Introduction to Frequency-Domain Analysis of Continuous-Time, Linear and Time-Invariant Systems

• Time-domain analysis of transient response
• Fourier series of periodic Dirichlet signals
• Bode plots of system frequency-response
• Bilateral Fourier transform for zero-state response
• Unilateral Laplace transform for transient response

Time-domain analysis of continuous-time LTI systems

• Signals: properties, operations, construction, important signals
• Single-input, single-output systems: properties
• The zero-input response (ZIR): characteristic values and modes
• The zero (initial) state response (ZSR): the impulse response, convolution
• System stability
• The (zero state) system transfer function and eigenresponse
• Resonance and step-response of second-order tuned circuits
A function \( x \) is a mapping of elements of a domain \( A \) to a range \( B \), i.e.,

\[
x : A \to B,
\]

in such a way that for all elements \( t \) of \( A \), \( x(t) \) is a single element of \( B \), i.e.,

\[
\forall t \in A, \ x(t) \in B.
\]

• The strict range of \( x \) is \( \{ x(t) \mid t \in A \} \subset B \).

• For example \( x(t) = t^2, t \in \mathbb{R} \), has strict range the non-negative real numbers,

\[
\mathbb{R}^{\geq 0} : = \{ z \in \mathbb{R} \mid z \geq 0 \} =: [0, \infty)
\]

where “\( a := b \)” means true by definition of \( a \).

• In this course, we will use the words “function” and “signal” interchangeably.

• Also, we will consider a one-dimensional, ordered domain \( A \) for our signals, and associate that domain with time.

• Note that in image processing, signals (images) will have two-dimensional spatial domains.

• Video signals are three-dimensional (two-dimensional images as a function of time).

Signal/function types - discrete and continuous time (cont)

• If the domain \( A \) has “countably” many values, then \( x : A \to B \) is said to be discrete time.

• A set is countable if there is a one-to-one mapping between it and a subset of the set of integers, \( \mathbb{Z} \).

• In this course, we will consider only continuous time-sIGNALS \( x \) whose domains are subsets of the real line, \( \mathbb{R} \), typically either \( A = \mathbb{R} = (-\infty, \infty) \) or \( A = \mathbb{R}^{\geq 0} = [0, \infty) \), and hence uncountably infinite in size.

• We will also consider signals that are either real or complex valued, i.e., their range \( B = \mathbb{R} \) or \( B = \mathbb{C} \).
A signal operation/transformation - time shift

- For all signals $x : \mathbb{R} \to \mathbb{R}$ and scalars $\tau \in \mathbb{R}$, define a new signal $\Delta_\tau x$ as
  $$\forall t \in \mathbb{R}, \quad (\Delta_\tau x)(t) = x(t - \tau),$$
  i.e., the signal $x$ delayed by $\tau$ seconds (equivalently, advanced by $-\tau$ seconds).

- An example pulse $x$ is depicted in Figure (a).
- $\Delta_{-2} x$ ($x$ delayed by -2 seconds, or advanced by 2 seconds) is depicted in Figure (b); note that $2 = (\Delta_{-2} x)(-1) = x(-1 - (-2)) = x(1)$.
- $\Delta_2 x$ ($x$ delayed by 2 seconds) is depicted in Figure (c); note that $2 = (\Delta_2 x)(3) = x(3 - 2) = x(1)$.

Periodic and aperiodic signals

- For the following introductory discussion, we will use examples where $x : \mathbb{R} \to \mathbb{R}$.
- A signal is said to be periodic if there is a fixed real number $T \neq 0$ such that
  $$\forall t \in \mathbb{R}, \quad x(t) = x(t - T).$$
- The smallest such $T > 0$ of a periodic signal $x$ is said to be the period of $x$.
- We take constant (DC) signals, i.e.,
  $$\exists c \in \mathbb{R} \text{ such that } \forall t \in \mathbb{R}, \quad x(t) = c,$$
  to be periodic with “fundamental frequency” $1/T = 0$ (which motivates us to choose their period $T = \infty$ by definition).
- A periodic signal with period $T$ will be called $T$-periodic.
- A signal that is not periodic is said to be aperiodic.
Periodic signals - examples

- For example, the sinusoid \( x(t) = \sin(5t) \), \( t \in \mathbb{R} \), has period \( 2\pi/5 \).

- The signal (a) in the following figure has period 4.

- The truncated exponential signal (b) is aperiodic.

Expressing periodicity with the delay operator

- Let’s recall the time-delay operator \( \Delta \) so as to restate the definition of periodicity in functional form.

- A signal \( x \) is said to be periodic if there is a real \( T \neq 0 \) such that

\[
x = \Delta_T x,
\]

recalling that this statement expresses functional equivalence, i.e.,

\[
\forall t \in \mathbb{R}, \quad x(t) = (\Delta_T x)(t) \equiv x(t - T).
\]

- So if \( x \) is \( T \)-periodic (for \( T > 0 \)) then \( x \) is an “eigenfunction” of \( \Delta_nT \) for all \( n \in \mathbb{Z} \).
Sums of periodic signals

- A periodic signal with period $T > 0$ "repeats itself" every $nT$ seconds, for any integer $n > 0$;
- that is, the pattern of the signal is repeated every consecutive interval of duration $nT$ seconds.
- Suppose $x$ and $y$ are periodic signals with periods $T_x > 0$ and $T_y > 0$, respectively.
- If there are positive integers $n_x > 0$ and $n_y > 0$ such that $n_x T_x = n_y T_y =: T$, then the sum of the signals, $x + y$, will repeat itself every $T$ seconds, where to be clear,
  \[ \forall t \in \mathbb{R}, \ (x + y)(t) := x(t) + y(t); \]
  i.e., $x + y$ will also be a periodic signal with period $\leq T$.

- The converse statement is also true, leading to:
- If $x$ and $y$ are periodic, then: $x + y$ is periodic if and only if $T_x/T_y$ is rational ($= n_y/n_x$ for integers $n_x, n_y > 0$).

Sums of periodic signals - examples
• Consider the following periodic signals: for \( t \in \mathbb{R} \),
\[
x_1(t) = 5 \cos(7t + 1), \quad x_2(t) = 13 \sin(12t - 1),
\]
\[
y_1(t) = 4 \sin(3\pi t + \pi/2), \quad y_2(t) = \sqrt{2} \sin(5\pi t - \pi/6)
\]
• \( x_1 + x_2 \) is periodic because the period of \( x_1 \) is \( 2\pi/7 \) and that of \( x_2 \) is \( 2\pi/12 \); so the ratio of their periods, \( 7/12 \) (or \( 12/7 \)), is rational.

• \( y_1 + y_2 \) is periodic because the period of \( y_1 \) is \( 2\pi/(3\pi) = 2/3 \) and that of \( y_2 \) is \( 2\pi/(5\pi) = 2/5 \); so, the ratio of their periods, \( 5/3 \) (or \( 3/5 \)), is rational.

• In these cases, the period of the sum is the least common integer multiple of the components’ periods, e.g., the period of \( x_1 + x_2 \) is \( 2\pi \), while that of \( y_1 + y_2 \) is \( 2 \).

• However, \( x_1 + y_2 \) is aperiodic because the ratio of their periods is \( (2\pi/7)/(2/5) = 5\pi/7 \) which is not rational (\( \pi = 3.1415... \) is not a rational number).

• Adding a constant to a periodic signal results in a periodic signal, i.e., if \( x \) is \( T \)-periodic and \( c \in \mathbb{R} \), then \( x + c \) is \( T \)-periodic where \( (x + c)(t) := x(t) + c, \ t \in \mathbb{R} \).

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**Energy signals**

• A signal \( x : \mathbb{R} \to \mathbb{R} \) is said to be an “energy signal” if
\[
0 < E_x := \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty.
\]

• Note that if \( x \) is either the voltage or current signal associated with a resistive load, then \( E_x \) would be proportional to the energy dissipated by the load.

• For example, for parameters \( A, T, b \in \mathbb{R} \) with \( b > 0 \), if
\[
x(t) = \begin{cases} 
A e^{-b(t-T)} & \text{if } t \geq T \\
0 & \text{if } t < T,
\end{cases}
\]
then
\[
E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = A^2 \int_T^{\infty} e^{-2b(t-T)} dt = A^2/(2b);
\]
since \( E_x = A^2/(2b) < \infty \), \( x \) is an energy signal.

• The exponential signal \( x(t) = -7e^{-5t}, \ t \in \mathbb{R} \), is not an energy signal because \( E_x = \infty \).
Power signals

- A signal $x : \mathbb{R} \to \mathbb{R}$ is said to be a “power signal” if the following limit exists and

$$0 < P_x := \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 dt < \infty,$$

i.e., the $P_x$ is average energy per unit time, or the power, of $x$.

- Note that if $x$ was an energy signal, then $P_x = 0$.

- Conversely, if $P_x > 0$, then $E_x = \infty$.

- If a signal $x$ is periodic with period $T$, then

$$P_x = \frac{1}{T} \int_{0}^{T} |x(t)|^2 dt$$

where $\int_{T}$ denote integration over any interval of $\mathbb{R}$ of length $T$, i.e., over any period of $x$.

- To see why, note that if $\tau/T$ is an integer, then the numerator of $P_x$ can be written as a sum of $\tau/T$ identical, consecutive integrals over periods of length $T$.

- Power signals are not necessarily periodic, though all of our examples will be.

Power versus energy signals - examples

- Fig. (b) is an energy signal with

$$E_g = \int_{-\infty}^{\infty} g^2(t) dt = \int_{-1}^{\infty} 4e^{-2(t+1)} dt = -2e^{-2(t+1)} \bigg|_{-1}^{\infty} = 2.$$

- Fig. (a), a wave with period $T = 4$, is a power signal with

$$P_f = \frac{1}{4} \int_{-2}^{2} f^2(t) dt = \frac{1}{4} \left( \int_{-2}^{0} 1^2 dt + \int_{0}^{2} (1-t)^2 dt \right) = \frac{t^3}{12} \bigg|_{-2}^{0} - \frac{1}{12} (1-t)^3 \bigg|_{0}^{2} = \frac{2}{3}.$$

- Adding a non-zero constant $c \in \mathbb{R}$ to (i.e., spatially shifting) any energy signal $x$ gives a power signal $x + c$ with $P_x = c^2$.

- Also, for any $T$-periodic, power-signal $x$ and non-zero constant $c \in \mathbb{R}$, $P_{x+c} = P_x + c^2$ only if $x$ has zero mean, i.e., $\int_{x} x(t) dt = 0$, cf. Parseval’s theorem.
Power versus energy signals - examples

- Consider a real-valued sinusoid of canonical form (referenced to cosine):
  \[ x(t) = A \cos(w_0 t + \phi), \quad t \in \mathbb{R}, \]
  where the parameter \( A > 0, A \in \mathbb{R} \) (i.e., \( A \in \mathbb{R}^>0 \)) is the amplitude, period \( T = 2\pi/w_0 \in \mathbb{R}^>0 \), and phase \( \phi \in \mathbb{R} \) (since cosine has period \( 2\pi \), we can take \( \phi \in [-\pi, \pi) \)).

- A sinusoid is a power signal with
  \[
  P_x = \frac{1}{2\pi/w_0} \int_0^{2\pi/w_0} A^2 \cos^2(w_0 t + \phi) dt \\
  = \frac{A^2 w_0}{2\pi} \left[ \frac{1}{2} \cos(2w_0 t + 2\phi) + \frac{1}{2} \right]_{0}^{2\pi/w_0} \\
  = \frac{A^2}{2}.
  \]

- Thus, the root mean-square (RMS) value of a sinusoid \( x \) is \( \sqrt{P_x} = A/\sqrt{2} \).

Periodic extensions of aperiodic signals of finite support

- The support of a signal \( x \) is the interval of time over which it is not zero.
- A signal of finite support is sometimes called a pulse.

- In this example, the pulse \( x \) has support of duration 4.
- For \( T \geq 4 \), the \( T \)-periodic extension of \( x \) depicted in this figure is
  \[
  x_T(t) := \sum_{n=-\infty}^{\infty} x(t - nT) = \sum_{n=-\infty}^{\infty} (\Delta_{nT} x)(t), \quad t \in \mathbb{R},
  \]
  i.e., superposition of time-shifted pulses \( x \) by all integer \( n \) multiples of \( T \).
- If \( x \) is an energy signal then \( x_T \) will be a power signal with \( P_{x_T} = E_x/T \).
Neither all signals of finite support are energy signals nor all periodic signals are power signals

- Consider the signal
  \[ x(t) = \begin{cases} \frac{1}{1-t} & 0 \leq t < 1 \\ 0 & \text{else} \end{cases} \]
  - The support of this signal is of duration 1.
  - But note that \( \lim_{t \to 1} x(t) = \infty \), i.e., the signal has finite escape time.
  - This signal is not an energy signal because \( E_x = \infty \).
  - Also, any periodic extension of this signal would not be a power signal.

