = 3\) given (real) initial conditions: \(y[-3], y[-2], y[-1]\).
• By the Euler-De Moivre identity for the previous example,
\[ y_{ZI}[k] = c_1(-1)^k + (c_2 + c_3)2^k \cos(k\pi/3) + j(c_2 - c_3)2^k \sin(k\pi/3) = c_1(-1)^k + 2\text{Re}\{c_2\}2^k \cos(k\pi/3) - 2\text{Im}\{c_2\}2^k \sin(k\pi/3) \]

• Again, because all initial conditions are real and \( Q \) has real coefficients, \( y_{ZI} \) is real valued and so \( c_3 = \overline{c_2} \Rightarrow c_2 + c_3, j(c_2 - c_3) \in \mathbb{R} \).

• In general, consider two complex conjugate characteristic values \( v \pm jq \) corresponding to two complex-valued characteristic modes \( |z|^k e^{\pm jk\angle z} \), where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).

• One can use Euler’s identity to show that the corresponding real-valued characteristic modes are
\[ |z|^k \cos(k\angle z), |z|^k \sin(k\angle z) \]

ZIR - the case of repeated characteristic values

• Consider the case where at least one characteristic value is of order > 1, i.e., there are repeated roots of the characteristic polynomial, \( Q \).

• For example, \( Q(z) = (z + 0.75)^3(z - 0.5) \) has a triple (twice repeated) root at -0.75 and a single root at 0.5.

• Again, \( \{(-0.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})(-0.75)^k \equiv 0 \) follows from
\[ (\Delta^{-1} + .75)(-0.75)^k = \Delta^{-1}(-.75)^k + .75(-.75)^k = (-.75)^{k+1} + .75(-.75)^k = 0. \]

• Similarly, \( (0.5)^k \) is a characteristic mode since \( (\Delta^{-1} - 0.5)(0.5)^k \equiv 0 \).

• Also, \( \{k(-.75)^k\} \) is a characteristic mode because \( Q(\Delta^{-1})k(-.75)^k \equiv 0 \) follows from
\[ (\Delta^{-1} + .75)^2k(-.75)^k = (\Delta^{-2} + 1.5\Delta^{-1} + (.75)^2)k(-.75)^k = \Delta^{-2}k(-.75)^k + 1.5\Delta^{-1}k(-.75)^k + (.75)^2k(-.75)^k = (k + 2)(-.75)^{k+2} + 1.5(k + 1)(-.75)^{k+1} + (.75)^2k(-.75)^k = (-.75)^{k+2}((k + 2) - 2(k + 1) + k) = 0. \]

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• Similarly, \( k^2(-.75)^k \) is also a characteristic mode because
\[
(\Delta^{-1} + .75)^3 k^2(-.75)^k = 0.
\]

• Note that without three such linearly independent characteristic modes
\[
\{(−.75)^k, k(−.75)^k, k^2(−.75)^k; k \geq 0\}
\]
for the twice-repeated (triple) characteristic value -.75, the initial conditions will create an “overspecified” set of \( n \) equations involving fewer than \( n \) “unknown” coefficients \( (c_k) \) of the linear combination of modes forming the ZIR.

• For this example,
\[
yZI[k] = c_0(-0.75)^k + c_1 k(-0.75)^k + c_2 k^2(-0.75)^k + c_3 (0.5)^k, \quad k \geq -4.
\]

• If the given initial conditions are, say,
\[
y[-4] = 12, \quad y[-3] = 6, \quad y[-2] = -5, \quad y[-1] = 10,
\]
the four equations to solve for the four unknown coefficients \( c_k \) are:
\[
\begin{align*}
yZI[-4] &= (-.75)^4 c_0 + (-4)(-.75)^4 c_1 + (-4)^2(-.75)^4 c_2 + (.5)^4 c_3 = 12 \\
yZI[-3] &= (-.75)^3 c_0 + (-3)(-.75)^3 c_1 + (-3)^2(-.75)^3 c_2 + (.5)^3 c_3 = 6 \\
yZI[-2] &= (-.75)^2 c_0 + (-2)(-.75)^2 c_1 + (-2)^2(-.75)^2 c_2 + (.5)^2 c_3 = -5 \\
yZI[-1] &= (-.75)^1 c_0 + (-1)(-.75)^1 c_1 + (-1)^2(-.75)^1 c_2 + (.5)^1 c_3 = 10
\end{align*}
\]

ZIR - the case of repeated, non-zero characteristic values

• In general, a set of \( r \) linearly independent modes corresponding to a non-zero characteristic value \( z \in \mathbb{C} \) repeated \( r - 1 \) times are
\[
k^{r-1}z^k, \quad k^{r-2}z^k, \quad \ldots, \quad kz^k, \quad z^k, \quad \text{for} \quad k \geq 0.
\]

• Also, if \( v \pm jq \) are characteristic values repeated \( r - 1 \) times, with \( v, q \in \mathbb{R} \) and \( q \neq 0 \), we can use the \( 2k \) real-valued modes
\[
k^{a|z|^k} \cos(k\angle z), \quad k^{a|z|^k} \sin(k\angle z), \quad \text{for} \quad a \in \{0, 1, 2, \ldots, r - 1\},
\]
where \( |z| = \sqrt{v^2 + q^2} \) and \( \angle z = \arctan(q/v) \).
ZIR - when some characteristic values are zero

- Again let \( n' \leq n \) be the number of non-zero roots of \( Q \) (characteristic values),
- \( i.e., r := n - n' \geq 0 \) is the order (1+repetition) of the characteristic value 0, and
- \( r \geq 0 \) is the smallest index such that (the coefficient of \( Q \)) \( a_r \neq 0 \).
- So, there is a polynomial \( \tilde{Q} \) such that \( Q(z) = z^r \tilde{Q}(z) \) and \( \tilde{Q}(0) \neq 0 \).
- Because the constant signal zero cannot be a characteristic mode, we add \( r = n - n' \) time-advanced unit-pulses:
  \[
  \forall k \geq -n, \ y_{ZI}[k] = \sum_{i=1}^{r} C_i \delta[k + i] + y_n[k] = C_r \delta[k + r] + C_{r-1} \delta[k + r - 1] + \ldots + C_1 \delta[k + 1] + y_n[k]
  \]
  where \( y_n \) is a “natural response” (linear combination of \( n' \) characteristic modes).
- The \( n \) initial conditions are then met by the \( r \) coefficients \( C_i \) of the advanced unit pulses together with the \( n' = n - r \) coefficients of the characteristic modes in \( y_n \).

ZIR - when some characteristic values are zero - example

- Consider a fourth-order system with characteristic polynomial
  \( Q(z) = z^2(z + 1)^2 \).
- Thus the poles are 0, -1 each repeated and the (non-zero) characteristic modes are \((-1)^k, k(-1)^k\).
- So, the ZIR is, for \( k \geq -4 \):
  \[
  y_{ZI}[k] = C_2 \delta[k + 2] + C_1 \delta[k + 1] + c_1 (-1)^k + c_2 k(-1)^k
  \]
- That is, the ZIR has four unknown coefficients \( C_2, C_1, c_1, c_2 \) to account for the four (given) initial conditions \( y[-4], y[-3], y[-2], y[-1] \).
Zero State Response - the unit-pulse response

- Recall the LTIC system

  \[ \sum_{r=0}^{n} a_r \Delta^{-r} y =: Q(\Delta^{-1})y = P(\Delta^{-1})f := \sum_{r=0}^{m} b_r \Delta^{-r} f \]

  with \( a_n = 1, \ a_0 \neq 0 \) or \( b_0 \neq 0, \ m \leq n \).

- We can express any input signal

  \[ f[k] = \sum_{r=0}^{\infty} f[r] \delta[k - r] \ \forall k \geq 0, \ i.e., \ \forall f, \ f = f * \delta. \]

- So the unit pulse \( \delta \) is the identity of the convolution operator in discrete time.

- Thus, by LTI, the ZSR \( y_{ZS} \) is the convolution of input \( f \) and ZSR \( h \) to unit pulse \( \delta \),

  \[ y_{ZS}[k] = \sum_{r=0}^{\infty} f[r] h[k - r] = (f * h)[k], \ \forall k \geq 0, \]

- \( h \) is called the **unit-pulse response** of the LTIC system, i.e.,

  \[ Q(\Delta^{-1})h = P(\Delta^{-1})\delta \ \text{s.t.} \ h[k] = 0 \ \forall k < 0. \]

### Computing an LTIC system’s unit-pulse response, \( h \)

- For the LTIC system in standard form, if \( a_0 \neq 0 \) then

  \[ h = (b_0 / a_0) \delta + y_N u \]

  where \( y_N \) is a natural response of the system (linear combination of characteristic modes).

- Note that \( h[k] = 0 \) for all \( k < 0 \) owing to the unit step \( u \).

- The \( n \) scalars of the natural response \( y_N \) component of \( h \) are solved using

  \[ (Q(\Delta^{-1})h)[k] = (P(\Delta^{-1})\delta)[k] \ \text{for} \ k \in \{-n, -n + 1, ..., -2, -1\} \]
Unit-pulse response when zero is a characteristic value

- If \( r \geq 0 \) is the smallest index such that \( a_r \neq 0 \) (0 is a char. mode of order \( r \)), then may need to add \( r \) delayed unit-pulse terms to \( h \):

\[
h = \sum_{i=0}^{r-1} A_i \Delta^i \delta + \left( b_0 / a_r \right) \Delta^r \delta + y_0 u,
\]

where

- by definition of the standard form of the difference equation, if \( r > 0 \), \( a_0 = 0 \) so \( b_0 \neq 0 \), and
- \( r \leq n \) since \( 0 \neq a_n := 1 \).

- So if \( r = 0 \) (i.e., \( a_0 \neq 0 \)), then \( A_0 = b_0 / a_0 \) as above, where \( \sum_{i=0}^{r-1}(...) := 0 \).

- **Exercise:** Prove \( A_r = b_0 / a_r \) for \( 0 \leq r \leq n \).

- Thus, zero is a characteristic value of degree \( r \geq 0 \), and

- there are \( r \) characteristic modes that will all be zero.

- The additional unit-pulse terms introduce \( r \) degrees of freedom in the form of the coefficients \( A_0, A_1, ..., A_{r-1} \) to accommodate the \( n = r + n' \) initial conditions of the unit-pulse response: \( h[-n] = h[-n+1] = ... = h[-2] = h[-1] = 0 \).

Computing the ZSR - example 1

- Recall that the difference equation \( y = 7f - 3\Delta y \) corresponds to the above system; in standard form:

\[
\forall k \geq -1, \quad y[k+1] + 3y[k] = 7f[k+1].
\]

with \( Q(z) = z + 3 \), \( P(z) = 7z \) and \( n = 1 = m \).

- Since the system characteristic value is \(-3\) and \( b_0 = 0 \), the (zero state) unit-pulse response has the form \( h[k] = c(-3)^k u[k] \).

- The scalar \( c \) is solved by evaluating the above difference equation at time \( k = -1 \):

\[
(Q(\Delta^{-1})h)[-1] = (P(\Delta^{-1})\delta)[-1]
\]

\[
i.e., \quad h[0] + 3h[-1] = 7\delta[0]
\]

\[
\Rightarrow c + 3 \cdot 0 = 7 \cdot 1, \quad c = 7
\]
Computing the ZSR - example 1 (cont)

• So, \( h[k] = 7(-3)^k u[k] \).

• If the input is \( f[k] = 4(0.5)^k u[k] \), the system ZSR is, for all \( k \geq 0 \),

\[
\begin{align*}
y_zs[k] &= \sum_{r=0}^{k} h[r] f[k-r] \\
&= \sum_{r=0}^{k} 7(-3)^r 4(0.5)^{k-r} \\
&= 28(0.5)^k \sum_{r=0}^{k} (-6)^r \\
&= 28(0.5)^k \frac{(-6)^{k+1} - 1}{-6 - 1} u[k] \\
&= (24(-3)^k + 4(0.5)^k) u[k].
\end{align*}
\]

• Note how the ZIR \( y_{z1} \) has a term that is a characteristic mode (excited by the input \( f \)) and a term that is proportional to the input \( f \) (this forced response is an eigenresponse).

• Exercise: For the difference equation, \( y[k+1] + 3y[k] = 7f[k] \ \forall k \geq -1 \): draw the block diagram, show that \( h[k] = 21(-3)^{k-1} u[k] + (7/3)\delta[k] \), and find the ZSR to the above input \( f \).

• Exercise: Read “sliding tape” method to compute convolution in Lathi, p. 595.

Computing the unit pulse response - example 2

• Find the ZSR of the following system to input \( f[k] = 2(-5)^k u[k] \):

\[
\begin{align*}
1.5 & \quad \Delta \\
+ & \quad \Delta \\
3 & \quad -6 \\
& \quad f \\
& \quad y
\end{align*}
\]

• Exercise: show the difference equation for this system (in direct canonical form) is:

\[
\forall k \geq 0, \quad y[k+2] - 5y[k+1] + 6y[k] = 1.5f[k+1]
\]

• That is, \( Q(z) = z^2 - 5z + 6 = (z - 3)(z - 2) \) and \( n = 2, \ P(z) = 1.5z \) and \( m = 1 \).