Bounded signals

- A signal \( x \) is bounded if there exists a real \( M < \infty \) such that
  \[ \forall t \in \mathbb{R}, \ |x(t)| \leq M. \]
  - Obviously, the signal of the previous example is not bounded.
- The sinusoid \( x(t) = A \cos(w_o t + \phi), \ t \in \mathbb{R} \) is bounded by \( M = A > 0 \).
- The signal \( x(t) = c + A \cos(w_o t + \phi), \ t \in \mathbb{R}, \) where the constant \( c \in \mathbb{R} \), is bounded by \( M = \max\{A + c, | - A + c|\} \) (note that \( c \) may be negative).
- Note that the signal \( x(t) = 1/\sqrt{t} \) for \( t \geq 1 \), and \( x(t) = 0 \) else, is a bounded signal \( (M = 1) \) with \( \lim_{t \to \infty} x(t) = 0 \), but it is
  - not an energy signal since \( E_x = \lim_{\tau \to \infty} \int_1^\tau t^{-1} dt = \lim_{\tau \to \infty} \log(\tau) = \infty \), and
  - not a power signal since \( P_x = \lim_{\tau \to \infty} \tau^{-1} \log(\tau/2) = 0 \).
Signal types - discussion

- We will define other types of signals in the following as needed, e.g., causal signals, Dirichlet signals.

- Such definitions can be extended to $\mathbb{C}$-valued signals by, e.g.,
  - simultaneously applying the definition to the both the real and complex components of the signal, e.g., for a $\mathbb{C}$-valued signal to be periodic, both its real and imaginary parts need to be periodic with rational ratio of their periods, or
  - using modulus instead of absolute value, i.e., for $x \in \mathbb{C}$,

  \[
  |x|^2 = |\text{Re}\{x\}|^2 + |\text{Im}\{x\}|^2.
  \]

Signal operations - time scaling

- Recall the time-delay operator $\Delta_{\tau}$ for any fixed $\tau \in \mathbb{R}$: $(\Delta_{\tau} x)(t) := x(t-\tau) \forall t \in \mathbb{R}$.

- A signal $x : \mathbb{R} \to \mathbb{R}$ time-dilated by factor $a > 0$ (equivalently, time- contracted by factor $1/a$) is the signal

  \[
  (S_a x)(t) := x(t/a), \quad t \in \mathbb{R}.
  \]

- In the example for $x(t/a)$, the rectangular pulse portion ends at $t/a = 2$, i.e., at $t = 2a = 4$ for $a = 2$, and $t = 2a = 1$ for $a = 1/2$.

- Also, note how the y-intercept does not change since, obviously, $0 = 0/a$ for all $a > 0$. 
Time inversion

• A signal \( x : \mathbb{R} \rightarrow \mathbb{R} \) time-inverted (or time-reflected) is the signal

\[
(S_{-1} x)(t) := x(-t), \quad t \in \mathbb{R}.
\]

• Basically, the time-inverted signal of \( x \) is \( x \)'s reflection in the y-axis, as the following example illustrates.

![Time inversion example](image)

Combinations of temporal (domain) operations

• **Example:** Given a signal \( x : \mathbb{R} \rightarrow \mathbb{R} \), how would one plot \( \{ x(-2t + 2), \ t \in \mathbb{R} \} \)?

• Note that

\[
x(-2t + 2) = (\Delta_{-2} x)(-2t) = (S_{1/2} \Delta_{-2} x)(-t) = (S_{-1} S_{1/2} \Delta_{-2} x)(t), \quad t \in \mathbb{R}.
\]

• So, beginning with the signal \( x \) (Fig. (a) below), do the following:

  (b) time delay by -2 (advance by 2) giving \( x(t + 2) := r(t) \), then

  (c) time dilate by 1/2 (contract by 2), \( r(2t) = x(2t + 2) := v(t) \), then

  (d) time reflect \( v(-t) = x(-2t + 2) \).

• To verify one point: by Fig. (d) \( 1 = x(-2t + 2) \big|_{t=1} = x(0) = 1 \) by Fig. (a).
Combinations of temporal (domain) operations (cont)

- It’s easy to verify that time dilation and reflection (S) operations generally commute; for example
  \[ S_{-1} S_{0.5} = S_{0.5} S_{-1} = S_{-0.5} \, . \]
- But time-dilation/reflection S and time-shift Δ operations generally do not commute.
- In the previous example, Fig. (e) depicts \( \Delta_{-2} S_{-0.5} x \) which is obviously different from \( S_{-0.5} \Delta_{-2} x \) depicted in Fig. (d), this is simply because \(-2(t + 2) \neq -2t + 2\).
- We can also deal with combined temporal operations by first factoring, e.g.,
  \[ x(-2t + 2) = x(-2(t - 1)) = (S_{-0.5} x)(t - 1) = (\Delta_1 S_{-0.5} x)(t), \quad t \in \mathbb{R}. \]
- So, we would do the time-scale/reflection operations before the time-shift, but obviously with different parameters than used without first factoring as above.
- In summary for this example, we’ve shown
  \[ \Delta_1 S_{-0.5} = S_{-0.5} \Delta_{-2} \neq \Delta_{-2} S_{-0.5} \, . \]

Spatial (range) operations

- For a signal \( x : \mathbb{R} \rightarrow \mathbb{R} \):
  - A spatial increase (upward shift) by constant \( b \in \mathbb{R} \) leads to the signal
    \( (x + b)(t) = x(t) + b, \quad t \in \mathbb{R}. \)
  - A spatial amplification (dilation) by factor \( a \in \mathbb{R} \) leads to the signal
    \( (ax)(t) = a \cdot x(t), \quad t \in \mathbb{R}. \)
  - A spatial reflection (in the time axis) leads to the signal
    \( -x(t), \quad t \in \mathbb{R}. \)
- As with temporal operations, spatial shift and spatial dilation operations generally do not commute.
- If combinations of spatial and temporal operations are in play, then any spatial operation will commute with any temporal operation.
Spatial (range) operations - example

- For the example pulse \( x \) of Fig. (a), \(-3x(t) + 5\) can be plotted by
  (b) first spatially reflecting and amplifying \( x \) by 3 to get \(-3x(t)\),
  (c) then shift up by 5.

- Alternatively, since \(-3x + 5 = -3(x - 5/3)\), one can first shift \( x \) down by 5/3, then spatially reflect and amplify by 3.

- It would be a mistake to first shift by 5 then scale by -3 as this would result in \(-3(x+5) = -3x - 15 \neq -3x + 5\).

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Even and odd signals

- With the notion of time-inversion, we can define other signal attributes.
- A signal \( x : \mathbb{R} \rightarrow \mathbb{R} \) is even if \( x(t) = x(-t) \ \forall t \in \mathbb{R} \).
- A signal \( x \) is odd if \( x(t) = -x(-t) \ \forall t \in \mathbb{R} \).
- Clearly, \( \{ \cos(t), t \in \mathbb{R} \} \) is an even signal, \( \{ \sin(t), t \in \mathbb{R} \} \) is an odd signal, and \( \{ 3e^{-5t}, t \in \mathbb{R} \} \) is neither.
- Any signal can be written as a sum of even and odd signals through the identity
  \[
  x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2} \ \forall t \in \mathbb{R}, \text{ i.e., } x = \frac{x + S_{-1}x}{2} + \frac{x - S_{-1}x}{2},
  \]
  where \( x_e(t) := \frac{1}{2}(x(t) + x(-t)) \) is even and \( x_o(t) := \frac{1}{2}(x(t) - x(-t)) \) is odd.
- For example, we can write
  \[
  3e^{-5t} = 3\frac{e^{-5t} + e^{5t}}{2} + 3\frac{e^{-5t} - e^{5t}}{2} = 3\cosh(5t) + (-3\sinh(5t)) \ \forall t \in \mathbb{R},
  \]
  where \( \sinh \) (hyperbolic sine) is odd \( \Rightarrow -3\sinh \) is odd and \( \cosh \) is even.
Even and odd signal decomposition - graphical example

For the sub-domain $-1 \leq t \leq 1$:
\[ x(t) = 1 + t, \quad x(-t) = 1 - t, \]
\[ x_e(t) = \frac{(1 + t + 1 - t)}{2} = 1, \quad \text{and} \quad x_o(t) = \frac{(1 + t - (1 - t))}{2} = t. \]

For the sub-domain $1 \leq t \leq 2$:
\[ x(t) = 2, \quad x(-t) = 0, \quad x_e(t) = \frac{(2 + 0)}{2} = 1, \quad \text{and} \quad x_o(t) = \frac{(2 - 0)}{2} = 1. \]

Even and odd signal - properties

- It’s easy to show that:
  - If $x$ is odd (respectively, even) then $cx$ is odd (respectively, even) for all scalars $c \in \mathbb{R}$.
  - The sum of odd functions is odd.
  - The sum of even functions is even.
  - The product of even functions is even.
  - The product of odd functions is even.
  - The product of an even and an odd function is odd.

- For example, if $f$ and $g$ are arbitrary odd functions, then $\forall t \in \mathbb{R},$
\[ (fg)(-t) = f(-t)g(-t) = (-f(t))(-g(t)) = f(t)g(t) = (fg)(t), \]
\[ i.e., \ fg \text{ is even.} \]
Some important signals - polynomials

- This course will focus on certain families of signals and combinations made from them.
- $x$ is a polynomial of degree or order $n \in \{0, 1, 2, \ldots\}$ if $\forall t \in \mathbb{R}$,
  \[ x(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 \]

  where we will take the scalar coefficients $a_k \in \mathbb{R} \ \forall k \in \{0, 1, 2, \ldots, n\}$.
- By the fundamental theorem of algebra, $x$ has $n$ roots in $\mathbb{C}$, i.e., there are $n$ solutions $t_k$ to $x(t) = 0$ (allowing for multiple, identical roots) so that
  \[ x(t) = a_n (t - t_1)(t - t_2)\ldots(t - t_n) \]

  \[ = a_n \prod_{k=1}^{n} (t - t_k). \]
- When all coefficients are real, all roots of $x$ are either real or come in complex-conjugate pairs.
- Obviously, the constant signal $x = a_0$ is a polynomial of order zero.

Exponential signals

- Real-valued exponential signals have the form $x(t) = Ae^{bt}$, $t \in \mathbb{R}$, where $A, b \in \mathbb{R}$.
- Periodic, complex-valued exponential signals have the form $x(t) = Ae^{j(wt+\phi)}$, $t \in \mathbb{R}$, where $j := \sqrt{-1}$ and $A > 0, w, \phi \in \mathbb{R}$ and the period is $2\pi/w$.
- In electrical engineering, we don’t use $i$ for $\sqrt{-1}$ as $i$ typically symbolizes electrical current.
- Since the signal $\{e^{jv}, v \in \mathbb{R}\}$ has period $2\pi$, we can take the phase $\phi \in [-\pi, \pi]$.
- By the Euler-De Moivre identity,
  \[ x(t) = Ae^{j(wt+\phi)} = A \cos(wt + \phi) + jA \sin(wt + \phi), \ t \in \mathbb{R}. \]
- Here, the constant (or "direct current" (DC)) signal obviously corresponds to $w = 0$ (again, infinite period).
- Also, here $w$ is the real-valued frequency – we will consider complex-valued frequencies ($s \in \mathbb{C}$) when we study transient response by unilateral Laplace transform.
Exponential signals (cont)

• For the frequency $s = \sigma + jw \in \mathbb{C}$, consider the complex exponential function $x : \mathbb{R} \to \mathbb{C}$:
  $$x(t) = Ae^{(\sigma + j\phi)t} = Ae^{\sigma t}e^{j(\omega t + \phi)}, \quad t \in \mathbb{R}.$$  

• Note the real part of this signal is a sinusoid with exponential envelope:
  $$\text{Re}\{x\}(t) = Ae^{\sigma t} \cos(\omega t + \phi), \quad t \in \mathbb{R}.$$  

• In the figure, $e^{-2t}\cos(10\pi t)$ is plotted with its envelope $\pm e^{-2t}$, i.e., $w = 10\pi$ so that the period $T = 2\pi/(10\pi) = 0.2$, and $\sigma = -2 < 0$.

The unit-step function, $u$

• The unit-step will be denoted by $u$:
  $$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{else } (t < 0) \end{cases}$$  

• Note that $u(0) = 1$ is indicated by the solid period in the figure; and just before time 0, $u(0^-) = 0$.