• So, the \( n = 2 \) characteristic values are \( z = 3, 2 \) and \( b_0 = 0 \) so the unit-pulse response

\[ h[k] = (c_1 3^k + c_2 2^k) u[k]. \]
• To find the constants, evaluate the difference equation at \( k = -1 \):

\[
2h[1] - 10h[0] + 12h[-1] = 3\delta[0]
\]
\[
\Rightarrow 2h[1] - 10h[0] = 3
\]
\[
\Rightarrow (2 \cdot 3 - 10 \cdot 1)c_1 + (2 \cdot 2 - 10 \cdot 1)c_2 = 3
\]
\[
\Rightarrow -4c_1 + -6c_2 = 3
\]

and at \( k = -2 \):

\[
2h[0] - 10h[-1] + 12h[-2] = 3\delta[-1] \Rightarrow 12h[0] = 0 \Rightarrow h[0] = 0
\]
\[
\Rightarrow c_1 + c_2 = 0.
\]

• Thus, \( c_2 = -1.5 = -c_1 \) so that \( h[k] = (-1.5(3)^k + 1.5(2)^k)u[k] \) and for \( k \geq 0 \)

\[
y_{ZS}[k] = (h \ast f)[k] = \sum_{r=0}^{k} h[r]f[k - r].
\]

• **Exercise:** Write the ZSR as a sum of system modes \( 2^k \) and \( 3^k \) and a (force) term like the input, here taken as \( f[k] = 4(-5)^k u[k] \).

---

**Convolution - other important properties**

• Again, for a LTI system with impulse response \( h \) and input \( f \), the ZSR is \( y_{ZS} = f \ast h \), where

\[
(f \ast h)[k] = \sum_{r=-\infty}^{\infty} f[r]h[k - r]
\]

• By simply changing the dummy variable of summation to \( r' = h - r \), can show convolution is commutative: \( f \ast h = h \ast f \).

• One can directly show that convolution \( f \ast h \) is a bi-linear mapping from pairs of signals \((f, h)\) to signals \((y_{ZS})\), consistent with convolution’s commutative property and the (zero state) system with impulse response \( h \) being LTI;

• that is, \( \forall \) signals \( f, g, h \) and scalars \( \alpha, \beta \in \mathbb{C} \),

\[
(\alpha f + \beta g) \ast h = \alpha(f \ast h) + \beta(g \ast h)
\]

• By changing order of summation (Fubini’s theorem), one can easily show that convolution is associative, i.e., \( \forall \) signals \( f, g, h \),

\[
(f \ast g) \ast h = f \ast (g \ast h).
\]
Convolution - other important properties (cont)

- We’ll use these properties when composing more complex systems from simpler ones.
- By just changing variables of integration, we can show how to exchange time-shift with convolution, i.e., ∀ signals \( f, h : \mathbb{Z} \to \mathbb{C} \) and times \( k \in \mathbb{Z} \),
  \[
  \Delta^k f \ast h = \Delta^k (f \ast h);
  \]
  recall how convolution represents the ZSR of linear and time-invariant systems.
- By the ideal sampling property, recall that the identity signal for convolution is the unit pulse \( \delta \), i.e., ∀ signals \( f \),
  \[
  f \ast \delta = \delta \ast f = f
  \]
- Exercise: Adapt the proofs of these properties in continuous time to this discrete-time case.
- Exercise: In particular, show that if \( f \) and \( h \) are causal signals, then \( y = f \ast h \) is causal; i.e., if the unit-pulse response \( h \) of a system is a causal signal, then the system is causal.

System stability - ZIR - asymptotically stable

- Consider a SISO system with input \( f \) and output \( y \).
- Recall that the ZIR \( y_{ZI} \) is a linear combination of the system’s characteristic modes, where the coefficients depend on the initial conditions, possibly including some initial unit-pulse terms if zero is a characteristic value (system pole).
- A system is said to be asymptotically stable if for all initial conditions,
  \[
  \lim_{k \to \infty} y_{ZI}[k] = 0.
  \]
- So, a system is asymptotically stable if and only if all of its characteristic values have magnitude less than 1.
• If the characteristic polynomial \( Q(z) = (z - 0.5)(z^2 + 0.0625) \), then

• the system’s characteristic values (roots of \( Q \)) are 0.5, ±0.25j each with magnitude less than one,

• and the ZIR is of the form,

\[
y_{ZI}[k] = (c_1(0.5)^k + c_2(0.25j)^k + 2\text{Re}\{c_2\}(0.25)^k\cos(k\pi/2) - 2\text{Im}\{c_2\}(0.25)^k\sin(k\pi/2)) u[k],
\]

• recalling that \( j^k = e^{jk\pi/2} \).

• So, \( y_{ZI}[k] \to 0 \) as \( k \to \infty \) for all \( c_1, c_2 \) (i.e., for all initial conditions), and

• hence is asymptotically stable.

### System stability - bounded signals

• A signal \( y \) is said to be bounded if

\[
\exists M < \infty \text{ s.t. } \forall k \in \mathbb{Z}, \ |y[k]| \leq M;
\]

otherwise \( y \) is said to be unbounded.

• For example, \( y[k] = 0.25(\frac{1+j\sqrt{3}}{2})^k u[k] \) is bounded (can use \( M = 0.25 \)).

• Also, \( 3\cos(5k) \) is bounded (can use \( M = 3 \)).

• But both \( 2^k\cos(5k) \) and \( 3 \cdot (-2)^k \) are unbounded.
System stability - ZIR - marginally stable

- A system is said to be marginally stable if it is not asymptotically stable but \( y_{ZI} \) is always (for all initial conditions) bounded.

- A system is marginally stable if and only if
  - it has no characteristic values with magnitude strictly greater than 1,
  - it has at least one characteristic value with magnitude exactly 1, and
  - all magnitude-1 characteristic values are not repeated.

- That is, a marginally stable system has
  - some characteristic modes of the form \( \cos(\Omega k) \) or \( \sin(\Omega k) \),
  - while the rest of the modes are all of the form \( k^r|z|^k \cos(\Omega k) \) or \( k^r|z|^k \sin(\Omega k) \),
    with \( |z| < 1 \) and integer degree \( r \geq 0 \).
  - Exercise: Explain why we can take \( \Omega \in (-\pi, \pi] \) without loss of generality.
  - Note: the dimension of \( \Omega \), [\( \Omega \)] = radians.

System stability - ZIR - marginally stable: Example

- The characteristic polynomial is \( Q(z) = z(z^2 + 1)(z - 0.25) \) gives characteristic values \( 0, 0.25, \pm j \).

- then the system is marginally stable with modes \( (0.25)^k, \cos(k\pi/2), \sin(k\pi/2) \),

- the last two of which are bounded but do not tend to zero as time \( k \rightarrow \infty \).
System stability - ZIR - unstable

- A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.

- For example, the system with \( Q(z) = (z^2 - 0.5)(z + 3) \) is unstable owing to the characteristic value \(-3\) with unbounded mode \((-3)^k\).

- For another example, if the characteristic polynomial is \( Q(z) = (z^2 + 1)^2(z - 0.5) \) then the purely imaginary characteristic values \( \pm j \) are repeated, and hence the two additional modes \( k\sin(k\pi/2), k\cos(k\pi/2) \) are unbounded, so this system is unstable.

- Similarly, if \( Q(z) = (z^2 - 1)^2(z - 0.5) \) then the characteristic values \( \pm 1 \) are repeated and the modes \( k \) and \( k(-1)^k \) are unbounded, so this system is unstable too.

---

ZIR stability - stability of poles

---
System stability - ZSR - BIBO stable

• A SISO system is said to be *Bounded Input, Bounded Output* (BIBO) stable if ∀ bounded input signals \( f \), the ZSR \( y_{ZS} \) is bounded.

• A sufficient condition for BIBO stability is absolute summability of the unit-pulse response,

\[
\sum_{k=0}^{\infty} |h[k]| < \infty.
\]

• To see why: If the input \( f \) is bounded (by \( M_f \) with \( 0 \leq M_f < \infty \)) then ∀ \( k \geq 0 \):

\[
|y_{ZS}[k]| = |(f * h)[k]|
= \left| \sum_{r=0}^{k} f[k-r]h[r] \right|
\leq \sum_{r=0}^{k} |f[k-r]h[r]| \quad \text{(by the triangle inequality)}
\leq \sum_{r=0}^{k} M_f|h[r]|
\leq M_f \sum_{r=0}^{\infty} |h[r]| =: M_y < \infty,
\]

System stability - ZSR - BIBO stable

• The condition of absolute summability of the unit-pulse response,

\[
\sum_{r=0}^{\infty} |h[r]| < \infty,
\]

is also necessary for, and hence equivalent to, BIBO stability.

• If any component characteristic mode of \( h \) is unbounded, then \( h \) will not to be absolutely summable.

• Thus, if the system (ZIR) is asymptotically stable it will be BIBO stable; the converse is also true.
ZSR - the transfer function, $H$

- Recall that for any polynomial $Q$ and $z \in \mathbb{C}$ (including $s = jw$, $w \in \mathbb{R}$),
  $$Q(\Delta^{-1})z^k = Q(z)z^k, \quad \forall k \geq 0.$$

- So, if we guess that a “particular” solution of the system $Q(\Delta^{-1})y = P(\Delta^{-1})f$ with input $f[k] = Az^k u[k]$ is of the form $y_0[k] = AH(z)z^k = H(z)f[k]$, $k \geq 0$, then we get by substitution that $\forall k \geq 0, z \in \mathbb{C}$,
  $$(Q(\Delta^{-1})y_0)[k] = (P(\Delta^{-1})f)[k] \Rightarrow AH(z)Q(z)z^k = AP(z)z^k \Rightarrow H(z) = P(z)/Q(z).$$

- The “rational polynomial” $H = P/Q$ is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

- So, the ZSR (forced response + characteristic modes) would be of the form:
  $$y_{ZS}[k] = (AH(z)z^k + \text{linear combination of char. modes})u[k].$$

- Recall that for the example with $Q(z) = z + 3$ and $P(z) = 7z$, we computed the unit-pulse response $h[k] = 7(-3)^k u[k]$ and the ZSR to input $f[k] = 4(0.5)^k u[k]$ as $y_{ZS}[k] = (24(-3)^k + 4(0.5)^k)u[k]$.

- Here, note that $H(0.5) = P(0.5)/Q(0.5) = 1$, i.e., the forced response component of $y_{ZS}$ is $H(0.5)f[k] = 1 \cdot 4(0.5)^k u[k] = 4(0.5)^k u[k]$.

ZSR - unit-pulse response $h$, transfer function $H$, and eigenresponse

- $y_{ZS}[k] = (H(z)Az^k + \text{linear combination of char. modes})u[k]$ is the ZSR to input $f[k] = Az^k u[k]$, where $H(z) = P(z)/Q(z)$.

- The eigenresponse is a special case of the forced response for exponential inputs.

- If $|z| = 1$, i.e., $z = e^{j\Omega}$ for some $\Omega \in (-\pi, \pi)$ (w.l.o.g.), and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:
  $$y[k] \rightarrow AH(e^{j\Omega})e^{j\Omega k} \text{ as } k \rightarrow \infty.$$

- Since $y = h * f$, we get that as $k \rightarrow \infty$ for a LTIC and asymptotically stable system,
  $$y_{ZS}[k] = \sum_{r=0}^{k} h[r]Ae^{j\Omega(k-r)} = Ae^{j\Omega k} \sum_{r=0}^{k} h[r]e^{-j\Omega r} \rightarrow Ae^{j\Omega k}H(e^{j\Omega}),$$
  $$\Rightarrow \sum_{r=0}^{\infty} h[r]e^{-j\Omega r} = H(e^{j\Omega}), \quad \forall \Omega \in (-\pi, \pi).$$
ZSR - transfer function $H$ and eigenresponse (cont)

The LTI system transfer function $H$ is the Discrete-Time Fourier Transform (DTFT) of the system unit-pulse response $h$:

$$\forall \Omega \in \mathbb{R}, \quad H(e^{j\Omega}) = \sum_{r=0}^{\infty} h[r] e^{-j\Omega r}.$$ 

• Note that $H(e^{j\Omega})$ is periodic since $H(e^{j\Omega}) \equiv H(e^{j\Omega + 2\pi k})$ for any integer $k$.

• For the $z$-transform (and DTFS) we will use this notation for $H$, but for the DTFT we will instead write $H(\Omega)$.