• The unit step obviously has a single jump discontinuity of unit height at the time-origin.
The unit impulse (Dirac delta function), $\delta$

- Suppose we wish to differentiate the unit-step function, $u$, i.e., compute $u = \frac{du}{dt} = D u$, where we have defined the time-derivative operator

$$D := \frac{d}{dt}$$

- Clearly, $(D u)(t) = 0$ for all $t \neq 0$, but what happens at the jump discontinuity, $t = 0$?

- Consider approximating the derivative $(D u)(0)$ by a difference quotient $\delta_\varepsilon = \frac{u - \Delta u}{\varepsilon}$:

$$\delta_\varepsilon(t) := \frac{u(t) - u(t - \varepsilon)}{\varepsilon}, \quad t \in \mathbb{R},$$

As $\varepsilon \to 0$, $\delta_\varepsilon$ is taller and thinner but with constant area $= 1$, i.e.,

$$\forall \varepsilon > 0, \quad \int_{-\infty}^{\infty} \delta_\varepsilon(t) = 1.$$

That is, as $\varepsilon \to 0$, the support of $\delta_\varepsilon$ disappears, but its area remains constant as its value at the origin diverges $\uparrow \infty$.

So as $\varepsilon \to 0$, $\delta_\varepsilon$ converges to the unit impulse $\delta$, graphically depicted as

$\uparrow$
The unit impulse, $\delta$ (cont)

- Since the unit step is constant except at the origin,
  \[ Du = \delta, \]
  i.e., the unit impulse is the derivative at a unit jump discontinuity.

- We will see later how this peculiar function is crucial to our understanding of linear systems.

- For now, we will examine some of its important properties.

- We've already ascertained that
  \[ \forall t \neq 0, \quad \delta(t) = 0 \]
  \[ \forall t \in \mathbb{R}, \quad \int_{-\infty}^{t} \delta(\tau) d\tau = u(t) \]
  where the second fact follows from $\delta = Du$ and also implies that, for all intervals $I \subset \mathbb{R}$ containing zero in their interior,
  \[ \int_{I} \delta(\tau) d\tau = 1. \]

---

Unit impulse, $\delta$ - ideal sampling property

- Since $\forall t \neq 0, \delta(t) = 0$, if the signal $x$ is continuous at the origin, then
  \[ (x\delta)(t) = x(t)\delta(t) = x(0)\delta(t) \]
  where $x(0)\delta(t)$ is an impulse but a unit impulse only if $x(0) = 1$.

- By definition, the impulse "$\alpha\delta$" for scalar $\alpha \in \mathbb{R}$ satisfies
  \[ \int_{-\infty}^{\infty} \alpha\delta(t) dt = \alpha. \]

- The ideal sampling property of the unit impulse is: For all signals $x$ continuous at the origin,
  \[ \int_{-\infty}^{\infty} x(t)\delta(t) dt = x(0). \]

- By a simple change of variables of integration: for all signals $x$ continuous at time $T \in \mathbb{R}$,
  \[ \int_{-\infty}^{\infty} x(t)\delta(t-T) dt = x(T). \]
Signal construction - example

• For $\tau > \sigma$, note the function:
\[
(\Delta_\sigma u - \Delta_\tau u)(t) = u(t - \sigma) - u(t - \tau) = \begin{cases} 1 & \text{if } \sigma \leq t < \tau \\ 0 & \text{else} \end{cases}
\]

• That is, $\Delta_\sigma u - \Delta_\tau u$ is a rectangular pulse of unit height and support $[\sigma, \tau)$; as such, it is a "window" function on this interval.

• This signal $x$ is linear ($x(t) = t + 1$) just on the time-interval $[-1, 1]$, and a constant ($x(t) = 2$) just on $[1, 2]$.

• So, $x(t) = (t + 1)(u(t + 1) - u(t - 1)) + 2(u(t - 1) - u(t - 2))$, $t \in \mathbb{R}$. 
Dirichlet signals

- In this course, we will focus on signals that are Dirichlet so as to study signals and linear systems in a frequency domain.

- A signal \( x : \mathbb{R} \to \mathbb{R} \):
  - Satisfies the weak Dirichlet condition if it is absolutely integrable, i.e., if
    \[
    \int_{-\infty}^{\infty} |x(t)|dt < \infty,
    \]
or just absolutely integrable over any period \( T \) if \( x \) is periodic, i.e.,
    \[
    \int_{T} |x(t)|dt < \infty.
    \]
  - Satisfies the strong Dirichlet conditions if over any finite interval of time: (i) it has a finite number of jump discontinuities, and (ii) it has a finite number of minima and maxima.

- Note how the strong Dirichlet condition (ii) is not possessed by the bounded signal \( x(t) = \cos(1/t) \), \( t \in \mathbb{R} \) as it has infinitely many extrema in any interval containing the origin \( (t = 0) \).

Systems - single input, single output (SISO)

- In the figure, \( f \) is an input signal that is being transformed into an output signal, \( y \), by the depicted system (box).

- To emphasize this functional transformation, and clarify system properties, we will write the output signal (i.e., system “response” to the input \( f \)) as
  \[
  y = Sf,
  \]
where, again, we are making a statement about functional equivalence:
  \[
  \forall t \in \mathbb{R}, \quad y(t) = (Sf)(t).
  \]

- Again, \( Sf \) is not \( S \) “multiplied by” \( f \), rather a functional transformation of \( f \).
SISO systems (cont)

- The \( n \) signals \( x_k, k \in \{1, 2, ..., n\} \), are the internal states of the system.

- We will see that states can be taken as "outputs of integrators," e.g., the states of a linear circuit can be taken as the currents through inductors and voltages across capacitors.

- In discussing the following properties of systems, we will generally consider Dirichlet, continuous-time, complex-valued input signals, \( f: \mathbb{R} \to \mathbb{C} \).

- But if there is a finite origin of time (e.g., \( f: [0, \infty) \to \mathbb{C} \)) the following properties pertain to systems with state variables that are initially zero (\( \forall k, x_k(0) = 0 \)).

Linear systems

- A linear system \( S \) has the following two properties:
  - (Homogeneity/Scaling) For all input signals \( f \) and scalars \( a \),
    \[
    S(af) = aSf,
    \]
    where we remind that, by definition, \( \forall t \in \mathbb{R}, (af)(t) = a \cdot f(t) \) and "\( \cdot \)" is just scalar multiplication.
  - (Superposition) For all signals \( f_1 \) and \( f_2 \),
    \[
    S(f_1 + f_2) = Sf_1 + Sf_2,
    \]
    where we remind that, by definition, \( \forall t \in \mathbb{R}, (f_1 + f_2)(t) = f_1(t) + f_2(t) \) and "\( + \)" on the right-hand-side is just scalar summation.

- Implicitly, when defining such system properties, we are restricting our attention to signals \( f \) for which \( Sf : \mathbb{R} \to \mathbb{C} \) is a well-defined (output) signal, restricting our attention to Dirichlet signals in particular.
• Suppose a system is defined by the input-output relationship,
\[
\forall t \in \mathbb{R}, \quad y(t) = 3 \int_{-\infty}^{t} f(\tau)d\tau = (Sf)(t).
\]

• This system is linear owing to the linearity property of the integration operator (and commutativity of scalar multiplication and addition).

• Scaling: For all input signals \( f \), scalars \( a \), and time \( t \in \mathbb{R} \),
\[
(Saf)(t) = 3 \int_{-\infty}^{t} af(\tau)d\tau = a3 \int_{-\infty}^{t} f(\tau)d\tau = a(Sf)(t),
\]
i.e., \( S(af) = aSf \).

• Superposition: For all input signals \( f_1, f_2 \), and time \( t \in \mathbb{R} \),
\[
(S(f_1 + f_2))(t) = 3 \int_{-\infty}^{t} (f_1(\tau) + f_2(\tau))d\tau = 3 \int_{-\infty}^{t} f_1(\tau)d\tau + 3 \int_{-\infty}^{t} f_2(\tau)d\tau = (Sf_1 + Sf_2)(t),
\]
i.e., \( S(f_1 + f_2) = Sf_1 + Sf_2 \) - by the absolute integrability property of Dirichlet signals, the integral of the sum (of integrands) is the sum of the integrals (of each integrand).

Linear systems - examples (cont)

• Suppose a system is defined by the input-output relationship,
\[
\forall t \in \mathbb{R}, \quad y(t) = 3f(t) + 7;
\]
or just \( y = 3f + 7 = Sf \) where "\( T \)" here is understood to be the constant signal \( = 7 \) for all time.

• Clearly, in general,
\[
S(f_1 + f_2) = 3(f_1 + f_2) + 7 \neq 3f_1 + 7 + 3f_2 + 7 = Sf_1 + Sf_2.
\]

• So, this is not a linear system because it does not have the superposition property.

• One can easily check that the scaling property also fails, i.e., that:
for all signals \( f \neq 0 \) and scalars \( a \) such that
\[
S(af) \neq aSf.
\]

• This system is said to be affine.

• For a signal, we will use the adjectives affine and linear interchangeably.
Disproving system linearity - discussion

- The logical negation of a universal statement is an existential one (i.e., there is a counter-example).

- Also recall De Morgan’s rule: for all statements $p$ and $q$,
  \[ \sim (p \text{ and } q) = (\sim p) \text{ or } (\sim q), \]
  where $\sim$ here represents logical negation.

- So, to disprove linearity, we either need only find a scalar and input-signal for which the scaling property does not hold or a pair of input signals for which the superposition property does not hold.

- In the previous example of an affine system, both properties more generally do not hold, i.e., not just for a single counter-example.

---

Linear systems - examples (cont)

- Consider the system whose (zero initial-state) output $y$ to input $f$ are related by
  \[ 3Dy + y = f^2 \]
  To check whether this system is linear, consider arbitrary input signals $f_1$ and $f_2$ and scalars $\alpha_1$ and $\alpha_2$.

- For $k \in \{1, 2\}$, let $y_k$ be the system output to $f_k$, i.e., $3Dy_k + y_k = f_k^2$.

- Thus, by linearity of the derivative $D$ operator,
  \[ 3D(\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 (3Dy_1 + y_1) + \alpha_2 (3Dy_2 + y_2) = \alpha_1 f_1^2 + \alpha_2 f_2^2 \neq (\alpha_1 f_1 + \alpha_2 f_2)^2, \]
  i.e., $\alpha_1 f_1^2 + \alpha_2 f_2^2 \neq (\alpha_1 f_1 + \alpha_2 f_2)^2$ for arbitrary signals $f_k$ and scalars $\alpha_k$.

- For example, if $f_1 \equiv 1 \equiv f_2$ (the constant signal 1) and $\alpha_1 = 1 = \alpha_2$, then $\alpha_1 f_1^2 + \alpha_2 f_2^2 = 2 \neq 4 = (\alpha_1 f_1 + \alpha_2 f_2)^2$.

- So, this system is not linear.
Time-invariant systems

• Recall the time-shift (delay) operator $\Delta_{\tau} f$,
  \[ \forall t \in \mathbb{R}, \quad (\Delta_{\tau} f)(t) = f(t - \tau). \]

• A system $S$ is time-invariant if for all $\tau \in \mathbb{R}$ and Dirichlet $f : \mathbb{R} \to \mathbb{C}$,
  \[ S(\Delta_{\tau} f) = \Delta_{\tau}(Sf), \]
  i.e., if $S$ and $\Delta_{\tau}$ commute.

• That is, if $y = Sf$, then the system $S$ is time-invariant if for all signals $f$ and delays $\tau$,
  the response to $\Delta_{\tau} f$ (i.e., "$f(t - \tau)"$) is $\Delta_{\tau} y$ (i.e., "$y(t - \tau)"$).

• Or even more simply, a delay in the input results in the same delay in the output.

Time-invariant systems - examples

• For example, consider again the integrator system
  \[ \forall t \in \mathbb{R}, \quad (Sf)(t) = \int_{-\infty}^{t} f(r)dr. \]

• For all Dirichlet $f$ and $\tau \in \mathbb{R}$:
  \[ \forall t \in \mathbb{R}, \quad (S(\Delta_{\tau} f))(t) = \int_{-\infty}^{t} (\Delta_{\tau} f)(r)dr \]
  \[ = \int_{-\infty}^{t} f(r - \tau)dr \]
  \[ = \int_{-\infty}^{t-\tau} f(r')dr', \quad r' := r - \tau \]
  \[ = (Sf)(t - \tau) \]
  \[ = (\Delta_{\tau}Sf)(t). \]

• Thus, this system is time invariant.
Consider again the affine system defined by \( y = Sf = 3f + 7 \).