Frequency-domain methods for discrete-time signals

• Discrete-Time Fourier Series (DTFS) of periodic signals

• Discrete-Time Fourier Transform (DTFT)

• sampled data systems

* DFT & FFT

• $z$-transform for (complete) transient response

• eigenresponse

• canonical system realization of a difference equation
Discrete-time Fourier series of periodic signals

• For all \( r, N \in \mathbb{Z} \), note that the signal \( \{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z}\} \) “repeats itself” every \( N > 0 \) units of (discrete) time \( k \), in particular
  \[
  \forall r \in \mathbb{Z}, \ e^{jr \frac{2\pi}{N} k} \big|_{k=0} = 1 = e^{jr \frac{2\pi}{N} k} \big|_{k=N}
  \]
• Also the signals \( \{ \exp(jr \frac{2\pi}{N} k) \mid k \in \mathbb{Z}\} \equiv \{ \exp(jr' \frac{2\pi}{N} k) \mid k \in \mathbb{Z}\} \) whenever \( r' = r \mod N \).

• Suppose \( N \) is the period of periodic signal \( x = \{ x[k] \mid k \in \mathbb{Z}\} \) and \( \Omega_o = 2\pi/N \) be the fundamental “frequency” of \( x \) (recall \( [\Omega_o] = \) radians).

• We can write \( x \) as a Discrete-Time Fourier Series (DTFS):
  \[
  \forall k \in \mathbb{Z}, \quad x[k] = \sum_{r=0}^{N-1} D_r e^{jr \Omega_o k}.
  \]

• If we show that these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are orthogonal then
  – linear independence follows
  – the \( r^{th} \) coordinate \( D_r \) (DTFS coefficients) is found by simply projecting \( x \) onto the vector \( \xi_r \):
    \[
    D_r = \langle x, \xi_r \rangle / ||\xi_r||^2.
    \]

Discrete-time Fourier series of periodic signals (cont)

• Consider the \( N \) signals \( \xi_r[k] := e^{j\Omega_o k} \) over any time-interval \( A \) of length \( N \).

• Equivalently consider these \( N \) signals \( \xi_r \) as \( N \)-vectors in \( \mathbb{R}^N \), i.e., the \( k^{th} \) entry of vector \( \xi_r \) is \( \xi_r[k] \).

• If these signals/vectors \( \{ \xi_r \}_{r=0}^{N-1} \) are linearly independent, then they will form a basis spanning all other signals \( x : A \to \mathbb{R} \), equivalently all other vectors \( x \in \mathbb{R}^N \),

• i.e., any such \( x \) can be written as a linear combination of the \( \{ \xi_r \}_{r=0}^{N-1} \) giving the DTFS of \( x \):
  \[
  x_r = \sum_{r=0}^{N-1} D_r \xi_r.
  \]

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• Consider any period of $x: \mathbb{Z} \rightarrow \mathbb{R}$, say $\{0, 1, 2, ..., N-1\}$.

• First note that for any $v \in \mathbb{Z}$ that is not a multiple of $N$ (so $e^{jv\Omega_o} = e^{jv(2\pi/N)} \neq 1$), the geometric series

$$\sum_{k=0}^{N-1} e^{jv\Omega_o k} = \sum_{k=0}^{N-1} \left( e^{j2\pi/N} \right)^k = \frac{e^{jv(2\pi/N)N} - e^{jv(2\pi/N)0}}{e^{j2\pi/N} - 1} = 0.$$  

• Thus, for any $r \neq v \in \mathbb{Z}$ such that $N \not| (v - r)$, the inner product $\langle \xi_r, \xi_v \rangle =$

$$\langle \{e^{jr(2\pi/N)k}\}, \{e^{jv(2\pi/N)k}\} \rangle := \sum_{k=0}^{N-1} e^{jr(2\pi/N)k}e^{jv(2\pi/N)k} = \sum_{k=0}^{N-1} e^{j(r-v)(2\pi/N)k} = 0,$$

recalling that the inner product is conjugate-linear in the second argument so that $\langle x, x \rangle = ||x||^2$ when $x$ is $\mathbb{C}$-valued.

• So, these signals are orthogonal and the DTFS coefficients of $N$-periodic $x$ are

$$D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k]e^{-jr\Omega_o k} = \frac{\langle x, \{e^{jr\Omega_o k}\} \rangle}{||\{e^{jr\Omega_o k}\}||^2}, \quad \Omega_o = \frac{2\pi}{N}.$$  

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DTFS - checking coefficients

• Let’s now compute the inner product of $\xi_v$, for any $v \in \{0, 1, ..., N-1\}$, with the DTFS of $N$-periodic $x$:

$$\langle x, \{e^{jv\Omega_o k}\} \rangle = \sum_{k=0}^{N-1} x[k]e^{-jv\Omega_o k} = \sum_{k=0}^{N-1} \sum_{r=0}^{N-1} D_r e^{jr\Omega_o k} e^{-jv\Omega_o k}$$  

$$= \sum_{r=0}^{N-1} D_r \sum_{k=0}^{N-1} e^{j(r-v)\Omega_o k} = \sum_{r=0}^{N-1} D_r N \delta(r - v) = D_v N$$

• Again, we have verified the DTFS coefficients is

$$D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k]e^{-jr\Omega_o k} = \frac{\langle x, \{e^{jr\Omega_o k}\} \rangle}{||\{e^{jr\Omega_o k}\}||^2}, \quad \Omega_o = \frac{2\pi}{N}.$$  

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Problem:
Identify the DTFS coefficients (if they exist) for
\[
x[k] = 7 \sin(5.7\pi k) + 2 \cos(3.2\pi k), \quad k \in \mathbb{Z}.
\]

Solution:

- First note that the two components of \( x \) are periodic, so their sum is periodic. (Why is this so in discrete time?)
- Since \( \sin \) and \( \cos \) have period \( 2\pi \), we can subtract integer multiples of \( 2\pi \) to get
  \[
x[k] = 7 \sin(1.7\pi k) + 2 \cos(1.2\pi k).
  \]
- \( 1.7\pi k \) is an integer multiple of \( 2\pi \) when (integer) \( k = 20 \), and when \( k = 5 \) for \( 1.2\pi k \), so least common multiple of these periods is \( k = 20 \).
- (Show that one can alternatively find the greatest common divisor of the component frequencies.)

Thus, the period of \( x \) is \( N = 20 \) and the fund. frequ. is \( \Omega_0 = 2\pi/N = 0.1\pi \).

By Euler’s identity and adding \( 2\pi k \) to the negative exponents,
\[
x[k] = \frac{7}{2j} e^{j1.7\pi k} - \frac{7}{2j} e^{-j1.7\pi k} + e^{j1.2\pi k} + e^{-j1.2\pi k}
  = -3.5je^{j1.7\pi k} + 3.5je^{j0.3\pi k} + e^{j1.2\pi k} + e^{j0.8\pi k}.
\]

So, the DTFS of \( x[k] = \sum_{r=0}^{19} D_r e^{jr0.1\pi k} \) with
\[
D_{17} = -3.5j = 3.5e^{-j\pi/2}, \quad D_3 = 3.5j = 3.5e^{j\pi/2}, \quad D_{12} = 1, \quad \text{and} \quad D_8 = 1;
\]
else \( D_r = 0 \) (incl. the fundamental \( r \in \{1, 19\} \) & DC \( r = 0 \) components).
DTFS - example and exercise

- **Example:** The DTFS of an even rectangle wave with period \( N = 6 \) and duty cycle 3:

\[
x[k] = \sum_{\ell=-\infty}^{\infty} \Delta^6 \ell (\Delta^{-1} u - \Delta^2 u)[k] = \sum_{\ell=-\infty}^{\infty} (u[k + 1 - 6\ell] - u[k - 2 - 6\ell])
\]

is

\[
D_r e^{jr\Omega_o k},
\]

where the fund. freq. \( \Omega_o = 2\pi / 6 \) and, \( \forall r \in \mathbb{Z} \),

\[
D_r = \frac{1}{6} \sum_{k=-3}^{2} x[k] e^{-jr\Omega_o k} = \frac{1}{6} \sum_{k=-1}^{1} 1 \cdot e^{-jr(2\pi/6)k} = \frac{1}{6} (1 + 2 \cos(r(2\pi/6)k)).
\]

- **Exercise:** Plot \( x[k] \) as a function of time \( k \) and plot its (periodic) spectrum:

\[
\forall r \in \{0, 1, 2, ..., 5\}, \ell \in \mathbb{Z},
\]

\[
\tilde{X}(r2\pi/6 + 2\pi\ell) = D_r.
\]

DTFS - Parseval’s theorem

- The average power of the \( N \)-periodic discrete-time signal \( x \) is

\[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \overline{x[k]};
\]

equivalently, the sum could be taken over any interval of length \( N \in \mathbb{Z}^>0 \).

- Substituting the Fourier series of \( x \) separately for \( x[k] \) and \( \overline{x[k]} \) (using a different summation-index variable for each substitution), leads to Parseval’s theorem

\[
P_x = \sum_{r=0}^{N-1} |D_r|^2.
\]

Parseval’s theorem can be used to determine the amount of periodic signal \( x \)’s power resides in a given frequency band \([\Omega_1, \Omega_2] \subset [0, 2\pi]\) radians:

1. determine the harmonics \( r\Omega_o \) of \( x \) that reside in this band, i.e., integers \( r \in \left[\Omega_1/\Omega_o, \Omega_2/\Omega_o\right] \) where \( x \)’s fundamental frequency \( \Omega_o = 2\pi / N \).
2. sum just over these harmonics to get the answer, \( \sum_{[\Omega_1/\Omega_o] \leq r \leq [\Omega_2/\Omega_o]} |D_r|^2 \).
DTFS - Parseval’s theorem example

- Find the fraction of $x$’s average power in the “frequency” band $[0.4\pi, 1.1\pi]$ radians where
\[
\forall k \in \mathbb{Z}, \; x[k] = \sum_{v=-\infty}^{\infty} (3\delta[k - 4v] - 4\delta[k - 1 - 4v])
\]

- **Solution:** $x$ has period $N = 4$ and average power
\[
P_x = \frac{1}{N} \sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{4} \sum_{k=0}^{3} |x[k]|^2 = \frac{1}{4} (3^2 + (-4)^2 + 0^2 + 0^2) = \frac{25}{4}
\]

- $x$ has fundamental frequency $\Omega_o = 2\pi/N = \pi/2$ radians and discrete-time Fourier coefficients
\[
D_r = \frac{1}{N} \sum_{k=0}^{N-1} x[k]e^{-j(r\Omega_o)k} = \frac{1}{4} \left(3 - 4e^{-j\pi/2}\right), \quad 0 \leq r \leq N - 1 = 3.
\]

- The harmonics $r$ of $x$ that reside in $[0.4\pi, 1.1\pi]$ satisfy $0.4\pi \leq r\Omega_o = r\pi/2 \leq 1.1\pi$, i.e., $r \in \{1, 2\}$.

- So, by Parseval’s theorem, the answer is $(|D_1|^2 + |D_2|^2)/P_x$.

---

**Periodic extensions**

- Consider signal $x: \mathbb{Z} \rightarrow \mathbb{R}$ having finite support $\{-M, -M + 1, ..., 0, ..., M - 1, M\}$ for $0 < M < \infty$; i.e., $\forall |k| > M, \; x[k] = 0$.

- For $N \geq M$, define $2N$-periodic $x^{(N)}$ such that
\[
x^{(N)}[k] = \begin{cases} 
  x[k] & \text{if } |k| \leq M \\
  0 & \text{if } M < |k| \leq N
\end{cases}
\]

- $x^{(N)}$ is a periodic extension of the finite-support signal $x$, where again $x^{(N)}$’s period is $2N$ and
\[
\lim_{N \rightarrow \infty} x^{(N)} = x.
\]
DTFS of periodic extension leading to DTFT

• For \( r \in \{-N + 1, -N + 2, ..., N - 1, N\} \), the DTFS of \( x^{(N)} \) has coefficients

\[
D_r^{(N)} = \frac{1}{2N} \sum_{k=-N}^{N} x^{(N)}[k] e^{-j\frac{2\pi}{2N} k r}
\]

\[
= \frac{1}{2N} \sum_{k=-M}^{M} x[k] e^{-j\frac{2\pi}{2N} k r}
\]

\[
= \frac{1}{2N} \sum_{k=-\infty}^{\infty} x[k] e^{-j\frac{2\pi}{2N} k r}
\]

\[
= \frac{1}{2N} X \left( \frac{2\pi}{2N} \right),
\]

where the Discrete-Time Fourier Transform (DTFT) of (aperiodic) \( x : \mathbb{Z} \to \mathbb{R} \) is \( X : \mathbb{R} \to \mathbb{C} \):

\[
X(\Omega) := \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k} =: (F x)(\Omega), \quad \Omega \in \mathbb{R}
\]

• Note that Fourier integrals (spectra of discrete-time signals) are periodic, repeating themselves every \( 2\pi \) radians: \( \forall \Omega \in \mathbb{R}, \ell \in \mathbb{Z}, \)

\[
X(\Omega) = X(\Omega + \ell 2\pi).
\]

Inverse DTFT by Fourier Integral

• Thus, \( \forall k \in \mathbb{Z}, \)

\[
x[k] = \lim_{N \to \infty} x^{(N)}[k]
\]

\[
= \lim_{N \to \infty} \sum_{r=-N}^{N} D_r^{(N)} e^{j\frac{2\pi}{2N} k r}
\]

\[
= \lim_{N \to \infty} \sum_{r=-N}^{N} X \left( \frac{2\pi}{2N} \right) e^{j\frac{2\pi}{2N} k r} \frac{1}{2N} \frac{2\pi}{2N}
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega
\]

where the last equality is by Riemann integration with \( 2\pi/(2N) \to d\Omega. \)

• Thus, we have derived the inverse DTFT by Fourier integral of \( X \) giving (aperiodic) \( x, \)

\[
\forall k \in \mathbb{Z}, \quad x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega k} d\Omega =: (F^{-1} X)[k].
\]

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DTFT Examples - exponential signal

- If \( x = \delta \) then obviously \( X \equiv 1 \).
- The geometric signal \( x[k] = \gamma^k u[k] \) for scalar \( \gamma \) s.t. \( |\gamma| < 1 \) has DTFT
  \[ X(\Omega) = \sum_{k=0}^{\infty} \gamma^k e^{-j\Omega k} = \sum_{k=0}^{\infty} (\gamma e^{-j\Omega})^k = \frac{1}{1 - \gamma e^{-j\Omega}} = \frac{1}{(1 - \gamma) \cos(\Omega) + j\gamma \sin(\Omega)} \]
- Note that
  \[ |X(\Omega)| = \frac{1}{(1 - \gamma \cos(\Omega))^2 + \gamma^2 \sin^2(\Omega)} = \frac{1}{1 + \gamma^2 - 2\gamma \cos(\Omega)} \]
  \[ \angle X(\Omega) = -\arctan \left( \frac{\gamma \sin(\Omega)}{1 - \gamma \cos(\Omega)} \right) \]

DTFT Examples - exponential signal (cont)

- The plots above are for \( \gamma = 0.5 \).
- Note how \( X \) has period \( 2\pi \).
- **Exercise**: What are the maximum and minimum values of \( |X| \), i.e., how would this plot depend on \( \gamma > 0 \)? Plot \( x \) and \( \angle X \). How do these plots differ when \(-1 < \gamma < 0\)?
- **Exercise**: Find the DTFT of anticausal signal \( x[k] = \gamma^k u[-k] \) for scalar \( \gamma \) s.t. \( |\gamma| > 1 \).
- **Exercise**: Find the DTFT of \( x[k] = \gamma^{|k|}, k \in \mathbb{Z} \), for scalar \( \gamma \) s.t. \( |\gamma| < 1 \).
DTFT Examples - Square and Triangle Pulse

• For \( T \in \mathbb{Z}^0 \), the even rectangle pulse with support \( 2T + 1 \),
  \[ x = \Delta^{-T}u - \Delta^{T+1}u \] (i.e., \( x[k] = u[k + T] - u[k - (T + 1)] \)), has DTFT
  \[ X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k} = 1 + 2 \sum_{k=1}^{T} \cos(k\Omega), \quad \Omega \in \mathbb{R}. \]

• Exercise (even rectangle pulse in frequency domain):
  Show that for fixed \( \Omega' \) s.t. \( 0 < \Omega' < \pi \),
  \[ \mathcal{F}^{-1}\{ \Delta_{-\Omega} u - \Delta_{\Omega'} u \}[k] = \frac{\Omega'}{\pi} \text{sinc}(\Omega'k), \quad k \in \mathbb{Z}. \]

• For \( T \in \mathbb{Z}^0 \), the odd triangle pulse with support \( 2T + 1 \),
  \( x[k] \equiv k(\Delta^{-T}u[k] - \Delta^{T+1}u[k]) \) has DTFT
  \[ X(\Omega) = \sum_{k=-T}^{T} k e^{-j\Omega k} = -2j \sum_{k=1}^{T} k \sin(k\Omega), \quad \Omega \in \mathbb{R}. \]

DTFT Examples - exponential sinusoid

• For fixed time \( K_0 \), clearly
  \[ \mathcal{F}\{ \delta[k - K_0] \}(\Omega) = e^{jK_0\Omega}, \]  
  where here \( \delta \) is the unit pulse.

• Note that \( e^{jK_0\Omega} \) is a sinusoidal function of \( \Omega \) with period \( 2\pi/K_0 \) (frequency \( K_0 \)).

• Exercise: For fixed frequency \( \Omega_0 \), show that
  \[ \mathcal{F}\{ e^{-j\Omega_0 k} \}(\Omega) = 2\pi \sum_{v=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi v), \]
  where here \( \delta \) is the Dirac impulse (in the frequency domain \( \Omega \in \mathbb{R} \)). Hint: work with \( \mathcal{F}^{-1} \).

• So, the DTFT of a \( N \)-periodic signal with Fourier series
  \[ \sum_{r=0}^{N-1} D_r e^{j\frac{2\pi r}{N} k} \overset{\mathcal{F}}{\rightarrow} \sum_{r=0}^{N-1} \sum_{v=-\infty}^{\infty} D_r \delta(\Omega - r \frac{2\pi}{N} + 2\pi v). \]
DTFT - Time shift and frequency shift properties

• If fixed $K_0 \in \mathbb{Z}$ and $X = \mathcal{F}\{x\}$ then

$$
\mathcal{F}\{\Delta^{K_0}x\}(\Omega) = \sum_{k=-\infty}^{\infty} (\Delta^{K_0}x)[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} x[k-K_0]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} x[k']e^{-j(k+K_0)\Omega} = e^{-jk_0\Omega}X(\Omega),
$$

i.e., shift in time by $K_0$ corresponds to product with sinusoid of period $2\pi/K_0$ (linear phase shift) in frequency domain.

• Exercise: Prove the dual property that if fixed $\Omega_0 \in \mathbb{R}$ and $X = \mathcal{F}\{x\}$ then

$$
\mathcal{F}\{x[k]e^{j\Omega_0 k}\}(\Omega) = X(\Omega - \Omega_0),
$$

i.e., modulation (multiplication by a sinusoid) in time domain results in frequency shift.

---

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DTFT - convolution properties

• Let $X_r = \mathcal{F}\{x_r\}$ for $r \in \{1, 2\}$.

$$
\mathcal{F}\{x_1 \ast x_2\}(\Omega) := \sum_{k=-\infty}^{\infty} (x_1 \ast x_2)[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k-l]e^{-j(k-l)\Omega}e^{-jl\Omega} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_1[l]x_2[k']e^{-jk'\Omega}e^{-j\Omega} \text{ where } k' = k - l
$$

$$
= \sum_{l=-\infty}^{\infty} x_1[l]e^{-jl\Omega} \sum_{k=-\infty}^{\infty} x_2[k']e^{-j\Omega} =: X_1(\Omega)X_2(\Omega)
$$

• Exercise: Prove the dual property that

$$
\mathcal{F}\{x_1x_2\}(\Omega) = \frac{1}{2\pi} (X_1 \ast X_2)(\Omega) := \frac{1}{2\pi} \int_{2\pi} X_1(v)X_2(\Omega - v)dv.
$$

• Exercise: Use the convolution properties to prove the time and frequency shift properties. Hint: $(\Delta^{K_0}) \ast x = \Delta^{K_0}x$.

• Exercise: Show that DTFT is a linear operator.

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DTFT - Parseval’s Theorem

• The energy of a signal DT \( x \) is

\[
E_x := \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-\infty}^{\infty} x[k]x[k] = \sum_{k=-\infty}^{\infty} (\mathcal{F}^{-1}X)[k](\mathcal{F}^{-1}X)[k]
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')e^{j\Omega'k}d\Omega' \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega k}d\Omega
\]

\[
= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega')e^{j\Omega'k}d\Omega' \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)e^{-j\Omega k}d\Omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \int_{-\pi}^{\pi} \overline{X(\Omega)} \frac{1}{2\pi} \left( \sum_{k=-\infty}^{\infty} e^{j\Omega k}e^{-j\Omega k} \right) d\Omega d\Omega'
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega') \overline{X(\Omega')} d\Omega' = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega')|^2 d\Omega', \text{ recalling that for fixed } \Omega': \mathcal{F}^{-1}\{2\pi\delta(\Omega - \Omega')\}[k] = e^{-j\Omega k} \& \int_{-\pi}^{\pi} \overline{X(\Omega)}\delta(\Omega - \Omega')d\Omega = X(\Omega').
\]

DTFT - Parseval’s Theorem - example

• The even rectangle pulse with support \( 2T + 1 \), \( x = \Delta^{-T}u - \Delta^{T+1}u \) has energy

\[
E_x = \sum_{k=-\infty}^{\infty} |x[k]|^2 = \sum_{k=-T}^{T} 1^2 = 2T + 1.
\]

• Recall its DTFT is \( X(\Omega) = \sum_{k=-T}^{T} e^{-j\Omega k} \), so

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega)\overline{X(\Omega)} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-T}^{T} e^{-j\Omega k} \sum_{k=-T}^{T} e^{+j\Omega k} d\Omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k=-T}^{T} 1 + \sum_{k<k'} e^{j(k-k')\Omega} \right) d\Omega
\]

\[
= \sum_{k=-T}^{T} \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 d\Omega + \sum_{k<k'} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(k-k')\Omega} d\Omega = \sum_{k=-T}^{T} 1 + 0 = 2T + 1.
\]
DTFT - Parseval’s Theorem - exercises

- **Exercise:** Repeat this calculation using $X(\Omega) = 1 + 2 \sum_{k=1}^{T} \cos(\Omega k)$.

- **Exercise:** Compute amount of energy of $x$ in the frequency band $[-\pi/6, \pi/6]$, i.e.,

$$\frac{1}{2\pi} \int_{-\pi/6}^{\pi/6} |X(\Omega)|^2 d\Omega$$

Analysis of Stable DT LTI Systems in Steady-State

- Consider a SISO, DT-LTIC system described by the difference equation

$$Q(\Delta^{-1}) y = P(\Delta^{-1}) f,$$

where $f$ is the input and $y$ is the ZSR (output).

- Recall that by the time-shift property,

$$Q(e^{j\Omega}) Y_{ZS}(\Omega) = P(e^{j\Omega}) F(\Omega) \Rightarrow Y_{ZS}(\Omega) = H(\Omega) F(\Omega).$$

- We now re-derive from first principles the eigenresponse by first recalling that the ZSR $y_{ZS} = f \ast h$ where $h$ is the unit-pulse response.

- Taking DTFTs, $Y_{ZS} = H F$ where $H = F h$ is the transfer function.

- Suppose the system is BIBO/asymptotically stable, i.e., the $n$ roots of $Q$ (system char. modes/poles) $z$ all have modulus $|z| < 1$.

- The ZSR will consist of a forced response plus characteristic modes, where the latter will → 0 over time (our stability assumption) so that the forced response becomes the steady-state response.
Analysis of Stable DT LTI Systems in Steady-State (cont)

- The forced response to a persistent sinusoidal input
  \[ f[k] = A_f e^{j(\Omega, k + \phi_f)} \]
  will be of the form
  \[ y_{ss}[k] = A_y e^{j(\Omega, k + \phi_y)} \]
  where (for \( k \geq 0 \)),
  \[ Q(e^{j\Omega})y_{ss}[k] = (Q(\Delta^{-1})y_{ss})[k] = (P(\Delta^{-1})f)[k] = P(e^{j\Omega})f[k]. \]
  \[ \Rightarrow y_{ss}[k] = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})}f[k] \]
- Also, the ZSR \( y_{ZS} = h * f \), i.e., for all time \( k \geq 0 \):
  \[ y_{ZS}[k] = \sum_{v=0}^{k} h[v] A_f e^{j(\Omega, (k-v) + \phi_f)} = f[k] \sum_{v=0}^{k} h[v] e^{-j\Omega,v} \]
  \[ \rightarrow f[k] \cdot H(\Omega_0) \Rightarrow: y_{ss}[k] \text{ as } k \to \infty. \]

Transfer Function and Eigenresponse in Discrete Time (cont)

- Equating the forced responses (steady-state response for a stable system), we again get that the system transfer function is
  \[ H(\Omega) = \frac{P(e^{j\Omega})}{Q(e^{j\Omega})} = (Fh)(\Omega). \]
- Note that \( \forall k \in \mathbb{Z}, H(\Omega) = H(\Omega + 2\pi k). \)
- Also, we write \( H(\Omega) \) not \( H(e^{j\Omega}) \) for the DTFT.
- So, the eigenresponse of a BIBO/asymptotically stable SISO, DT-LTIC system is the steady-state response to a sinusoid:
  \[ f[k] = A_f e^{j(\Omega, k + \phi_f)} \rightarrow H(\Omega_0)f[k] = A_y e^{j(\Omega, k + \phi_y)} \Rightarrow: y_{ss}[k] \]
- The system magnitude response (gain) is \( |H(\Omega)| = |P(e^{j\Omega})|/|Q(e^{j\Omega})| \), i.e., \( A_y = A_f |H(\Omega_0)| \).
- The system phase response is \( \angle H(\Omega) = \angle P(e^{j\Omega}) - \angle Q(e^{j\Omega}) \), i.e., \( \phi_y = \phi_f + \angle H(\Omega_0) \).
Eigenresponse - example

- **Problem:** For the system \(2y[k] = 0.6y[k - 1] - 7f[k]\) find the steady-state response (if it exists) to \(f[k] = 4\cos(5k)u[k]\).