Clearly, for all signals \( f : \mathbb{R} \to \mathbb{C} \) and delays \( \tau \in \mathbb{R} \): \( \forall t \in \mathbb{R} \),
\[
S(\Delta \tau f)(t) = 3(\Delta \tau f)(t) + 7 = 3f(t - \tau) + 7 = \Delta \tau (Sf)(t) = y(t - \tau),
\]
where \( y := Sf \).

So, this system is time-invariant.

For the system \( \forall t \in \mathbb{R}, (Sf)(t) = 3f(t) + 7t \) if \( \tau \neq 0 \) then
\[
S(\Delta \tau f)(t) = 3(\Delta \tau f)(t) + 7t = 3f(t - \tau) + 7t \neq 3f(t - \tau) + 7(t - \tau) = (\Delta \tau Sf)(t);
\]

That is, there are cases (in fact, all but \( \tau \neq 0 \)) where time-shift in the input does not lead to the same-time-shift in the output.

So, this system is time varying, i.e., not time invariant.

One can similarly show that the system \( \forall t \in \mathbb{R}, (Sf)(t) = 3tf(t) + 7 \) is also time-varying.

Consider the system whose output to input \( f \) is \( y(t) = (Sf)(t) = f(t^2), \forall t \in \mathbb{R} \).

To check for time-invariance, take an arbitrary input \( f \) and scalar \( T \) and let \( g = \Delta T f \), i.e., \( g(t) = f(t - T), \forall t \in \mathbb{R} \).

The output to \( g \) is \( (Sy)(t) = g(t^2) = f((t - T)^2), \forall t \in \mathbb{R} \).

Also, \( (\Delta Ty)(t) = f((t - T)^2), \forall t \in \mathbb{R} \).

So, this system is not time-invariant because for arbitrary \( f, T \),
\[
(S\Delta Tf)(t) = f(t^2 - T) \neq f((t - T)^2) = (\Delta T Sf)(t), \forall t \in \mathbb{R}.
\]

For example, if \( f(t) = t, \forall t \in \mathbb{R} \), and \( T = 1 \), then
\[
(S\Delta Tf)(t) = t^2 - 1 \neq (t - 1)^2 = (\Delta T Sf)(t), \forall t \neq 1.
\]
Causal signals and systems

- A signal $f$ is causal if $f(t) = 0$ for all $t < 0$.
- Note that for any signal $x : \mathbb{R} \to \mathbb{R}$, $f = xu$ is a causal signal.
- A system $S$ is said to be causal if: for all causal input signals $f$, the output signal (response) $Sf = y$ is also causal.
- As we will be focusing on time-invariant (and linear) systems, the choice of time origin will be arbitrary, and we will typically choose $0$.
- If the system has non-zero response at negative time, it is said to be anticipating the causal signal which does not "arrive" till time $0$.
- So, non-causal systems are said to be anticipative.

Causal systems - examples

- Consider again, the integrator system $\forall t \in \mathbb{R}$, $(Sf)(t) = \int_{-\infty}^{t} f(r)dr$.
- Take arbitrary causal $f$, $f(t) = 0$ for all $t < 0$.
- Thus, for all $t < 0$, $(Sf)(t) = \int_{-\infty}^{t} f(r)dr = \int_{-\infty}^{t} 0dr = 0$.
- So, this system is causal.
- Now consider the example $\forall t \in \mathbb{R}$, $(Sf)(t) = 3f^{2}(t + 2)$.
- If $f$ is causal, then for $t = -1$ note that $(Sf)(-1) = 3f^{2}(-1 + 2) = 3f^{2}(1)$
- So, $(Sf)(-1) = 0$ only if $f(1) = 0$, which may not be the case for an arbitrary causal signal.
- So, for arbitrary causal $f$ and $t < 0$, $(Sf)(t) \neq 0$ (e.g., when $f(1) \neq 0$).
- Hence this system is not causal.
- One can easily show that this system is also not linear, but it is time invariant.
Causal systems - examples (cont)

- Consider the system \( \forall t \in \mathbb{R}, (Sf)(t) = 3tf(t) + 7 \).
- If \( f \) is causal, then for all \( t < 0 \),
  \[ (Sf)(t) = 3tf(t) + 7 = 3t \cdot 0 + 7 = 7 \neq 0. \]
- So this system is also not causal.
- Exercise: Is this system linear? time-invariant?
- Exercise: Is the system \( (Sf)(t) = 3f(t) + 7t, t \in \mathbb{R} \), causal? linear? time-invariant?

Stateless/memoryless systems

- In a memoryless (or stateless) system, \( \forall t \in \mathbb{R} \) and input signals \( f \), the output at time \( t \), \( y(t) \), depends on the input \( f \) only through \( f(t) \).
- For example, the system \( \forall t \in \mathbb{R}, y(t) = (Sf)(t) = 3tf(t) + 7 \) is obviously a memoryless system.
- But in the integrator system \( \forall t \in \mathbb{R}, y(t) = (Sf)(t) = \int_{-\infty}^{t} f(r)dr \), clearly the output \( y(t) \) generally depends on \( f(r) \) for \( r \neq t \) (here, for all \( r \leq t \)).
- So this integrator system is stateful.
- We will see that stateful systems have associated internal state signals (again, we will denote them with \( x \)).
In this course, we focus on linear and time-invariant (LTI) SISO systems $S$ in continuous time.

If $y = Sf$ represents the response of the system $y$ to input $f$, then we will represent the system with the following ordinary differential equation (ODE) relating $y$ and $f$:

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m f}{dt^m} + b_{m-1} \frac{d^{m-1} f}{dt^{m-1}} + \cdots + b_1 \frac{df}{dt} + b_0 f,$$

with integers $n, m \geq 0$, and the scalar coefficients $a_k, b_k \in \mathbb{R}, \forall k$.

Note that we can take the coefficient $a_n = 1$ without loss of generality, i.e., one can divide a system equation by the coefficient of $\frac{d^n y}{dt^n}$ if it is not 1 to get a system equation in the above form.

Also note that this system equation is a functional equivalence in that it holds for all time, i.e., one can stipulate "$(t)$" for all participating signals giving an equation that would hold for all $t \in \mathbb{R}$.

We will write such system equations more compactly in two ways.

First, using the time-derivative operator

$$D := \frac{d}{dt},$$

we can write the system equation as

$$D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 D y + a_0 y = b_m D^m f + b_{m-1} D^{m-1} f + \cdots + b_1 D f + b_0 f.$$

Note that $D^0$ is just the identity map: $\forall f, \ D^0 f = f$.

Furthermore, we can define the system polynomials (with real coefficients, $a_n = 1$),

$$Q(s) := \sum_{k=0}^{n} a_k s^k \quad \text{and} \quad P(s) := \sum_{k=0}^{m} b_k s^k, \quad \forall s \in \mathbb{C}.$$

So, the system equation can be written simply as

$$Q(D)y = P(D)f$$

where $P$ is a polynomial of degree $m$ and $Q$, the characteristic polynomial of the system, has degree $n$. 

This SISO system is LTIC

- The system with input \( f \) and output \( y \) (the response to \( f \)) satisfying
  \[
  D^n y + a_{n-1} D^{n-1} y + \ldots + a_1 D y + a_0 y = b_m D^m f + b_{m-1} D^{m-1} f + \ldots + b_1 D f + b_0 f
  \]
  is Linear, Time-Invariant and Causal (LTIC) because:
  - the time-derivative operators \( D^k \) are LTIC for all integers \( k \geq 0 \), and
  - the coefficients \( b_k, a_k \) of the polynomials \( P \) and \( Q \) are all assumed to be constant, i.e., no time-varying coefficients.

- In the following, we will explicitly "solve" this system (for \( y \) in terms of \( f \)) and more clearly demonstrate that it is LTIC.

SISO Canonical ODE - resistive circuit example

- Consider this simple, ideal resistive circuit.

- By Kirchhoff’s current law (KCL) at node \( y \) (i.e., zero is the sum of the currents referenced into the node between the resistors) and Ohm’s law (current in terms of node voltages),
  \[
  \frac{f - y}{R_1} + \frac{0 - y}{R_2} = 0,
  \]
  where “0” on the left-hand-side is the ground voltage.

- In canonical form, the input-output relationship is the voltage divider:
  \[
  y = \frac{R_2}{R_1 + R_2} f.
  \]

- Obviously, this system is memoryless/stateless.
In this passive RC circuit, the KCL nodal equation at $y$ is

$$\frac{f - y}{R} + C D(0 - y) + \frac{0 - y}{R} + C D(0 - y) = 0.$$ 

In canonical form,

$$D y + \frac{1}{RC} y = \frac{1}{2RC} f.$$ 

We now solve the system $D y + (RC)^{-1} y = (2RC)^{-1} f$, i.e., we find the output $y$ explicitly in terms of the input $f$ and initial conditions.

Using the co-factor method, assume the output is of the form

$$y(t) = A(t)e^{-t/(RC)}$$

and substitute it into the system equation: $\forall t \geq 0$,

$$(D A)(t)e^{-t/(RC)} - \frac{1}{RC} y + \frac{1}{RC} y = \frac{1}{2RC} f(t)$$

$$\Rightarrow (D A)(t) = f(t) \frac{1}{2RC} e^{t/(RC)}$$

$$\Rightarrow A(t) = A(0-)+ \int_{0}^{t} f(r) \frac{1}{2RC} e^{r/(RC)} \, dr$$

$$\Rightarrow y(t) = A(0-)e^{-t/(RC)} + \int_{0}^{t} f(r) \frac{1}{2RC} e^{-(t-r)/(RC)} \, dr$$

where $0$ is the origin of time, and clearly this system is stateful.
• If the initial condition \( y(0^-) \) is given just before the origin of time, then clearly \( A(0^-) = y(0^-) \), i.e.,

\[
\forall t \geq 0, \quad y(t) = y(0^-)e^{-t/(RC)} + \int_{0^-}^{t} f(r) \frac{1}{2RC}e^{-(t-r)/(RC)} \, dr.
\]

• We will see how this form of the solution is important for transient analysis as it separates out

– the portion of the output \( \{y(t) \mid t \geq 0\} \) due only to the initial condition, here simply the constant signal \( y(0^-) \), is called the Zero-Input Response (ZIR), and

– the portion due only to the input, here \( \int_{0^-}^{t} f(r) \frac{1}{2RC}e^{-(t-r)/(RC)} \, dr, \ t \geq 0 \) - this is called the Zero-State Response (ZSR) or zero initial-state response.

• Note for the above example, if input \( f \equiv 0 \), then the voltage source is a short circuit and the energy stored in the capacitor will eventually dissipate, i.e., \( \text{ZIR} \to 0 \) over time.

• For the example of Fig. 1.26 of Lathi p. 78:
check that the system equation is \( D^2 y = R D f + f/C \) so that 0 is the characteristic value - see equ. (1.36a). Here if \( f \equiv 0 \), then the current source is an open circuit so the capacitor’s energy does not dissipate, i.e., ZIR is constant - see (1.42).

SISO Canonical ODE - active RC circuit example

• Recall that with an op-amp (having a power supply not shown), a relatively low-powered input signal can control a separately high-powered output signal.

• Recall that an ideal op-amp

– draws zero current from its two (\( \pm \)) input leads, and

– has zero differential input voltage,

– i.e., has infinite input impedance.
SISO Canonical ODE - active RC circuit example (cont)

- So, for this circuit, the voltage at the negative input terminal of the op-amp is also \( f \).
- Thus, KCL nodal equation at the “−” input terminal is
  \[
  \frac{v_A - f}{R_1} + C_1 D(y - f) = 0
  \]
  \[\Rightarrow v_A = f + R_1 C_1 D f - R_1 C_1 D y.\]
- Also, KCL at node \( A \) is
  \[
  \frac{f - v_A}{R_1} + \frac{y - v_A}{R_2} + C_2 D(0 - v_A) = 0.
  \]
- Substituting \( v_A \) into KCL at \( A \) and then rewriting in canonical form gives:
  \[
  D^2 y + \frac{1}{C_2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) D y + \frac{1}{R_2 C_2 R_1 C_1} y
  = D^2 f + \left( \frac{1}{R_1 C_1} + \frac{1}{C_2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right) D f + \frac{1}{R_2 C_2 R_1 C_1} f.
  \]

Time-Domain Analysis of LTIC SISO Systems - Introduction

- Consider a LTIC SISO system with input \( f : [0, \infty) \to \mathbb{R} \) and output \( y : [0, \infty) \to \mathbb{R} \), i.e., with time origin 0.
- We will now study solutions of general LTIC SISO systems of the form
  \[ Q(D) y = P(D) f, \]
  with initial conditions \((D^{n-1} y)(0−), (D^{n-2} y)(0−), \ldots, (D y)(0−), y(0−)\), where
  - \( D \) is the time derivative operator;
  - \( Q \) is the system’s characteristic polynomial of degree \( n \geq 1 \) with real coefficients: \( Q(D) = \sum_{k=0}^{n} a_k D^k \), \( a_n = 1 \);
  - \( P \) is a polynomial of degree \( m \) with real coefficients: \( P(D) = \sum_{k=0}^{m} b_k D^k \);
- By “solution” we mean to find the entire signal \( y : [0, \infty) \to \mathbb{R} \) given:
  - a causal input signal \( f \) with zero initial conditions, \((D^k f)(0−) = 0\) for all integers \( k \geq 0 \) (so \( f \) typically specified with a step-function \( u \) factor);
  - the \( n \) initial conditions; and
  - the model of the system itself, \( P, Q \).
Approach to solution: ZIR and ZSR

- The total solution $y$ to $P(D)f = Q(D)y$ and the given initial conditions is a sum of two parts:
  - the ZSR, $y_{ZS}$, which solves
    $P(D)f = Q(D)y_{ZS}$
    with zero initial conditions, i.e., $(D^k y)(0^-) = 0 \forall$ integers $k \geq 0$; and
  - the ZIR, $y_{ZI}$, which solves
    $0 = Q(D)y_{ZI}$
    with the given initial conditions.