- **Solution:** The difference equation in standard form is
  \[
  (Q(z)y)[k] = y[k + 1] - 0.3y[k] = -3.5f[k + 1] = (P(z)f)[k],
  \]
  where \(Q(z) = z - 0.3\) and \(P(z) = -3.5z\).

- The sole system characteristic value (root of \(Q\), system pole) is \(0.3\), hence the system is BIBO/asymptotically stable.

- By linearity, the eigenresponse is therefore
  \[
  2H(5)e^{5k} + 2H(-5)e^{(-5)k},
  \]
  where \(H(\Omega) = P(e^{j\Omega})/Q(e^{j\Omega}) = -3.5e^{j\Omega}/(e^{j\Omega} - 0.3) = H(-\Omega)\), so that
  \[
  |H(\Omega)| = \frac{3.5}{\sqrt{(\cos(\Omega) - .3)^2 + \sin^2(\Omega)}},
  \]
  \[
  \angle H(\Omega) = \pi + \Omega - \arctan\left(\frac{\sin(\Omega)}{\cos(\Omega) - .3}\right)
  \]

- **Exercise:** Show that the eigenresponse is also simply \(|H(5)|4\cos(5k + \angle H(5))\).

---

2D Image Processing Example

- Apply 1-dimensional filtering to a 2-dimensional (2D) image by separately performing row and column operations.

- For \(256 \times 256\) pixel (2D) image,
  \[
  f = \begin{bmatrix}
  f[1, 1] & f[1, 2] & \ldots & f[1, 256] \\
  \vdots & \vdots & \ddots & \vdots \\
  f[256, 1] & f[256, 2] & \ldots & f[256, 256] 
  \end{bmatrix}
  \]

- If \(f[k, i]\) represents the 8-bit (grey) intensity of the pixel in row \(k\) and column \(i\) (i.e., 8 bits per pixel or bpp), then the "raw" image size will be \(256^3\)bits = 16Mb = 2MB.

- Each of \(f\)'s rows of pixels can be processed by a system with unit-pulse response \(h\) to obtain a new row of pixels, and thus a new image \(y\):
  \[
  \forall k, f[k, \cdot] \rightarrow [h] \rightarrow y[k, \cdot]
  \]

- Alternatively, each of \(f\)'s columns of pixels can be processed by a system with unit-pulse response \(h\) to obtain a new column of pixels, and thus a new image \(y\):
  \[
  \forall i, f[\cdot, i] \rightarrow [h] \rightarrow y[\cdot, i]
  \]
Image Processing: High-Pass and Low-Pass Filtering

- The system $h$ may have a specific signal processing objective.
- The output pixels $y[k, i]$ may be quantized to fewer bpp than those of the input, thus achieving image compression.
- The simple low-pass filter (L)
  \[
  h[k] = \frac{1}{2}(\delta[k] + \delta[k - 1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] + f[k - 1])
  \]
can capture shading and texture in the image.
- The simple high-pass filter (H)
  \[
  h[k] = \frac{1}{2}(\delta[k] - \delta[k - 1]) \quad \Rightarrow \quad y[k] = \frac{1}{2}(f[k] - f[k - 1])
  \]
can capture edges in the image.
- Typically more compression possible in higher-frequency bands (H).

Image Processing: Tandem Row and Column Filtering

- Define $y_{LH}$ as the output of
  \[
  f \rightarrow \text{row filtering} \rightarrow \text{column filtering} \rightarrow y
  \]
- Similarly define $y_{LL}$, $y_{HH}$ and $y_{HL}$.
- The $y$ images are downsampled by a factor of four (two in each direction).
- The $y_{LL}$ image will have a lot of energy while $y_{HH}$ will have the least energy.
- This motivates non-uniform quantization (bit allotment per pixel) of these images.
- Together with a coding strategy for the quantized images (particularly for the regions of zero pixel-values), this is the basic approach used in JPEG leading to very good compression, e.g., from 8 bpp to 0.2-0.5 bpp.
Sampling Continuous-Time Signals (A/D)

- Consider continuous-time signal $x$ with $X = \mathcal{F}x$.

- Recall that by sampling at period $T$ with impulses in continuous time $t \in \mathbb{R}$, we get

$$x_T(t) := \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} x(kT)e^{-jkTw} =: X_T(w),$$

equivalently,

$$X_T(w) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( w - \frac{2\pi}{T}v \right).$$

- Now define the sampled process in discrete-time $k \in \mathbb{Z}$ and its DTFT,

$$\hat{x}[k] := x(kT) \xrightarrow{\mathcal{F}} \hat{X}(\Omega) = \sum_{k=-\infty}^{\infty} \hat{x}[k]e^{-j\Omega k}.$$  

- Substituting $w = \Omega/T$ we get

$$\hat{X}(\Omega) = X_T \left( \frac{\Omega}{T} \right) = \frac{1}{T} \sum_{v=-\infty}^{\infty} X \left( \frac{\Omega - v2\pi}{T} \right).$$

- **Exercise:** Read decimation (downsampling) and interpolation (upsampling) of Lathi Figs. 8.17 & 10.9.

Sampling Continuous-Time Signals - example

- We are particularly interested in the case where
  - the continuous-time signal $x$ is band-limited, i.e., $\exists w' > 0$ s.t. $X(w) = 0$ for $|w| > w'$, and
  - the sampling frequency is greater than Nyquist’s, i.e., $2\pi/T > 2w' \Rightarrow w'T < \pi$.

- **Example:** For fixed $w' > 0$, consider the cts-time signal $x(t) = A\text{sinc}(w't)$ with FT

$$X(w) = \frac{A\pi}{w'}(u(w + w') - u(w - w')).$$

- Sampling $x$ at period $T < \pi/w'$ we get the discrete-time signal $x[k] = A\text{sinc}(w'kT)$.

- Using inverse DTFT, recall that we can easily check that the DTFT of $x$ is,

$$\hat{X}(\Omega) = \sum_{v=-\infty}^{\infty} \frac{A\pi}{w'}(u(\Omega + w'T - 2\pi v) - u(\Omega - w'T - 2\pi v))$$

$$= \sum_{v=-\infty}^{\infty} \frac{1}{T} X \left( \frac{\Omega - 2\pi v}{T} \right),$$

noting $\forall T > 0, u\left(\frac{\Omega}{T} \pm w' \right) = u\left(\frac{1}{T}(\Omega \pm w'T)\right) = u(\Omega \pm w'T), \Omega := \Omega - 2\pi v$. 

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Suppose the signal \( f \) is sampled every \( T_s \) seconds, i.e., at sampling frequency \( w_s := \frac{2\pi}{T_s} \).

Recall Poisson’s identity (the Fourier series of the picket-fence function)

\[
p_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}
\]

Let’s rederive the relationship between the spectrum of a sampled continuous-time signal and its discrete-time counterpart by first defining the discrete-time signal

\[
\forall k \in \mathbb{Z}, \quad \hat{f}[k] = f(kT_s).
\]

We want to relate the (continuous-time) Fourier transform of \( f \) to the (discrete-time) Fourier transform of \( \hat{f} \),

\[
F(\Omega) := \sum_{k=-\infty}^{\infty} \hat{f}[k]e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} f(kT_s)e^{-j\Omega k}.
\]
Sampled Data Systems: A/D (cont)

• To this end, recall

\[ f(t)p_T(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} f(t)e^{jkwT_s} \Rightarrow \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w-kw_s), \text{ and also} \]

\[ f(t)p_T(t) = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t-kT_s) \Rightarrow \sum_{k=-\infty}^{\infty} f(kT_s)e^{-jkT_s} = \hat{F}(wT_s). \]

• Equating these two expressions for \( \mathcal{F}\{f p_T\} \) we get,

\[ \hat{F}(wT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w-kw_s). \]

• Substituting \( w = \Omega/T_s \) we get,

\[ \hat{F}(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F\left(\frac{\Omega - k2\pi}{T_s} \right). \]

Sampled Data Systems: D/A (digital to analog conversion)

• Now consider a discrete time signal \( \hat{y}[k] \).

• We implement at D/A with a \( T_s \)-second hold, i.e., construct the continuous-time signal

\[ y(t) := \sum_{k=-\infty}^{\infty} \hat{y}[k]r_T(t-kT_s), \text{ where} \]

\[ r_T(t) := u(t) - u(t-T_s) \Rightarrow r_T\text{sinc}(wT_s/2)e^{-jwT_s/2} =: R_T(w). \]

• Note that \( y \) is in the form of a convolution, so:

\[ Y(w) = \sum_{k=-\infty}^{\infty} \hat{y}[k]R_T(w)e^{-jkwT_s} = R_T(w)\hat{Y}(wT_s) \]

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Consider a digital system \( \hat{H}(\omega) \) (or \( \hat{H}(e^{j\omega}) \) depending on notation), whose (ZS) output is \( \hat{y} \) when the input is \( \hat{f} \), i.e., \( \hat{Y} = \hat{H}\hat{F} \).

The equivalent continuous-time transformation of the tandem system

\[
\begin{align*}
f & \rightarrow \text{A/D (}T_s\text{-sample)} \rightarrow \hat{H}(\omega) \rightarrow \text{D/A (}T_s\text{-hold)} \rightarrow y
\end{align*}
\]

with input \( f \) has (ZS) output

\[
Y(\omega) = R_T(w)\hat{Y}(wT_s) = R_T(w)\hat{H}(wT_s)\hat{F}(wT_s)
\]

\[
= R_T(w)\hat{H}(wT_s)\frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s).
\]

Exercise: Show that if \( f \) is band-limited by \( w_s/2 \) (i.e., \( w_s \) is greater than \( f \)'s Nyquist frequency) and the previous sampled data system is followed by an ideal low-pass filter with bandwidth \( w_s/2 \), then the equivalent (continuous-time) transfer function is

\[
H(\omega) = \hat{H}(wT_s)T_s^{-1}R_T(w)(u(w + w_s/2) - u(w - w_s/2))
\]

Note that the term in the transfer function \( H \),

\[
T_s^{-1}R_T(w)(u(w + w_s/2) - u(w - w_s/2)) = \text{sinc}(\Omega/2)(u(\Omega + \pi) - u(\Omega - \pi))
\]

is not a constant function of \( \Omega = wT_s \).

This distortion due to the hold function \( R \) can be reduced by putting in tandem with \( \hat{H} \) an equalizer system with transfer function approximately

\[
\hat{R}^{-1}(\Omega) := \sum_{k=-\infty}^{\infty} \frac{u(\Omega + \pi - k2\pi) - u(\Omega - \pi - k2\pi)}{\text{sinc}((\Omega - k2\pi)/2)}
\]

i.e.,

\[
\hat{H}(\Omega) \rightarrow \hat{R}^{-1}(\Omega)
\]
Sampled Data Systems: equalization of hold sinc(\(\Omega/2\)) by \(\hat{R}^{-1}(\Omega)\)

- the hold (at left, \(R\)) distorts the signal by attenuating its higher frequency components
- the equalizer (at right, \(R^{-1}\)) amplifies at the higher frequencies to cancel out this distortion

DFT and FFT - Reading Exercise on Computational Issues

- Read Lathi Sec. 5.2 and 5.3 re. continuous-time FS, FT
- Read Lathi Sec. 10.6 re. DTFS, DTFT
Transient analysis in discrete time by unilateral z-transform

- z-transform definition and region of convergence.
- Basic z-transform pairs and properties.
- Inverse z-transform of rational polynomials by Partial Fraction Expansion (PFE).
- Total transient response of SISO DT LTIC systems $Q(\Delta^{-1})y = P(\Delta^{-1})f$.
- The steady-state eigenresponse revisited.
- System composition and canonical realizations.

The unilateral z-transform & region of convergence

- The z-transform of a signal $x = \{x[k]\}_{k \geq 0}$ is

$$X(z) = (Zx)(z) = \sum_{k=0}^{\infty} x[k]z^{-k} := \lim_{K \to \infty} \sum_{k=0}^{K} x[k]z^{-k},$$

where $z \in \mathbb{C}$.

- If the signal $x$ is bounded by an exponential (geometric), i.e.,

$\exists M, \gamma \in \mathbb{R}^+ \quad \text{such that} \quad \forall k \in \mathbb{Z}^+, \quad |x[k]| \leq M\gamma^k \quad (i.e., \quad -M\gamma^k \leq x[k] \leq M\gamma^k)$

then the series $X(z)$ converges in the region outside of a disk centered $0 \in \mathbb{C}$,

$$\{z \in \mathbb{C} \mid |z| > \gamma\}.$$

- To see why bounded by an exponential suffices, recall absolute convergence $\Rightarrow$ convergence:

$$\forall k \geq 0, \quad |x[k]z^{-k}| = |x[k]| \cdot |z|^{-k} \leq M\gamma^k |z|^{-k} = M(\gamma/|z|)^k \Rightarrow \sum_{k=0}^{\infty} |x[k]z^{-k}| \leq M \sum_{k=0}^{\infty} (\gamma/|z|)^k \quad \text{which converges if} \quad \gamma/|z| < 1.$$
Basic $z$-transform pairs and RoCs

\[
\begin{align*}
\delta[k] & \xrightarrow{Z} 1, \quad z \in \mathbb{C} \\
u[k] & \xrightarrow{Z} \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \quad |z| > 1 \\
\beta^k u[k] & \xrightarrow{Z} \sum_{k=0}^{\infty} \beta^k z^{-k} = \frac{1}{1-\beta z^{-1}} = \frac{z}{z-\beta}, \quad |z| > |\beta| \\
\{\beta^{k-1} u[k-1]\}(z) & \xrightarrow{Z} \sum_{k=1}^{\infty} \beta^{k-1} z^{-k} = z^{-1} \sum_{k=0}^{\infty} \beta^k z^{-k'} = z^{-1} \frac{1}{1-\beta z^{-1}} = \frac{z}{z-\beta}, \quad |z| > |\beta| \\
e^{j\Omega k} u[k] & \xrightarrow{Z} \sum_{k=0}^{\infty} e^{j\Omega k} z^{-k} = \frac{1}{1-e^{j\Omega} z^{-1}}, \quad |z| > 1 \quad (\beta = e^{j\Omega}) \\
k\beta^k u[k] & \xrightarrow{Z} \sum_{k=0}^{\infty} k\beta^k z^{-k} = \beta \frac{d}{d\beta} \sum_{k=0}^{\infty} \beta^k z^{-k} = \beta \frac{d}{d\beta} \frac{1}{1-\beta z^{-1}} = \frac{\beta z^{-1}}{(1-\beta z^{-1})^2}, \quad |z| > |\beta| \\
\end{align*}
\]

Exercise: Find $\mathcal{Z}\{A \cos(\Omega, k + \phi) u[k]\}$ and $\mathcal{Z}\{A \sin(\Omega, k + \phi) u[k]\}$.

Basic $z$-transform properties: linearity

- The $z$-transform is a linear operator: for all scalars $a_1, a_2 \in \mathbb{C}$ and all signals $x_1, x_2 : \mathbb{Z}^\geq \to \mathbb{C}$ with respective ROCs $C_1, C_2 \subset \mathbb{C}$,
  \[
  (\mathcal{Z}\{a_1 x_1 + a_2 x_2\})(z) = a_1 (\mathcal{Z} x_1)(z) + a_2 (\mathcal{Z} x_2)(z), \quad z \in C_1 \cap C_2.
  \]
- Note that
  \[
  \{z \mid |z| > \gamma_1\} \cap \{z \mid |z| > \gamma_2\} = \{z \mid |z| > \max\{\gamma_1, \gamma_2\}\} \subset \mathbb{C}.
  \]
Basic $z$-transform properties: advance time shift

- Advance time shift (no change in RoC): Let $X = Zx$.

\[ \Delta^{-1}x \xrightarrow{Z} \sum_{k=0}^{\infty} x[k+1]z^{-k} = -zx[0] + \sum_{k=-1}^{\infty} x[k+1]z^{-k} \]

\[ = -zx[0] + z \sum_{k=-1}^{\infty} x[k+1]z^{-(k+1)} \]

\[ = -zx[0] + z \sum_{k=0}^{\infty} x[k']z^{-k'} \]

\[ = -zx[0] + zX(z) \]

- Exercise: For $v \in \mathbb{Z}^+$ show by induction that

\[ (Z\{\Delta^{-v}x\})(z) = -\sum_{k=1}^{v} z^k x[v-k] + z^v X(z) \]

Basic $z$-transform properties: delay time shift

- Delay time shift (no change in RoC): For $v \in \mathbb{Z}^+$,

\[ \Delta^v(xu) \xrightarrow{Z} \sum_{k=0}^{\infty} x[k-v]u[k-v]z^{-k} \]

\[ = \sum_{k=v}^{\infty} x[k-v]z^{-k} = \sum_{k=0}^{\infty} x[k']z^{-k'-v} \]

\[ = z^{-v}X(z). \]

- So in the “zero-state” (input-output) context (i.e., $x[k]u[k] = 0$ for $k < 0$), we identify multiplying by $z^{-1}$ in complex-frequency domain with the unit delay $\Delta$ in the time domain.

- Delay $v \in \mathbb{Z}^+$ of non-causal $x$:

\[ \Delta^v x \xrightarrow{Z} \sum_{k=0}^{\infty} x[k-v]z^{-k} = \sum_{k=-v}^{\infty} x[k']z^{-k'-v} \]

\[ = \sum_{k=-v}^{\infty} x[k']z^{-k'-v} + z^{-v}X(z). \]
Basic $z$-transform properties: frequency shift & convolution

- Let $X = \mathcal{Z}x$ with RoC $C(\gamma) := \{z \in \mathbb{C} \mid |z| > \gamma\}$.
  \[
  \beta^k x[k] \xrightarrow{Z} \sum_{k=0}^{\infty} \beta^k x[k] z^{-k} = \sum_{k=0}^{\infty} x[k] (z/\beta)^{-k} = X(z/\beta), \quad z \in C(\gamma/|\beta|).
  \]

- For signals $x_1, x_2 : \mathbb{Z}^\geq \rightarrow \mathbb{C}$ ($x_1[k], x_2[k] = 0$ for $k < 0$), with respective ROCs $C_1, C_2 \subset \mathbb{C}$,
  \[
  x_1 \ast x_2 \xrightarrow{Z} \sum_{k=0}^{\infty} (x_1 \ast x_2)[k] z^{-k} = \sum_{k=0}^{\infty} \sum_{v=0}^{k} x_1[v] x_2[k-v] z^{-(k-v)} z^{-v} \\
  = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=v}^{\infty} x_2[k-v] z^{-(k-v)} \\
  = \sum_{v=0}^{\infty} x_1[v] z^{-v} \sum_{k=0}^{\infty} x_2[k] z^{-k} \\
  = X_1(z) X_2(z), \quad z \in C_1 \cap C_2.
  \]

Basic $z$-transform properties: convolution, IVT & FVT

- So convolution in the time-domain is multiplication in the frequency domain.
- The converse is also true.
- Directly by definition of $X = \mathcal{Z}x$, we get the initial value theorem
  \[
  \lim_{z \rightarrow \infty} X(z) = x[0].
  \]
- There is also a "final value" theorem for $\lim_{k \rightarrow \infty} x[k]$.
Total response of SISO LTIC systems

- We now study transient analysis of LTI difference equations using $z$-transforms.

- Recall our system is defined given polynomials $P, Q$, input $f$ and initial conditions:
  - $Q(\Delta^{-1})y = P(\Delta^{-1})f$, where $y$ is the (total) output and
  - input $f[k] = 0$ for $k < 0$,
  - degree of polynomial $Q = n \geq m =$ degree of polynomial $P$ (causal system),
  - $Q(z) = z^n + \sum_{v=0}^{n-1} a_v z^v$ (i.e., $a_n = 1$) and $P(z) = \sum_{v=0}^{m} b_v z^v$,
  - $a_n \neq 0$ or $b_n \neq 0$ for poly’ls $Q, P$ of minimum degree,
  - $n$ initial conditions $y[-n], y[-n+1], ..., y[-2], y[-1]$.

- We can restate the difference equation in terms of delays by delaying both sides by $n$ time-units (i.e., applying with $\Delta^n$), to get

\[
\Delta^n Q(\Delta^{-1})y = \Delta^n P(\Delta^{-1})f
\]

\[
\Rightarrow \bar{Q}(\Delta)y := \sum_{v=0}^{n} a_v \Delta^{n-v}y = \sum_{v=0}^{m} b_v \Delta^{n-v}f =: \bar{P}(\Delta)f
\]

Total response of SISO LTIC systems (cont)

- So, taking the $z$-transform of the (delay) difference equation, we get by the (delay) time-shift and linearity properties that

\[
\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k] z^{-k-v} + \bar{Q}(z^{-1})Y(z) = \bar{P}(z^{-1})F(z)
\]

- So, solving for the total response $Y$ we get

\[
Y(z) = \frac{\bar{P}(z^{-1})}{\bar{Q}(z^{-1})} F(z) - \sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k] z^{-k-v} = Y_ZS(z) + Y_ZI(z)
\]

- where the ZIR and ZSR in the complex-frequency ($z$) domain respectively are

\[
Y_ZI(z) := -\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k] z^{-k-v} = -\sum_{v=0}^{n} a_v \sum_{k=-v}^{n-1} y[k] z^{n-k-v}
\]

\[
Y_ZS(z) := \frac{\bar{P}(z^{-1})}{\bar{Q}(z^{-1})} F(z) = \frac{P(z)}{Q(z)} F(z) = H(z) F(z)
\]

- $H$ is the system transfer function, and $Y$’s RoC is the intersection of those of its characteristic modes, i.e., determined by the characteristic value(s) of largest modulus.
Total response of SISO LTIC systems - example

**•** Suppose i.c. $y[-1] = -1$, input $f[k] = 2(-3)^k u[k]$ and output $y$ s.t.

$$\forall k \geq -1, \quad 2y[k + 1] + 2y[k] = 3f[k + 1] + 2f[k].$$

**•** To find the total response $y$, we take the $z$-transform of the equivalent system: $\forall k \geq 0$,

$$2y[k] + 2y[k - 1] = 3f[k] + 2f[k - 1] \quad \Rightarrow \quad 2Y(z) + 2z^{-1}Y(z) + y[-1] = 3F(z) + 2z^{-1}F(z).$$

**•** So by the delay property for non-causal signals ($y$), the total response

$$Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} F(z) + \frac{-2y[-1]}{2 + 2z^{-1}}$$

$$= H(z) F(z) + \frac{-y[-1]}{1 + z^{-1}}$$

$$= Y_{ZS}(z) + Y_{ZI}(z)$$

with RoC for $Y$ being the intersection of those of $F$ and $H$.

---

Total response of SISO LTIC systems - example

**•** Here $F(z) = \mathcal{Z}\{2(-3)^k\} = 2/(1 + 3z^{-1})$, $y[-1] = -1$, so

$$Y(z) = \frac{3 + 2z^{-1}}{2 + 2z^{-1}} \cdot \frac{2}{1 + 3z^{-1}} + \frac{1}{1 + z^{-1}}$$

$$= \frac{3 + 2z^{-1}}{1 + 3z^{-1} + 1 + z^{-1}} + \frac{1}{1 + z^{-1}}$$

where for the last equality see PFE below (here in $z^{-1}$).

**•** Understanding that the ZIR begins at $k = -1$ (initial condition) and the ZSR at time $k = 0$, we get:

$$\forall k \geq -1, \quad y[k] = (3.5(-3)^k - 0.5(-1)^k)u[k] + (-1)^k = y_{ZS}[k] + y_{ZI}[k],$$

where we minded the ambiguity $\mathcal{Z}x = \mathcal{Z}xf$.

**Exercise:** Verify this solution using time-domain methods, i.e.,

$y = y_{ZI} + y_{ZS} = y_{ZI} + h * f$, where $h$ and $y_{ZI}$ consist of char. modes.
Inverse $z$-transform of proper rational polynomials

- We now describe how to find $Z^{-1}X$ of causal signal $X$ that is rational polynomial in $z$, i.e., $X(z) = M(z)/N(z)$ where $M(z)$ and $N(z)$ are polynomials in $z$.

- If $\deg(M) = \deg(N) + 1$, we perform long division to write $X = c + \bar{M}/N$ where $\deg(N) = \deg(\bar{M})$ and $Z^{-1}X = c\delta + Z^{-1}\{\bar{M}/N\}$.

- If $\deg(M) = \deg(N)$ and $M(0) = 0$ (so $z^{-1}M(z)$ is a polynomial), we can factor $z$ from $M$ to get

$$X(z) = z^{-1}M(z)/N(z).$$

- We will find $Z^{-1}X$ using PFE of the strictly proper rational polynomial $z^{-1}M(z)/N(z)$.

- Alternatively, we could apply PFE on strictly proper rational polynomials in $z^{-1}$, $z^{-K}M(z)/(z^{-K}N(z))$ where $K := \deg(N)$, as in the previous example.

Partial Fraction Expansion (PFE) example in $z$ (not $z^{-1}$)

- For example, suppose

$$X(z) := \frac{z(3z + 2)}{z^2 - 0.64} = \frac{(3z + 2)}{(z + 0.8)(z - 0.8)}$$

$$= z \left( \frac{0.25}{z + 0.8} + \frac{2.75}{z - 0.8} \right) = 0.25 \frac{z}{z + 0.8} + 2.75 \frac{z}{z - 0.8}$$

where PFE (below) gave the numerators (residues) 0.25 and 2.75.

- So,

$$Z^{-1}X[k] = 0.25(-0.8)^k u[k] + 2.75(0.8)^k u[k]$$

- Note that the associated RoC of $X$ is $\{z \in \mathbb{C} \mid |z| > 0.8\}$. 
Partial Fraction Expansion (PFE) - preliminaries

- Let $K = \text{deg}(N) = \text{deg}(M)$ so that we can factor

\[N(z) = \prod_{k=1}^{K} (z - p_k),\]

where the $p_k$ are the roots of $N$ (poles of $M/N$).