- The total response $y$ of the system to $f$ and the given initial conditions is, by linearity,
  $$ y = y_{ZI} + y_{ZS} $$

- We will determine the ZIR by finding the characteristic modes of the system.
- We will determine the ZSR by convolution of the input with the impulse response.

Total response - example

- Recall that this RC circuit with input $f$ and output $y$ is described by
  $$ D y + \frac{1}{RC} y = \frac{1}{2RC} f $$
  i.e., $Q(D) = D + 1/(RC)$ with degree $n = 1$, and $P(D) = 1/(2RC)$ with degree $m = 0$. 
Also recall that the total-response is
\[ y(t) = y(0-)e^{-t/(RC)} + \int_{0-}^{t} f(r) \frac{1}{2RC} e^{-(t-r)/(RC)} \, dr \quad \forall t \geq 0, \]
i.e., \( \forall t \geq 0, \)
\[ y_{ZI}(t) = y(0-)e^{-t/(RC)}, \]
\[ y_{ZS}(t) = \int_{0-}^{t} f(r) \frac{1}{2RC} e^{-(t-r)/(RC)} \, dr. \]

Comment: Solution by homogeneous and particular components

- You may have learned how to find the total response by summing
  - any "particular" solution \( y_p \) satisfying \( P(D)f = Q(D)y_p \) irrespective of initial conditions and possibly including characteristic modes, and
  - a generic "homogeneous" solution \( y_h \) satisfying \( 0 = Q(D)y_h \) consisting only of \( n \) characteristic modes, with \( n \) free parameters/coefficients.
- The \( n \) free parameters of the homogeneous solution are determined by applying the \( n \) initial conditions to the total response, \( y = y_p + y_h \).
- The difference of this approach is simply that
  - \( y_p \) may be the ZSR together with a portion of the ZIR, and
  - \( y_h \) is the remaining portion of the ZIR.
- That is, the ZSR is the special "particular" solution with zero initial conditions, and the ZIR is the special "homogeneous" solution that accounts for the initial conditions by itself.
ZIR - the characteristic values

- To solve for the ZIR, i.e., solve
  \[ Q(D)y = 0 \]
given \( (D^n y)(0), (D^{n-1} y)(0), \ldots, (D y)(0), y(0) \), recall that exponential functions, \( \{e^{st} \mid t \in \mathbb{R}\} \) for frequency \( s \in \mathbb{C} \), are eigenfunctions of differential operators of the form \( Q(D) \) for a polynomial \( Q \).

- That is, for scalar \( s \in \mathbb{C} \), if we substitute \( y = \{e^{st} \mid t \in \mathbb{R}\} \), we get
  \[ \forall t \in \mathbb{R}, \ (Q(D)y)(t) = Q(D)e^{st} = Q(s)e^{st}. \]

- So, to solve \( Q(s)e^{st} = 0 \) for all time \( t \geq 0 \), where \( e^{st} \) is not always zero over this time period, we require
  \[ Q(s) = 0, \ the \ characteristic \ equation \ of \ the \ system. \]

- If \( s \) is a root of the characteristic polynomial \( Q \) of the system,
  - \( s \) would be a characteristic value of the system, and
  - the signal \( \{e^{st} \mid t \geq 0\} \) is a characteristic mode of the system, i.e., \( Q(D)e^{st} = 0 \) \( \forall t \geq 0 \).

- Since \( Q \) has degree \( n \), there are \( n \) roots of \( Q \) in \( \mathbb{C} \), each a system characteristic value.

---

ZIR - the characteristic values (cont)

- Though there may be some repeated roots of the characteristic polynomial \( Q \), there will always be \( n \) different, linearly independent characteristic modes, \( \mu_k \), i.e.,
  \[ \forall t \geq 0, \ \sum_{k=1}^{n} \gamma_k \mu_k(t) = 0 \Leftrightarrow \forall k, \ scalars \ \gamma_k = 0. \]

- So, by system linearity, we will be able to write
  \[ \forall t \geq 0, \ y_{ZI}(t) = \sum_{k=1}^{n} c_k \mu_k(t), \]
  for scalars \( c_k \in \mathbb{C} \) that are found by considering the given initial conditions
  \[ (D^l y)(0^-) = (D^l \sum_{k=1}^{n} c_k \mu_k)(0^-) = \sum_{k=1}^{n} c_k (D^l \mu_k)(0^-), \ for \ 0 \leq l \leq n - 1, \]
  i.e., \( n \) equations in \( n \) unknowns \( (c_k) \).

- The linear independence of the modes implies linear independence of these \( n \) equations in \( c_k \), and so they have a unique solution.

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ZIR - the case of \( n \) different and real characteristic values

- If there are \( n \) different roots of \( Q \) in \( \mathbb{R} \), \( s_1, s_2, \ldots, s_n \), then there are \( n \) characteristic modes: for \( k \in \{1, 2, \ldots, n\} \),
  \[
  \mu_k(t) = e^{s_k t}, \quad t \geq 0.
  \]
- Therefore,
  \[
  \forall t \geq 0, \quad y_{ZI}(t) = \sum_{k=1}^{n} c_k e^{s_k t}.
  \]
- Since \( \forall k \in \{1, 2, \ldots, n\}, l \geq 0 \),
  \[
  D^l \mu_k = s_k^l \mu_k \quad \text{and} \quad \mu_k(0) = 1,
  \]
  the \( n \) unknown scalars \( c_k \in \mathbb{R} \) can be solved using the \( n \) equations:
  \[
  (D^l y)(0^-) = \sum_{k=1}^{n} c_k s_k^l, \quad \text{for } 0 \leq l \leq n - 1.
  \]

ZIR - the case of \( n \) different and real characteristic values - example

- For this series RLC circuit with input \( f \) and output \( y \), KVL gives
  \[
  \forall t \geq 0, \quad R i(t) + L(D i)(t) + v_c(0) + \frac{1}{C} \int_0^t i(\tau) d\tau = f(t).
  \]
- We can differentiate this equation and substitute \( i = y/R \) to get
  \[
  D y + LR^{-1} D^2 y + (RC)^{-1} y = D f
  \]
  \[
  \Rightarrow \quad D^2 y + RL^{-1} D y + (LC)^{-1} y = RL^{-1} D f,
  \]
  i.e., \( P(s) = RL^{-1} s \) and the characteristic polynomial of this second-order \( (n = 2) \) circuit is
  \[
  Q(s) = s^2 + RL^{-1} s + (LC)^{-1}.
  \]
ZIR - the case of \( n \) different and real char. values - example (cont)

- For example, if \( R = 700\Omega, \, L = 0.1H, \) and \( C = 1\mu F \), then
  \[
  Q(s) = s^2 + 7000s + 10^7 = (s + 5000)(s + 2000).
  \]
- So, the characteristic values are -5000, -2000.
- The characteristic modes are \( \exp(-5000t), \, \exp(-2000t) \forall t \geq 0 \).
- The ZIR is of the form
  \[
  \forall t \geq 0-, \quad y_{ZI}(t) = c_1e^{-5000t} + c_2e^{-2000t}.
  \]
- If the given initial conditions are
  \[
  (D y)(0-) = -2000V/s, \quad y(0-) = 10V,
  \]
  then
  \[
  -2000 = -5000c_1 + -2000c_2 = (D y_{ZI})(0-) \]
  \[
  10 = c_1 + c_2 = y_{ZI}(0-)
  \]
- Thus, \( c_1 = -6 \) and \( c_2 = 16 \), \( i.e. \),
  \[
  \forall t \geq 0-, \quad y_{ZI}(t) = -6e^{-5000t} + 16e^{-2000t}
  \]

- The characteristic polynomial \( Q \) may have non-real roots, but such roots come in complex-conjugate pairs because \( Q \)'s coefficients \( a_k \) are all real.
- For example, if the characteristic polynomial is
  \[
  Q(s) = s^3 + 5s^2 + 11s + 15 = (s + 3)(s^2 + 2s + 5)
  \]
  then the characteristic values (\( Q \)'s roots) are
  \[-3, \quad -1 \pm 2j \quad \text{recalling} \quad j = \sqrt{-1}.
  \]
- Because we have three different characteristic values \( \in \mathbb{C} \), we can specify three corresponding characteristic modes,
  \[
  e^{-3t}, \quad e^{(-1+2j)t}, \quad e^{(-1-2j)t}
  \]
  and construct the ZIR as
  \[
  y_{ZI}(t) = c_1e^{-3t} + c_2e^{(-1+2j)t} + c_3e^{(-1-2j)t} \forall t \geq 0 -.
  \]
• By the Euler-De Moivre identity,
  \[ y_{ZI}(t) = c_1 e^{-3t} + 2(c_2 + c_3)e^{-t} \cos(2t) + 2j(c_2 - c_3)e^{-t} \sin(2t) \]

• Because all initial conditions are real and \( y_{ZI} \) is real valued, \( c_2 \) and \( c_3 \) will be complex conjugates, i.e.,

\[ c_2 = \overline{c_3} \in \mathbb{C}. \]

• So, we can write
  \[ y_{ZI}(t) = c_1 e^{-3t} + \tilde{c}_2 e^{-t} \cos(2t) + \tilde{c}_3 e^{-t} \sin(2t), \]
  where

\[ \tilde{c}_2 = (c_2 + c_3) = 2 \text{Re}\{c_2\} \in \mathbb{R} \]
\[ \tilde{c}_3 = 2j(c_2 - c_3) = -2 \text{Im}\{c_2\} \in \mathbb{R} \]

i.e., just two “degrees of freedom” with these two coefficients to accommodate the initial conditions, so not an “underspecified” scenario.

• Thus, instead of \( e^{(-1 \pm 2j)t} \), the following real-valued characteristic modes can be used
  \[ e^{-t} \cos(2t), \ e^{-t} \sin(2t) \]

---

ZIR - not-real char. values with real char. modes (cont)

• So for \( t \geq 0 \), if the ZIR is of the form
  \[ y_{ZI}(t) = c_1 e^{-3t} + \tilde{c}_2 e^{-t} \cos(2t) + \tilde{c}_3 e^{-t} \sin(2t), \]

\[ \Rightarrow (D y_{ZI})(t) = -3c_1 e^{-3t} + \tilde{c}_2[-e^{-t} \cos(2t) - 2e^{-t} \sin(2t)] + \tilde{c}_3[-e^{-t} \sin(2t) + 2e^{-t} \cos(2t)], \]
\[ \Rightarrow (D^2 y_{ZI})(t) = 9c_1 e^{-3t} + \tilde{c}_2[-3e^{-t} \cos(2t) + 4e^{-t} \sin(2t)] + \tilde{c}_3[-3e^{-t} \sin(2t) - 4e^{-t} \cos(2t)], \]

and the initial conditions are

\[ (D^2 y)(0-) = 3, \ (D y)(0-) = 10, \ y(0-) = 5, \]

then the three equations to solve for the three unknown constants \( c_1, \tilde{c}_2, \tilde{c}_3 \) are

\[ 5 = c_1 + \tilde{c}_2 = y_{ZI}(0-) \]
\[ 10 = -3c_1 - \tilde{c}_2 + 2\tilde{c}_3 = (D y_{ZI})(0-) \]
\[ 3 = 9c_1 - 3\tilde{c}_2 - 4\tilde{c}_3 = (D^2 y_{ZI})(0-) \]

whose solution is

\[ c_1 = 6, \ \tilde{c}_2 = -1, \ \tilde{c}_3 = 13.5. \]
In summary, for the case of a complex-conjugate pair of characteristic values \( \alpha \pm j\beta \), with \( \alpha, \beta \in \mathbb{R} \), we can use the real-valued modes (with two different associated real coefficients),

\[
e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t) \quad \forall t \geq 0,
\]

instead of complex-valued modes (with associated complex-conjugate coefficients)

\[
e^{(\alpha \pm j\beta)t} \quad \forall t \geq 0,
\]

in either case, the pair of coefficients give two degrees of freedom to meet the given initial conditions.