- We assume $M$ and $N$ have no common roots, i.e., no "pole-zero cancellation" issue to consider, so that the $p_k$ are the poles of $M/N$.

- Again, we assume $M(0) = 0$ (0 is a zero of $M/N$) and so $z^{-1}M(z)$ is a polynomial of degree $K - 1$.

- Note that the RoC for $M(z)/N(z)$ is $\{ z \in \mathbb{C} \mid |z| > \max_k |p_k| \}$.

PFE - the case of no repeated poles

- Suppose there are no repeated poles for $M/N$, i.e., $\forall k \neq l, \ p_k \neq p_l$.

- In this case, we can write the PFE of $z^{-1}M(z)/N(z)$ as

\[
\frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} \\
\Rightarrow \frac{M(z)}{N(z)} = \frac{z^{-1}M(z)}{N(z)} = \sum_{l=1}^{K} \frac{c_l}{z - p_l} = \sum_{l=1}^{K} \frac{1}{z - p_l} = \frac{z^{-1}M(z)}{N(z)}(z - p_l) \bigg|_{z = p_l}.
\]

where the scalars (Heaviside coefficients) $c_l \in \mathbb{C}$ are

\[
c_l = \left. \frac{z^{-1}M(z)}{\prod_{k \neq l}(z - p_k)} \right|_{z = p_l} = \lim_{z \to p_l} \frac{z^{-1}M(z)}{N(z)}(z - p_l) = \frac{z^{-1}M(z)}{N(z)}(z - p_l) \bigg|_{z = p_l}.
\]

- That is, to find the Heaviside coefficient $c_k$ over the term $z - p_k$ in the PFE, we have removed (covered up) the term $z - p_k$ from the denominator $N(z)$ and evaluated the remaining rational polynomial at $z = p_k$.

- This approach, called the Heaviside cover-up method, works even when $p$ is $\mathbb{C}$-valued.

- Given the PFE of $z^{-1}M/N$, $(z^{-1}M/N)[k] = \sum_{l=1}^{K} c_l p_l^k u[k]$. 

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To prove that the above formula for the Heaviside coefficient $c_l$ is correct, note that the claimed PFE of $z^{-1}M(z)/N(z)$ is

$$
\sum_{l=1}^{K} \frac{c_l}{z - p_l} = \frac{\sum_{l=1}^{K} c_l \prod_{k \neq l}(z - p_k)}{N(z)}
$$

Thus, the PFE equals $z^{-1}M(z)/N(z)$ if and only if the numerator polynomials are equal, i.e., iff

$$
z^{-1}M(z) = \sum_{l=1}^{K} c_l \prod_{k \neq l}(z - p_k) =: \hat{M}(z).
$$

Again, two polynomials are equal if their degrees, $L$, are equal and either:

– their coefficients are the same, or

– they agree at $L + 1$ (or more) different points, e.g., two lines ($L = 1$) are equal if they agree at 2 ($= L + 1$) points.

Since $z^{-1}M(z)$ is a polynomial of degree $< K$, it suffices to check that whether $z^{-1}M(z) = \hat{M}(z)$ for all $z = p_k$, $k \in \{1, 2, ..., K\}$, i.e., this would create $K$ equations in $< K$ unknowns (the coefficients of $\hat{M}$).

But note that any pole $p_r$ of $z^{-1}M(z)/N(z)$ is a root of all but the $r^{th}$ term in $\hat{M}$, so that

$$
\hat{M}(p_r) = c_r \prod_{k \neq r}(p_r - p_k)
$$

$$
= \left(\frac{z^{-1}M(z)}{\prod_{k \neq r}(z - p_k)}\right)_{z = p_r} \prod_{k \neq r}(p_r - p_k)
$$

$$
= \frac{p_r^{-1}M(p_r)}{\prod_{k \neq r}(p_r - p_k)} \prod_{k \neq r}(p_r - p_k)
$$

$$
= p_r^{-1}M(p_r).
$$

Q.E.D.
PFE - the case of no repeated poles - example

- To find the inverse \(z\)-transform of a proper rational polynomial \(X = M/N\) with \(M(0) = 0\), first factor its denominator \(N\) and factor \(z\) from \(M\), e.g.,

\[
X(z) = \frac{z^3 + 5z^2}{z^3 + 9z^2 + 26z + 24} = \frac{z^2 + 5z}{(z + 4)(z + 3)(z + 2)}, \text{ for } |z| > 4.
\]

- So, by PFE

\[
X(z) = z\left(\frac{c_4}{z + 4} + \frac{c_3}{z + 3} + \frac{c_2}{z + 2}\right) = z\frac{z^{-1}M(z)}{N(z)} \Rightarrow
\]

\[
z^{-1}M(z) = 1z^2 + 5z + 0 = c_4(z + 3)(z + 2) + c_3(z + 4)(z + 2) + c_2(z + 4)(z + 3) =: \tilde{M}(z).
\]

- We can solve for the 3 constants \(c_k\) by comparing the 3 coefficients of quadratic \(M\) and \(\tilde{M}\).

- The Heaviside cover-up method suggests we try \(z = -2, -3, -4\) to solve for \(c_2, c_3, c_4\):

\[
c_4 = \frac{z^2 + 5z}{(z + 3)(z + 2)} \bigg|_{z=-4} = -2, \quad c_3 = \frac{z^2 + 5z}{(z + 4)(z + 2)} \bigg|_{z=-3} = 6, \quad c_2 = \frac{z^2 + 5z}{(z + 4)(z + 3)} \bigg|_{z=-2} = -3
\]

- Thus, \(x[k] = (z^{-1}X)[k] = (-2(-4)^k + 6(-3)^k - 3(-2)^k)u[k]\).

PFE - the case of a non-repeated, complex-conjugate pair of poles

- Again, recall that for polynomials with all coefficients \(\in \mathbb{R}\), all complex poles will come in complex-conjugate pairs, \(p_1 = \bar{p}_2\).

- The case of non-repeated poles \(p_1, p_2 = \alpha \pm j\beta (\alpha, \beta \in \mathbb{R}, j := \sqrt{-1})\) that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients \(c_1, c_2\), i.e., \(c_1 = \bar{c}_2\).

- In the PFE, we can alternatively combine the terms

\[
\frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} = \frac{r_1z + r_0}{(z - \alpha)^2 + \beta^2}
\]

where by equating the two numerator polynomials’ coefficients,

\[
r_0 = -c_1p_2 - c_2p_1 = -2\text{Re}\{c_1p_2\} \in \mathbb{R} \quad \text{and} \quad r_1 = c_1 + c_2 = 2\text{Re}\{c_1\} \in \mathbb{R}.
\]

- Exercise: Show that

\[
2|c|\cdot |p|^k \cos(k\angle p + \angle c) \xrightarrow{z} \frac{cz}{z - p} + \frac{\bar{c}z}{z - \bar{p}}
\]
To find the inverse $z$-transform of
$$X(z) = \frac{3z^2 + 2z}{z^3 + 5z^2 + 10z + 12},$$
first factor the denominator and divide the numerator by $z$ to get
$$X(z) = \frac{3z + 2}{(z^2 + 2z + 4)(z + 3)}.$$

Note that the poles of $X$ are $-3$ and $-1 \pm j\sqrt{3}$ (so $X$’s RoC is $|z| > 3$).

So, we can expand $X$ to
$$X(z) = \frac{r_1 z + r_0}{z + 2z + 4} + \frac{c_3}{z + 3},$$
where by the Heaviside cover-up method,
$$c_3 = \frac{3z + 2}{z^2 + 2z + 4} \bigg|_{z=-3} = -1.$$

## PFE - non-repeated complex-conjugate pair of poles - example (cont)

To find $r_1, r_0$, we will compare coefficients of the numerator polynomials of $X$ (actually $z^{-1}X$) and its PFE, i.e.,
$$0z^2 + 3z + 2 = (r_1 z + r_0)(z + 3) + c_3(z^2 + 2z + 4) \quad (*)$$
$$= (r_1 - 1)z^2 + (3r_1 + r_0 - 2)z + 3r_0 - 4.$$

Thus, by comparing coefficients
$$0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4$$
we get
$$r_0 = 2 \quad \text{and} \quad r_1 = 1.$$

Note how $z = -3$ in $(*)$ gives $c_3 = -1$ as Heaviside cover-up did.
• Thus by substituting, we get

\[ X(z) = \frac{z^2 + 2}{z^2 + 2z + 4} + \frac{-1}{z + 3} \]

• Exercise: Show that

\[ x[k] = (Z^{-1}X)[k] = \left(\sqrt{4/3} \cdot 2^k \cos(k2\pi/3 - \pi/6) - (-3)^k \right) u[k]. \]

---

PFE - the general case of repeated poles

• If a particular pole \( p \) of \( z^{-1}M(z)/N(z) \) is of order \( r \geq 1 \), i.e., \( N(z) \) has a factor \((z - p)^r\), then the PFE of \( z^{-1}M(z)/N(z) \) has the terms

\[
\frac{c_1}{z - p} + \frac{c_2}{(z - p)^2} + \ldots + \frac{c_r}{(z - p)^r} = \sum_{k=1}^{r} \frac{c_k}{(z - p)^k} = \frac{z^{-1}M(z)}{N(z)} - \Phi(z)
\]

with \( c_k \in \mathbb{C} \forall k \in \{1, 2, ..., r\} \), where \( \Phi(z) \) represents the other PFE terms of \( z^{-1}M(z)/N(z) \) (i.e., corresponding to poles \( \neq p \)).

• Note that equating \( z^{-1}M(z)/N(z) \) to its PFE and multiplying by \((z - p)^r\) gives

\[
\frac{z^{-1}M(z)}{N(z)}(z - p)^r = c_r + \sum_{k=1}^{r-1} c_k(z - p)^{r-k} + \Phi(z)(z - p)^r
\]

\[
\Rightarrow \left. \frac{z^{-1}M(z)}{N(z)}(z - p)^r \right|_{z=p} = c_r,
\]

i.e., Heaviside cover-up (of the entire term \((z - p)^r\)) works for \( c_r \).
PFE - the general case of repeated poles (cont)

- To find \( c_{r-1} \), we differentiate the previous display to get

\[
\frac{d}{dz} \frac{z^{-1}M(z)}{N(z)} (z - p)^r = \sum_{k=1}^{r-1} c_k (r-k)(z-p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z-p)^r
\]

\[
= c_{r-1} + \sum_{k=1}^{r-2} c_k (r-k)(z-p)^{r-1-k} + \frac{d}{dz} \Phi(z)(z-p)^r
\]

\[
\Rightarrow c_{r-1} = \left. \left( \frac{d}{dz} \frac{z^{-1}M(z)}{N(z)} (z-p)^r \right) \right|_{z=p}
\]

- If we differentiate the original display \( k \in \{0, 1, 2, ..., r-1\} \) times and then substitute \( z = p \), we get (with 0! := 1)

\[
\left. \left( \frac{d^k}{dz^k} \frac{z^{-1}M(z)}{N(z)} (z-p)^r \right) \right|_{z=p} = k!c_{r-k}
\]

\[
\Rightarrow c_{r-k} = \frac{1}{k!} \left. \left( \frac{d^k}{dz^k} \frac{z^{-1}M(z)}{N(z)} (z-p)^r \right) \right|_{z=p}
\]

---

PFE - the general case of repeated poles - example

- To find the inverse \( z \)-transform of

\[
X(z) = \frac{z(3z + 2)}{(z+1)(z+2)^3}
\]

write the PFE of \( X \) as

\[
X(z) = z^{-c_{1}} + z^{-c_{2,1}} + z^{-c_{2,2}} \frac{c_{2,2}}{(z+2)^2} + z^{-c_{2,3}} \frac{c_{2,3}}{(z+2)^3}
\]

so clearly the RoC of causal \( X \) is \( |z| > 2 \).

- By Heaviside cover-up

\[
c_1 = \frac{3z+2}{(z+2)^3} \bigg|_{z=-1} = -1 \quad \text{and} \quad c_{2,3} = \frac{3z+2}{z+1} \bigg|_{z=-2} = 4.
\]
• Also,
\[ c_{2,2} = \frac{1}{1!} \left( \frac{d}{dz} \frac{3z + 2}{z + 1} \right) \bigg|_{z=-2} = \frac{1}{1! (z+1)^2} \bigg|_{z=-2} = 1 \]
\[ c_{2,1} = \frac{1}{2!} \left( \frac{d^2}{dz^2} \frac{3z + 2}{z + 1} \right) \bigg|_{z=-2} = \frac{1}{2! (z+1)^3} \bigg|_{z=-2} = 1 \]

• Thus,
\[ X(z) = \frac{z-1}{z+1} + z \frac{1}{z+2} + z \frac{1}{(z+2)^2} + z \frac{4}{(z+2)^3} \quad \forall |z| > 2 \]
\[ \Rightarrow x[k] = (Z^{-1}X)[k] = \left( (-1)^k + (-2)^k + k(-2)^{k-1} + 4 \frac{k(k-1)}{2} (-2)^{k-2} \right) u[k] \]

• **Exercise:** Show by induction and integration by parts that: \( \forall m \in \mathbb{Z}^{>0}, \)
\[ \left( \frac{k}{m} \right) \zeta^{k-m} u[k] \xrightarrow{z} \frac{z}{(z-\gamma)^m} \]

• **Exercise:** Find the ZSR \( y \) to input \( f[k] = 2^k u[k] = 2e^{ik\pi/2} u[k] \) of the marginally stable system \( H(z) = \frac{4}{(z^2 + 1)}. \)

### PFE of \( M/N \) when \( M(0) \neq 0 \)

• If \( M(0) \neq 0 \) (so cannot factor \( z \) from \( M(z) \)), then just perform long division if \( \deg(M) \geq \deg(N) \) to get a strictly proper rational polynomial, factor \( N \) to find the poles, and find the PFE as before.