Restricting the span to real-valued signals, we can use these two real-valued modes because

\[
\text{span}\{e^{(\alpha+j\beta)t}, e^{(\alpha-j\beta)t}\} = \text{span}\{e^{\alpha t} \cos(\beta t), e^{\alpha t} \sin(\beta t)\}, \quad t \in [0, \infty).
\]

Generally, linear combinations of characteristic modes span all possible real-valued ZIR signals.

Consider the case where at least one characteristic value is of order \( \geq 1 \), i.e., repeated roots of the characteristic polynomial, \( Q \).

For example, \( Q(D) = (D + 7)^3(D - 2) \) has a triple (twice repeated) root at \(-7\) and a single root at \(2\).

Again, \( \{e^{-7t}, t \geq 0\} \) is a characteristic mode because \( Q(D)e^{-7t} \equiv 0 \) follows from

\[
(D + 7)e^{-7t} = D e^{-7t} + 7e^{-7t} = -7e^{-7t} + 7e^{-7t} = 0 \quad \forall t \geq 0.
\]

Also, \( \{te^{-7t}, t \geq 0\} \) is a characteristic mode because \( Q(D)te^{-7t} \equiv 0 \) follows from

\[
(D + 7)^2te^{-7t} = (D^2 + 14D + 49)te^{-7t} = D^2te^{-7t} + 14Dte^{-7t} + 49te^{-7t} = \left(-14e^{-7t} + 49te^{-7t}\right) + 14(e^{-7t} - 7te^{-7t}) + 49te^{-7t} = (-14 + 14)e^{-7t} + (49 - 98 + 49)te^{-7t} = 0 \quad \forall t \geq 0.
\]
• Similarly, \( t^2e^{-7t} \) is also a characteristic mode because \((D + 7)^3t^2e^{-7t} = 0\).

• Note that without three such linearly independent characteristic modes \( \{e^{-7t}, te^{-7t}, t^2e^{-7t}; t \geq 0\} \) for the twice-repeated (triple) characteristic value -7, the initial conditions will create an “overspecified” set of \( n \) equations involving fewer than \( n \) “unknown” coefficients \( (c_k) \) of the linear combination of modes forming the ZIR.

• For this example,
  \[
y_{ZI}(t) = c_0e^{-7t} + c_1te^{-7t} + c_2t^2e^{-7t} + c_3e^{2t}, \quad t \geq 0.
  \]

• If the given initial conditions are, say,
  \[
  (D^3y)(0-) = 10, \quad (D^2y)(0-) = -5, \quad (Dy)(0-) = 6, \quad y(0-) = 12,
  \]
  the four equations to solve for the four unknown coefficients \( c_k \) are:
  \[
  y_{ZI}(0-) = c_0 + c_3 = 12 \\
  (Dy_{ZI})(0-) = -7c_0 + c_1 + 2c_3 = 6 \\
  (D^2y_{ZI})(0-) = 49c_0 - 14c_1 + 2c_2 + 4c_3 = -5 \\
  (D^3y_{ZI})(0-) = -343c_0 + 147c_1 - 42c_2 + 8c_3 = 10
  \]

---

**ZIR - general case of repeated characteristic values**

• In general, a set of \( k \) linearly independent modes corresponding to a characteristic value \( s \in \mathbb{C} \) repeated \( k - 1 \) times (i.e., of multiplicity \( k \)) are,
  \[
  t^{k-1}e^{st}, \quad t^{k-2}e^{st}, \ldots, \quad te^{st}, \quad e^{st}, \quad \text{for } t \geq 0.
  \]

• Also, if \( s = a \pm jw \) are characteristic values repeated \( k - 1 \) times, with \( a, w \in \mathbb{R} \) and \( w \neq 0 \), we can use the \( 2k \) real-valued modes
  \[
  te^{st}\cos(wt), \quad te^{st}\sin(wt), \quad \text{for } r \in \{0, 1, 2, \ldots, k-1\}.
  \]
Zero State Response - the impulse response

- Let \( h \) be a LTI system's ZSR to the unit impulse, \( \delta \).

- At this point, it's difficult to see why \( h \) should be a well-defined signal, let alone a crucial one to the study of LTI systems.

- In the following, we will derive \( h \) by inverse Fourier or inverse Laplace transform of the system transfer function \( H = P/Q \).

- For a procedure to compute \( h \) directly from \( P(D)\delta = Q(D)h \) see, for example, Sec. 2.3 of Lathi.

- For now, we will assume that \( h \) is known for a given LTI system and demonstrate how it can be used to derive the ZSR to an arbitrary (Dirichlet) input signal \( f \) by the process of convolution.

ZSR - LTI property and convolution with impulse response

- Consider a LTI system with impulse response \( h \) and recall the approximate impulse function,

\[
\delta_{\varepsilon} := \varepsilon^{-1}(u - \Delta_{\varepsilon}u),
\]

where \( 0 < \varepsilon \ll 1 \).

- For arbitrary integer \( k \), consider a short interval of time \( [k\varepsilon, (k+1)\varepsilon) \) and suppose over this interval the input signal \( f \), i.e., \( \{f(t), k\varepsilon \leq t < (k+1)\varepsilon\} \), is smooth enough to be well approximated by a constant \( f(k\varepsilon) \), i.e., for \( k\varepsilon \leq t < (k+1)\varepsilon \),

\[
f(t) \approx f(k\varepsilon)(u(t - k\varepsilon) - u(t - (k+1)\varepsilon)) = f(k\varepsilon)\delta_{\varepsilon}(t - k\varepsilon) \varepsilon,
\]

where \( \varepsilon f(k\varepsilon)\delta_{\varepsilon}(t - k\varepsilon) \) is zero outside of this short interval of time.
Piecing together these non-overlapping approximations leads to a Riemann approximation of $f$ over $\mathbb{R}$:

$$f(t) \approx \sum_{k=-\infty}^{\infty} f(k\varepsilon)\delta_{k}(t - k\varepsilon), \quad \forall t \in \mathbb{R}$$

Note that as $\varepsilon \downarrow 0$ (i.e., $\varepsilon \downarrow d\tau$, a differential), this approximation becomes more precise tending to a Riemann integral that is the sampling property of the unit impulse:

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t - \tau)d\tau, \quad \forall t \in \mathbb{R}.$$
• Again, we have constructed a Riemann approximation of the input signal as a linear combination of delayed approximations of the impulse:

\[ f \approx \sum_{k=-\infty}^{\infty} f(k\varepsilon) \Delta k \varepsilon \varepsilon \]

• For the (zero initial states) LTI system under consideration, let the ZSR of \( \delta_{\varepsilon} \) be \( h_{\varepsilon} \).

• So, by time-invariance, the zero-state response to \( \Delta k \varepsilon \delta_{\varepsilon} \) is \( \Delta k \varepsilon h_{\varepsilon} \).

• Thus, by linearity of the system, the approximate ZSR to \( f \) is

\[ y_{ZS} \approx \sum_{k=-\infty}^{\infty} f(k\varepsilon) \Delta k \varepsilon h_{\varepsilon} \varepsilon \]

i.e., \( \forall t \), \( y_{ZS}(t) \approx \sum_{k=-\infty}^{\infty} f(k\varepsilon)(\Delta k \varepsilon h_{\varepsilon})(t) \varepsilon \)

As \( \varepsilon \downarrow 0 \), this Riemann approximate integral converges to the precise ZSR to \( f \) is \( \forall t \),

\[ y_{ZS}(t) = \int_{-\infty}^{\infty} f(\tau)(\Delta \varepsilon h)(t) d\tau \]

\[ = \int_{-\infty}^{\infty} f(\tau)h(t-\tau) d\tau \]

\[ = (f \ast h)(t), \ \forall t \in \mathbb{R}, \]

i.e., the convolution of the input \( f \) and (zero state) impulse response

\[ h = \lim_{\varepsilon \downarrow 0} h_{\varepsilon}. \]

• Note how the convolution is a consequence of the LTI property (of the zero-state system).

• Also note that the convolution operator \( (\ast) \) is a bilinear mapping of two signals (here, \( f \) and \( h \)) to a third signal \( (y = f \ast h) \).
Recall that the total response $y$ to this (LTIC) circuit with causal input $f$ and initial condition $y(0^-)$ is $\forall t \geq 0$,

$$
y(t) = y(0^-)e^{-t/(RC)} + \int_{0^-}^{t} f(r) \frac{1}{2RC} e^{-t-r}/(RC) \, dr
$$

$$
= y(0^-)e^{-t/(RC)} + \int_{0^-}^{\infty} f(r) \frac{1}{2RC} e^{-(t-r)/(RC)} u(t-r) \, dr
$$

Clearly, the ZSR portion is $y_{ZS} = f \ast h$ with impulse response $h(t) = (2RC)^{-1}e^{-t/(RC)}u(t)$, $t \in \mathbb{R}$, and $f(t) = 0$ for $t < 0$ (i.e., $f = fu$) since $f$ is causal.

**Total response - example - circuit with switches thrown at 0**

- The circuit is operational for time $t \in \mathbb{R}$, i.e., including negative time when the switch on the right is closed and the one on the left is open.
- At time zero, the two switches are simultaneously thrown.
- Prior to time zero, the circuit was in DC steady state (DC battery of $A$ volts).
- So, $y(0^-) = 0$ and $v_c(0^-) = A$ (capacitor has infinite DC impedance).
- Since $v_c$ is a state of the circuit (it’s the output of an integrator (of $y$)), it’s continuous at 0 so that $v_c(0^-) = v_c(0) = A.$
Total response - example - circuit with switches thrown at 0 (cont)

- But \( y \) is not a state variable and at time 0, \( y(0) = (f(0) - v_c(0))/R \); so \( y \) will not be continuous at 0, i.e., \( y(0-) \neq y(0) \), unless \( f(0) = A \).

- By KVL, \( \forall t \geq 0, f(t) = v_c(t) + R y(t) = v_c(0) + C^{-1} \int_{0^-}^{t} y(\tau) d\tau + R y(t) \).
- Thus, \( R^{-1} D f = D y + (RC)^{-1} y \).

- **Exercise:** Show by the co-factor method and then IBP that the total response is:

\[
\forall t \geq 0, \quad y(t) = y(0-) e^{-t/(RC)} + \int_{0^-}^{t} R^{-1}(D f)(\tau) e^{-(t-\tau)/(RC)} d\tau
\]

\[
= y(0-) e^{-t/(RC)} + \int_{0^-}^{t} f(\tau) \left( \frac{1}{R} \delta(t-\tau) - \frac{1}{R^2 C} e^{-(t-\tau)/(RC)} \right) d\tau
\]

\[
= y(0-) e^{-t/(RC)} + \int_{0^-}^{\infty} f(\tau) h(t-\tau) d\tau,
\]

where the impulse response \( h(t) = \frac{1}{R} \delta(t) - \frac{1}{RC} e^{-t/(RC)} u(t) \).

- Note: The total response will be discontinuous at time 0 when \( f(0) \neq A \), i.e., \( y(0) = y(0-) + f(0)/R \), because \( h \) has an impulse component (again, \( f(0-) := 0 \)).

---

**Convolution of causal signals**

- Generally, if
  - the system is causal, so \( h \) is a causal signal, and
  - the origin of time is zero, so that we can also assume the input \( f(t) = 0 \) for \( t < 0 \) (i.e., \( f \) is a causal signal too),

then

\[
(f * h)(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau = \int_{0^-}^{t} f(\tau) h(t-\tau) d\tau, \quad \forall t \geq 0.
\]

- Here, since \( h \) is causal, the integrand is zero for \( \tau > t \).
- And since \( f \) is causal, is zero for \( \tau < 0 \).
- Here, we integrate from \( 0^- \) so that integrals involving impulses \( \delta \) at the origin are unambiguous, i.e., we’re not "splitting hairs".

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Discussion: the ZSR has zero initial conditions

- We’ve identified the ZSR $y_{ZS}$ to a causal input $f$ of a LTIC system with (causal) impulse response $h$ as the convolution,

$$y_{ZS}(t) = \int_{0-}^{t} f(\tau)h(t-\tau)d\tau, \quad t \geq 0.$$  

- Here, we assume that the input $f$ has zero "initial conditions,"

$$(D^k f)(0-) = 0 \quad \text{for all integers } k \geq 0.$$  

- So, clearly, $y_{ZS}(0-) = 0$ and by differentiating the above display, we get that $(D^k y_{ZS})(0-) = 0$ for all integers $k > 0$ as well.

- But $y_{ZS}$ may not be continuous at time 0.

ZSR and ZIR - op-amp circuit example

- For another example, consider the above circuit input $f$, output $y$, and $v_c(0) = 2V$.

- By KCL at \(\text{"+\}}\) and $v_c$, respectively,

$$\frac{y-f}{1} + \frac{v_c-f}{2} = 0 \quad \text{and} \quad \frac{f-v_c}{2} + 0.5D(0-v_c) = 0. \quad (1)$$

- So, $v_c = 3f - 2y$ and by substitution,

$$Q(D)y = Dy + y = \frac{3}{2}Df + f = P(D)f. \quad (2)$$
• The system characteristic polynomial is \( Q(D) = D + 1 \), so that the system characteristic-value (system pole, root of \( Q \)) is \(-1\), and characteristic mode is \( e^{-t} \).

• Thus, \( y_{ZS}(t) = y(0-)e^{-t}, \quad t \geq 0- \), where (by KCL at “-” op-amp input at time \( 0- \))

\[
y(0-) = \frac{3}{2} f(0-) - \frac{1}{2} v_c(0-) = \frac{3}{2} 0 - \frac{1}{2} 2 = -1;
\]

noting \( v_c(0-) = v_c(0) \) since \( v_c \) is a continuous state variable.

• For the ZSR, if the forced response \( y_F \) is known, one might be tempted to simply write

\[
y_{ZS}(t) = y_F(t) + ce^{-t}u(t),
\]

and solve for the coefficient \( c \) of the characteristic mode by using \( y_{ZS}(0) = 0 \).

• However, we’ll see later that when the degree of \( P \) equals that of \( Q \) (as in this example), and the input \( f(0) \neq 0 \), then \( y_{ZS} \) will be discontinuous at time 0 too.

---

**Op-amp circuit example: ZSR discontinuous at time 0**

• The “transfer function” for this op-amp circuit,

\[
H(s) = \frac{P(s)}{Q(s)} = \frac{1.5s + 1}{s + 1} = 1.5 - \frac{0.5}{s + 1},
\]

is proper but not strictly proper, where \( b_1 = 1.5 \) is the leading coefficient of \( P \).

• We’ll see later that the impulse response \( h \) of this system has an impulse component \( \delta \) (indicating direct coupling between input and output),

\[
h(t) = 1.5 \delta(t) - 0.5e^{-t}u(t).
\]

• Thus, \( y_{ZS} = h \ast f = b_n f + (h - b_n \delta) \ast f \),

- where \( h - b_n \delta \) corresponds to a “strictly proper” system \( \tilde{P}/Q = P/Q - b_n = H - b_n \),
- and so \((h - b_n \delta) \ast f\) is continuous at time 0 and so zero initial (at time 0) conditions apply,
- but \( y_{ZS} \) is discontinuous at time zero if \( f(0) \neq 0 \).
For example, if the input $f(t) = e^{-2t}u(t)$ then $f$ is discontinuous at time 0 and the forced response (or "eigenresponse" since $f$ is exponential) of this system is

$$y_f(t) = H(-2)f(t) = 2f(t).$$

- If we take $y_{ZS}(t) = 2f(t) + ce^{-t}u(t)$ and find $c$ using $y_{ZS}(t) = 0$, we get $c = -2$.
- However, $c = -0.5$ for the true ZSR.
- One can verify this by finding the true ZSR by convolution, $y_{ZS} = h \ast f$ (or by using the unilateral Laplace transform which we will later discuss in detail).

Indeed by linearity, for all $t \geq 0$,

$$y_{ZS}(t) = (f \ast h)(t)$$
$$= e^{-2t}u(t) \ast (1.5\delta(t) - 0.5e^{-t}u(t))$$
$$= 1.5e^{-2t} - 0.5 \int_{0}^{t} e^{-2\tau}e^{-(t-\tau)}d\tau = 1.5e^{-2t} - 0.5e^{-t}(1 - e^{-t})$$
$$= 2e^{-2t} - 0.5e^{-t}.$$

Graphical convolution - example 1

- We now describe how to compute convolutions with the help of graphical illustrations for the different cases of integration.
- For example, let's compute $f \ast h$ where $f(t) = 2e^{-t}u(t)$ and $h(t) = u(t) - u(t - 3), \ \forall t \in \mathbb{R}$.
- Again, $(f \ast h)(t) := \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau$.
- For this example, we will see graphically that there are three cases for time $t$ (which is fixed when integrating over the dummy variable of integration $\tau$).
Graphical convolution - example 1 - case 1

Case 1: When \( t < 0 \),

\[
f(\tau)h(t-\tau) = 0, \quad \forall \tau \in \mathbb{R}.
\]

So, simply because the integrand is constantly zero in this case,

\[
(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau) d\tau = 0.
\]

Graphical convolution - example 1 - case 2

Case 2: When \( t \geq 0 \) and \( t - 3 < 0 \) (i.e., \( 0 \leq t < 3 \)),

\[
(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau) d\tau
\]

\[
= \int_{0}^{t} f(\tau)h(t-\tau) d\tau
\]

\[
= \int_{0}^{t} 2e^{-\tau} \cdot 1 d\tau = 2 - 2e^{-t}.
\]
Case 3: When \( t - 3 \geq 0 \) (i.e., \( t \geq 3 \)),

\[
(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau
\]

\[
= \int_{t-3}^{t} f(\tau)h(t - \tau)d\tau
\]

\[
= \int_{t-3}^{t} 2e^{-\tau} \cdot 1d\tau = 2e^{-(t-3)} - 2e^{-t}.
\]

In summary,

\[
(f * h)(t) = \begin{cases} 
0 & t < 0 \\
2 - 2e^{-t} & 0 \leq t < 3 \\
2e^{-(t-3)} - 2e^{-t} & 3 \leq t 
\end{cases}
\]

Note that \( f * h \) is continuous even though both \( f \) and \( h \) have jump discontinuities.
Convolution is commutative

- Convolution is commutative, i.e.,
  \[ \forall f, h, \quad h * f = f * h, \]

- This is proved by simple change of variable of integration \((\tau' = t - \tau)\).

- To illustrate this, let’s recompute the previous example instead as
  \[ (h * f)(t) = \int_{-\infty}^{\infty} h(\tau) f(t - \tau) d\tau, \quad \forall t \in \mathbb{R}. \]

- Again, we have the same three cases in \(t\).

Graphical convolution - example 1b - case 1

- Case 1: When \(t < 0\),
  \[ h(\tau)f(t - \tau) = 0, \quad \forall \tau \in \mathbb{R}. \]

- Thus, \((h * f)(t) = 0\).
• Case 2: When $0 \leq t < 3$,

$$(h * f)(t) = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau$$

$$= \int_{0}^{t} h(\tau)f(t-\tau)d\tau$$

$$= \int_{0}^{t} 1 \cdot 2e^{-(t-\tau)}d\tau$$

$$= 2e^{-t} \int_{0}^{t} e^{\tau}d\tau = 2e^{-t}(e^{t} - 1) = 2 - 2e^{-t}.

• In summary, we confirm that $f * h = h * f$ for this example.

---

• Case 3: When $t \geq 3$,

$$(h * f)(t) = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau$$

$$= \int_{0}^{3} h(\tau)f(t-\tau)d\tau$$

$$= \int_{0}^{3} 1 \cdot 2e^{-(t-\tau)}d\tau = 2e^{-t} \int_{0}^{3} e^{\tau}d\tau = 2e^{3-t} - 2e^{-t}$$

• In summary, we confirm that $f * h = h * f$ for this example.
• Now consider a graphical convolution example where each signal is of finite span.

• Suppose we wish to compute \( f \ast h \) where

\[
    f(t) = (2 - t)(u(t) - u(t - 2)) \quad \text{and} \quad h(t) = u(t + 1) - u(t - 2), \quad \forall t \in \mathbb{R}.
\]

• Note that \( h \) is not causal.

Graphically, we will see that there are five cases for \( t \).

Graphical convolution - example 2 - case 1

• Case 1: When \( t + 1 < 0 \) (i.e., \( t < -1 \)), \( f(\tau)h(t - \tau) = 0, \quad \forall \tau \in \mathbb{R} \).

• Thus,

\[
(f \ast h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau) d\tau = 0.
\]
Graphical convolution - example 2 - case 2

- Case 2: When \(0 \leq t + 1 < 2\) (i.e., \(-1 \leq t < 1\)),

\[
(f \ast h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t - \tau)d\tau \\
= \int_{0}^{t+1} f(\tau)h(t - \tau)d\tau \\
= \int_{0}^{t+1} (2 - \tau) \cdot 1d\tau = 2 - 0.5(1 - t)^2
\]

Graphical convolution - example 2 - case 3

- Case 3: When \(t + 1 \geq 2\) and \(t - 2 < 0\) (i.e., \(1 \leq t < 2\)),

\[
(f \ast h)(t) = \int_{0}^{2} f(\tau)h(t - \tau)d\tau \\
= \int_{0}^{2} (2 - \tau) \cdot 1d\tau = 2
\]
Case 4: When $0 \leq t - 2 < 2$ (i.e., $2 \leq t < 4$),

$$\begin{align*}
(f * h)(t) &= \int_{t-2}^{2} f(\tau)h(t-\tau)d\tau \\
&= \int_{t-2}^{2} (2 - \tau) \cdot 1 d\tau \\
&= 0.5(4 - t)^2
\end{align*}$$

Case 5: When $t - 2 \geq 2$ (i.e., $t \geq 4$), $f(\tau)h(t - \tau) = 0 \forall \tau \in \mathbb{R}$.

Thus,

$$(f * h)(t) = 0.$$
In summary for this example, we again have a continuous $f * h$:

$$
(f * h)(t) = \begin{cases} 
0 & t < -1 \\
2 - 0.5(1 - t)^2 & -1 \leq t < 1 \\
2 & 1 \leq t < 2 \\
0.5(4 - t)^2 & 2 \leq t < 4 \\
0 & 4 \leq t 
\end{cases}
$$

Also note that in the previous example,

- the support of $f * h$ (time interval over which $f * h$ is non-zero) is of length 5,
- while that of $f$ is 2 and that of $h$ is 3.

Convolution has a general property that, $\forall f, h$, the length of support of $f * h$ is the sum of that of $f$ and that of $h$. 
Convolution - other important properties

- Recall ZSR of LTI systems by convolution with an impulse response.
- One can directly show that convolution is a commutative bi-linear mapping of pairs of (Dirichlet) signals to another, i.e., \( \forall \) signals \( f, g, h \) and scalars \( \alpha, \beta \in \mathbb{C} \),
  \[
  (\alpha f + \beta g) * h = \alpha (f * h) + \beta (g * h)
  \]
- By changing order of integration (Fubini’s theorem), one can easily show that convolution is associative, i.e., \( \forall \) signals \( f, g, h \),
  \[
  (f * g) * h = f * (g * h), \quad \text{cf. next slide.}
  \]
- We’ll use these properties when composing more complex systems from simpler ones.
- By the ideal sampling property, the identity signal for convolution is the impulse \( \delta \), i.e., \( \forall \) signals \( f \),
  \[
  f * \delta = \delta * f = f
  \]
- By just changing variables of integration, we can show how to exchange time-shift with convolution (time-invariance), i.e., \( \forall \) signals \( f, h : \mathbb{R} \to \mathbb{C} \) and times \( \tau \in \mathbb{R} \),
  \[
  (\Delta_\tau f) * h = \Delta_\tau(f * h) \quad \text{(or just associativity & commutativity with } \ g = \Delta_\tau \delta). \]

Convolution - proof of associative property

- \( \forall \) signals \( f, g, h \) and \( \forall t \in \mathbb{R} \),
  \[
  ((f * g) * h)(t) = \int_{-\infty}^{\infty} (f * g)(\tau) h(t - \tau) d\tau
  = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\sigma) g(\tau - \sigma) d\sigma h(t - \tau) d\tau
  = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau - \sigma) h(t - \tau) d\tau f(\sigma) d\sigma, \quad \text{by Fubini’s theorem}
  = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau') h((t - \sigma) - \tau') d\tau' f(\sigma) d\sigma, \quad \tau' = \tau - \sigma
  = \int_{-\infty}^{\infty} (g * h)(t - \sigma) f(\sigma) d\sigma
  = (f * (g * h))(t).
  \]
- Q.E.D.
Recall that for this series RLC circuit with $R = 700\,\Omega$, $L = 0.1\,\text{H}$, and $C = 1\,\mu\text{F}$:

\[
D^2 y + RL^{-1} D y + (LC)^{-1} y = RL^{-1} D f \\
(D + 2000)(D + 5000)y = 7000 D f \\
Q(D)y = P(D)f
\]

We'll show later that the impulse response for this circuit is

\[
h(t) = \left( -\frac{14000}{3} e^{-2000t} + \frac{35000}{3} e^{-5000t} \right) u(t), \quad t \in \mathbb{R}.
\]
• If the input is, say, \( f(t) = 5e^{-1000t}u(t) \), \( t \in \mathbb{R} \), then the ZSR \( y_{ZS} = f \star h \) is:

\[
\begin{align*}
\forall t < 0, \quad y_{ZS}(t) &= 0 \\
\forall t \geq 0, \quad y_{ZS}(t) &= \int_{0}^{t} h(\tau)f(t - \tau)d\tau \\
&= \int_{0}^{t} (-14000/3 e^{-2000\tau} + 35000/3 e^{-5000\tau})5e^{-1000(t-\tau)}d\tau \\
&= -7000/3 e^{-1000t} \int_{0}^{t} (2e^{-1000\tau} - 5e^{-4000\tau})d\tau \\
&= -7000/3 e^{-1000t} \left( \frac{2}{1000} (1 - e^{-1000\tau}) - \frac{5}{4000} (1 - e^{-4000\tau}) \right) \\
&= -7/4 e^{-1000t} + 70/3 e^{-2000t} - 175/12 e^{-5000t}
\end{align*}
\]

• Note how the ZSR consists of a forced response plus characteristic modes.