• When taking inverse \( z \)-transform, recall the \( z \)-transform pair
\[ \beta^{k-1}u[k-1] \xrightarrow{z} \frac{1}{z-\beta}, \quad |z| > |\beta| \]

• For example, to find the ZSR to \( f[k] = 2(-1)^k u[k] \) of the system \( y[k+2] - 4y[k + 1] = 5f[k+1] \), take \( z \)-transforms of the proper form \( y[k+1] - 4y[k] = 5f[k] \) to get
\[ Y(z) = H(z)F(z) = \frac{5}{z-4} F(z) = \frac{10z}{(z-4)(z+1)} \]
\[ = \frac{8}{z-4} + \frac{2}{z+1} \quad \text{(by PFE)} \]
\[ \Rightarrow y[k] = 8(4)^{k-1}u[k-1] + 2(-1)^{k-1}u[k-1] \]

• **Exercise:** Repeat for the system \( y[k+1] - 4y[k] = 5f[k + 1] \).

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The total response of a SISO LTI system to input $f$ is of the form

$$Y(z) = H(z)F(z) + \frac{P_1(z)}{Q(z)} = \frac{P(z)}{Q(z)}F(z) + \frac{P_1(z)}{Q(z)} = Y_{ZS}(z) + Y_{ZI}(z).$$

where $P_1$ depends on the initial conditions and the RoC is the intersection of that of input $F = Zf$ and the system characteristic modes.

Unlike for DTFT notation, we here write $H(z) = Q(z)/P(z) = (Zh)(z)$.

Suppose the system is BIBO/asymptotically stable and the input is a sinusoid at frequency $\Omega_o$, $f[k] = Ae^{j(\Omega_o k + \phi)}u[k] = Ae^{j\phi}(e^{j\Omega_o})^ku[k]$ with $A > 0$

$\Rightarrow F(z) = Ae^{j\phi}z/(z - e^{j\Omega_o})$ with RoC $|z| > 1$.

Since $e^{j\Omega_o}$ cannot be a system pole (owing to asymptotic stability all poles have modulus strictly less than one), we can use Heaviside cover-up on

$$Y_{ZS}(z) = H(z)F(z) = z\frac{P(z)}{Q(z)}(z - e^{j\Omega_o})Ae^{j\phi} \quad \text{to get}$$

$$Y_{ZS}(z) = \frac{1}{z - e^{j\Omega_o}}Ae^{j\phi} + \text{char. modes} = H(e^{j\Omega_o})F(z) + \text{char. modes}.$$

Thus, the total response of an asymptotically stable system to a sinusoidal input $f$ at frequency $\Omega_o$ is

$$y[k] = H(e^{j\Omega_o})f[k] + \text{linear combination of characteristic modes}.$$

So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as $k \to \infty$,

$$y[k] \to H(e^{j\Omega_o})f[k] = H(e^{j\Omega_o})Ae^{j(\Omega_o k + \phi)} = |H(e^{j\Omega_o})|Ae^{j(\Omega_o k + \phi + \angle H(e^{j\Omega_o}))},$$

where again,

- $H = P/Q$ is the system’s transfer function,
- $|H(e^{j\Omega_o})|$ is the system’s magnitude response at freq. $\Omega_o$, and
- $\angle H(e^{j\Omega_o})$ is the system’s phase response at freq. $\Omega_o$. 
PFE and eigenresponse for asymptotically stable systems (cont)

- Laplace’s approximation: the rate at which the total response converges to the eigenresponse response is according to the characteristic value of largest modulus,
  - which will be $< 1$ owing to the stability assumption,
  - i.e., giving the modes(s) that $\to 0$ slowest.

- In continuous-time systems, it’s the characteristic value of largest real part, which will be negative owing to stability assumption.

Canonical system-realizations - direct form

- Consider the proper ($m \leq n$) transfer function
  \[ H(z) = \frac{P(z)}{Q(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0} = \frac{Y(z)}{F(z)} \]

- The direct-form realization employs the interior system state $X := F/Q$, i.e., $F = QX$ and $Y = PX$ where the former implies (with $a_n = 1$),
  \[ F(z) = \sum_{r=0}^{n} a_r z^r X(z) \Rightarrow z^n X(z) = F(z) - \sum_{r=0}^{n-1} a_r z^r X(z). \]

- For $n = 2$, there are two "system states" (outputs of unit delays), $X$ and $zX$ (respectively, $x[k]$ and $(\Delta^{-1} x)[k] = x[k+1]$):

![Diagram](image-url)
• Now adding \( Y = PX \), we finally get the direct-form canonical system-realization of \( H \):

\[
\begin{align*}
\dot{x}_1 & = -b_1 x_1 + b_2 X + d_1 F \\
\dot{x}_2 & = -b_3 x_2 + b_4 X + d_2 F \\
Y & = b_0 X + b_4 X
\end{align*}
\]

• Again, state variables taken as outputs of unit delays, here: \( x, \Delta^{-1}x, \ldots, \Delta^{-(n-1)}x \).

• If \( b_n = b_2 \neq 0 \), there is direct coupling of input and output, \( H \) is proper but not strictly so, \( h = \mathcal{Z}^{-1}H \) has a unit-pulse component \( b_2 \delta \).

---

**Canonical system realizations - direct form (cont)**

• Note that this \( n = 2 \) example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for \( n = 2 \), \( H(z) = P(z)/Q(z) \) where

\[
Q(z) = z^2 + a_1 z + a_0 = (z - \alpha)^2 + \beta^2
\]

for \( \alpha, \beta \in \mathbb{R} \), so the poles are \( \alpha \pm j\beta \).
Canonical system realizations by PFE

- In the general case of a proper transfer function, we can use partial-fraction expansion
  - grouping the terms corresponding to a complex-conjugate pair of poles, i.e., a second-order denominator, and
  - using a direct-form realization for these terms.
- Besides the PFE-based and direct-form realizations, there are other (zero-state) system realizations, e.g., "observer" canonical.
- For proper rational-polynomial transfer functions \( H = P/Q \), all of these realizations involve \( n (\equiv \text{degree of } Q) \) unit delays, the output of each being a different interior state variable of the system.

Canonical system realizations by PFE - example

\[
H(z) = 0.3 \frac{z^2 - 0.1}{z^2 - 0.1z - 0.3} = 0.3 + \frac{0.3z - 0.01}{(z-0.6)(z+0.5)} = 0.3 + \frac{17/1.1}{z-0.6} + \frac{16/1.1}{z+0.5}
\]

Note that one cannot factor \( z \) from the numerator of \( H \).

Exercise: Find a realization for this transfer function \( H \) by

1. splitting/forking the input signal \( F \),
2. using the direct canonical form for each of these 3 terms of \( H \) found by long division and PFE, and
3. summing three resulting output signals to get the (ZS) output \( Y = HF \).
Digital Proportional-Integral (PI) system

- Consider a continuous-time signal \( x \) sampled every \( T \) seconds,
  \[
  \forall k \in \mathbb{Z}^+, \ x[k] = x(kT),
  \]
  and its integral \( y(t) = \int_0^t x(\tau) d\tau \).

- The sampled integral can be approximated, \( y(kT) \approx y[k] \), by the trapezoid rule,
  \[
  y[k] = y[k - 1] + \frac{x[k - 1] + x[k]}{2} T.
  \]

- In the complex-frequency domain,
  \[
  Y(z) = Y(z)z^{-1} + \frac{X(z)z^{-1} + X(z)T}{2}
  \]
  \[
  \Rightarrow \frac{Y(z)}{X(z)} = \frac{T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}.
  \]

Digital PI system (cont)

- So, a digital PI transfer function would be of the form,
  \[
  G(z) = K_p + \frac{K_i T}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}}.
  \]
  for constants \( K_p, K_i \).

- In practice, PID or PI systems \( G \) are commonly used to control a plant \( H \), where \( G \) may be in series with \( H \) or in the feedback branch.

- Exercises:
  - Draw the direct-form canonical realization for \( G \).
  - Draw the block diagram for the closed-loop system with negative feedback:
    \( Y = HX \) and \( X = F - GY \) where \( H \) is the (open-loop) system.
  - Find the closed-loop transfer function \( Y/F \).
Recursive Least Squares (RLS) Filter - Introduction

- Consider a LTI system with input \( f \) and output \( y \),
  \[
  y[k] = \sum_{r=0}^{K} h[k-r]f[r] + v[k], \quad k \in \mathbb{Z},
  \]
  where \( v \) is an additive noise process and \( K \) is the maximum system order.

- The system (unit-pulse response) \( h \) is not known.

- Past values of the output \( y \) are observed (known).

- At time \( k \), the objective is to forecast the next output \( \hat{y}[k+1] \), based on the assumed known/observed quantities:
  - the next input \( f[k+1] \),
  - the past \( R \) input-output pairs \( \{f[r], y[r]\}_{k-R+1 \leq r \leq k} \).

---

RLS objective and \( R \)th-order linear tap filter

- The output of an \( R \)th-order RLS tap-filter at time \( k \) is,
  \[
  \hat{y}_k[i] = \sum_{r=i-R+1}^{i} \eta_k[i-r]f[r], \quad i \leq k + 1.
  \]

- The objective of this filter at time \( k \) is to accurately estimate the system output \( y[k+1] \) with \( \hat{y}_k[k+1] \) by choosing the \( R \) filter coefficients
  \( \eta_k[k-R+1], \ldots, \eta_k[k-1], \eta_k[k] \)
  that minimize the following sum-of-square-error objective:
  \[
  \mathcal{E}_k = \sum_{r=k-R+1}^{k} \lambda^{k-r} |y[r] - \hat{y}_k[r]|^2 = \sum_{r=k-R+1}^{k} \lambda^{k-r} |e_k[r]|^2
  \]
  where
  - \( \lambda > 0 \) is a forgetting factor and
  - error \( e_k[r] := y[r] - \hat{y}_k[r] \).
So, to minimize $\mathcal{E}_k$, substitute $\hat{y}_k[r]$ into $\mathcal{E}_k$ and solve
\[
0 = \frac{\partial \mathcal{E}_k}{\partial \eta_k[i]} \quad \text{for} \quad i \in \{k - R + 1, \ldots, k - 1, k\}.
\]

That is, $R$ equations in $R$ unknowns: for $i \in \{k - R + 1, \ldots, k - 1, k\},$
\[
0 = \sum_{r=k-R+1}^{k} 2\lambda^{k-r} e_k[r] \frac{\partial e_k[r]}{\partial \eta_k[i]}
\]
\[
= \sum_{r=k-R+1}^{k} 2\lambda^{k-r} (y[r] - \hat{y}_k[r]) \left(-\frac{\partial \hat{y}_k[r]}{\partial \eta_k[i]}\right)
\]
\[
= \sum_{r=k-R+1}^{k} 2\lambda^{k-r} (\hat{y}_k[r] - y[r]) f[r - i]
\]

Exercise: Prove the last equality.

Substituting $\hat{y}_k[r]$, rewrite these equations to get the following $R$ equations in $R$ unknowns $\eta_k[i]$ that are $\mathcal{E}_k$-minimizing: for $i \in \{k - R + 1, \ldots, k - 1, k\},$
\[
\sum_{r=k-R+1}^{k} \lambda^{k-r} f[r - i] \sum_{\ell=r-R+1}^{r} f[\ell] \eta_k[r - \ell] = \sum_{r=k-R+1}^{k} \lambda^{k-r} y[r] f[r - i]
\]

Exercise: Prove the last equality and write it in matrix form.

Exercise: Research how the $\mathcal{E}_k$-minimizing filter parameters $\eta_k$ can be computed recursively, i.e., using $\eta_{k-1}$.

The filter order $R$ can also be “trial adapted” to discover the system order $K$ so that the error-minimizing filter parameters $\eta_k$ “track” the system unit-pulse response $h$ over time $k$.

Note the required initial “warm-up” period of $R$ time-units where the outputs of system $h$ are simply observed and recorded and no estimates are made.

Exercise: If there was no additive noise process $v$ and the system unit-pulse response $h$ had finite support (i.e., a FIR system with $K < \infty$), show how $h$ can be deduced from input-output $(f, y)$ observations.