• Re. the forced (eigen-) response, note that the transfer function

\[
H(-1000) = \frac{7}{4} \quad \text{where} \quad H(s) = \frac{P(s)}{Q(s)} = \frac{7000s}{(s + 2000)(s + 5000)}.
\]

Total response - second-order, series RLC circuit example (cont)

• Recall if the initial conditions are \((Dy)(0-) = -2000V/s\) and \(y(0-) = 10V\), then the ZIR is

\[
\forall t \geq 0-, \quad y_{ZI}(t) = 16e^{-2000t} - 6e^{-5000t}.
\]

• Combining with the the ZSR just derived, the total response of the system (to both initial conditions and input) is: \( \forall t \geq 0- \),

\[
y(t) = y_{ZI}(t) + y_{ZS}(t) \\
= \sum_{k=1}^{n} c_k \mu_k(t) + (h \star f)(t) \\
= 16e^{-2000t} - 6e^{-5000t} + \left( -\frac{7}{4} e^{-1000t} + \frac{70}{3} e^{-2000t} - \frac{175}{12} e^{-5000t} \right) u(t)
\]
Total response - discussion of existence and uniqueness

- The existence of the ZIR follows from the linear independence of characteristic modes.
- Similarly, the existence of the ZSR follows from the assumption that the input and impulse response are Dirichlet signals, so that the convolution is well defined.
- When the initial conditions are (respectively, input signal is) zero, the unique solution for the ZIR (respectively, ZSR) is the zero signal.
- Given this, it's easy to prove by contradiction the uniqueness of the ZIR and ZSR in general.
- For example, assume that \( y_{ZI} \) and \( \tilde{y}_{ZI} \) are two different ZIRs for the same system \((Q)\) and the same initial conditions:
  - Note that the signal \( y_{ZI} - \tilde{y}_{ZI} \) is the ZIR for \( Q \) with zero initial conditions.
  - That is, \( y_{ZI} - \tilde{y}_{ZI} = 0 \), a contradiction.
  - So, the original assumption of the existence of two different ZIRs is false.

System stability - ZIR - asymptotically stable

- Consider a SISO system with input \( f \) and output \( y \).
- Recall that the ZIR \( y_{ZI} \) is a linear combination of the system's characteristic modes, where the coefficients depend on the initial conditions.
- A system is said to be asymptotically stable if for all initial conditions,
  \[
  \lim_{t \to \infty} y_{ZI}(t) = 0.
  \]
- So, a system is asymptotically stable if and only if all of its characteristic values have strictly negative real part.
- For example, if the characteristic polynomial \( Q(s) = (s + 5)(s + 7) \), then
  - the system’s characteristic values (roots of \( Q \)) are \(-7, -5\) (strictly negative and real),
  - the ZIR is of the form \( y_{ZI}(t) = c_1 e^{-7t} + c_2 e^{-5t}, \ t \geq 0 \),
  - So, \( y_{ZI}(t) \to 0 \) as \( t \to \infty \) for all \( c_1, c_2 \), and
  - hence is asymptotically stable.
- If instead \( Q(s) = (s + 5)^2 \), then the characteristic modes would be \( e^{-5t} \), \( te^{-5t} \), both of which \( \to 0 \) as \( t \to \infty \), so this system is still asymptotically stable.
System stability - bounded signals

- A signal \( y \) is said to be bounded if
  \[ \exists M < \infty \text{ s.t. } \forall t \in \mathbb{R}, |y(t)| \leq M; \]
otherwise \( y \) is said to be unbounded.
- For example, \( y(t) = 3e^{-5t}u(t), \ t \in \mathbb{R} \) is bounded (can use \( M = 3 \)).
- Also, \( 3 \cos(5t), \ t \in \mathbb{R} \), is bounded (again, can use \( M = 3 \)).
- But both \( 3e^{2t} \cos(5t) \) and \( 3e^{2t} \) are unbounded; for the latter example, obviously
  \[ \lim_{t \to \infty} 3e^{2t} = \infty. \]

System stability - ZIR - marginally stable

- A system is said to be marginally stable if it is not asymptotically stable but \( y_{ZI} \) is always (for all initial conditions) bounded.
- A system is marginally stable if and only if
  - it has no characteristic values with strictly positive real part,
  - it has at least one purely imaginary characteristic value, and
  - all purely imaginary characteristic values are not repeated.
- That is, a marginally stable system
  - has some characteristic modes of the form \( \cos(zt) \) or \( \sin(zt) \),
  - while the rest are of the form \( t^k e^{-at} \cos(zt) \) or \( t^k e^{-at} \sin(zt) \),
  - for real \( a > 0 \), frequency \( \omega \geq 0 \) and integer degree \( k \geq 0 \) (and time \( t \geq 0 \)).
- For example, if the characteristic polynomial is \( Q(s) = s(s^2 + 2)(s + 3) \), with characteristic values \( 0, \pm j\sqrt{2} \) and \(-3\), then the system is marginally stable with modes \( 1, \cos(\sqrt{2}t), \sin(\sqrt{2}t) \) and \( e^{-3t} \), the first three of which are bounded but do not tend to zero as time \( t \to \infty \).
System stability - ZIR - unstable

- A system that is neither asymptotically nor marginally stable (i.e., a system with unbounded modes) is said to be unstable.

- For example, if the characteristic polynomial is \( Q(s) = (s^2 + 2)^2(s + 3) \) then the purely imaginary characteristic values \( \pm j\sqrt{2} \) are repeated, and hence the two additional modes \( t \sin(\sqrt{2}t), t \cos(\sqrt{2}t) \) are unbounded - so this system is unstable.

- Similarly, the system with \( Q(s) = s^2(s^2 + 2)(s + 3) \) is unstable owing to the repeated characteristic value at 0 with associated modes 1 and \( t \), the latter unbounded.

- For another example, the system with \( Q(s) = (s^2 + 2)(s - 3) \) is unstable owing to the (unstable) characteristic value at 3 with associated unbounded mode \( e^{3t} \).

System stability - ZIR - series RLC circuit example

- If the output signal is redefined as the capacitor voltage, then \( P(s) = (LC)^{-1} = w_o^2 \) but the characteristic polynomial remains
  \[
  Q(s) = s^2 + 2\zeta w_o s + w_o^2,
  \]
  where
  - resonant frequency \( w_o = 1/\sqrt{LC} \), and
  - damping factor \( \zeta = (R/2)\sqrt{C/L} \).
  - The characteristic values (roots of \( Q \)) of the system are
  \[
  -w_o \left( \zeta \pm \sqrt{\zeta^2 - 1} \right)
  \]

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Cases regarding the discriminant $\zeta^2 - 1$ (assuming fixed $L, C > 0$):

- $\zeta = 0$ ($R = 0$): two purely imaginary characteristic values $\pm jw_o$, i.e., marginally stable.

- $0 < \zeta < 1$ ($0 < R < 2\sqrt{L/C}$): two complex conjugate characteristic values of strictly negative real part and magnitude $w_o$, $c_{\pm} := -w_o(\zeta \pm j\sqrt{1-\zeta^2})$, i.e., asymptotically stable, underdamped.

- $\zeta = 1$ ($R = 2\sqrt{L/C}$): a repeated (double) real characteristic value at $-w_o < 0$, i.e., asymptotically stable, critically damped.

- $1 < \zeta < \infty$ ($R > 2\sqrt{L/C}$): two different real strictly negative char. values, $r_{\pm} := -w_o(\zeta \pm \sqrt{\zeta^2 - 1})$, i.e., asymptotically stable, overdamped.

---

**System stability - ZSR - BIBO stable**

- A SISO system is said to be *Bounded Input, Bounded Output* (BIBO) stable if $\forall$ bounded input signals $f$, the ZSR $y_{ZS}$ is bounded.

- A necessary condition for BIBO stability is absolute integrability of the impulse response,

$$\int_0^\infty |h(\tau)|d\tau < \infty.$$  

- To see why: For arbitrary bounded input $f$ (by $M_f$ with $0 \leq M_f < \infty$): $\forall t \geq 0$,

$$|y_{ZS}(t)| = |(f * h)(t)| = \left|\int_0^t f(t - \tau)h(\tau)d\tau\right| \leq \int_0^t |f(t - \tau)h(\tau)|d\tau \leq \int_0^t |f(\tau)|d\tau \leq M_f \int_0^\infty |h(\tau)|d\tau =: M_y < \infty,$$
The condition of absolute integrability of the impulse response,

\[ \int_0^\infty |h(\tau)| \, d\tau < \infty, \]

is also necessary for, and hence equivalent to, BIBO stability.

When we study LTI systems in the frequency domain and derive the impulse response, we will readily see how asymptotic stability (of ZIR) is equivalent to BIBO stability (of ZSR).

**Example:** If the system has char. values \( \pm jw_h \) of multiplicity 1, and the input is \( f(t) = c_f \cos(w_f t)u(t) \), then \( y_{ZS}(t) \) has the term

\[ \frac{c_h c_f}{2} \int_0^t \left( \cos((w_h - w_f)\tau + w_f t) + \cos((w_h + w_f)\tau - w_f t) \right) \, d\tau; \]

so if \( w_f \neq w_h \), then \( y_{ZS} \) is bounded, else \( y_{ZS}(t) \) has the term \( 0.5c_h c_f t \cos(w_f t) \) which is unbounded - i.e., marginally stable system is not BIBO.

For another example, see Ex. 2.6-4 of Lathi.

---

**ZSR - the transfer function, \( H \)**

Recall that for any polynomial \( Q \) and frequency \( s \in \mathbb{C} \) (including \( s = jw, \ w \in \mathbb{R} \)),

\[ Q(D)e^{st} = Q(s)e^{st}, \quad \forall t \geq 0. \]

So, if we guess that \( y_D(t) = H(s)e^{st}, \ t \geq 0, \) is a “particular” solution of \( Q(D)y = P(D)f \) in response to input \( f(t) = e^{st}u(t) \), we get by substitution that

\[ \forall t \geq 0, s \in \mathbb{C}, \quad H(s)Q(s)e^{st} = Q(D)y = P(D)f = P(s)e^{st} \]

\[ \Rightarrow H(s) = P(s)/Q(s). \]

The “rational polynomial” \( H = P/Q \) is known as the system’s transfer function and will figure prominently in our study of frequency-domain analysis.

So, the ZSR (forced response + characteristic modes) would be of the form:

\[ y_{ZS}(t) = H(s)e^{st} + \text{linear combination of char. modes, } t \geq 0, \]

again with \( (D^k y_{ZS})(0-) := 0 \ \forall \ \text{integers } k \geq 0. \)

Recall for the previous series RLC circuit example with input \( f(t) = 5e^{-1000t}, \ t \geq 0: \)

\[ H(s) = \frac{P(s)}{Q(s)} = \frac{7000}{(s + 2000)(s + 5000)}, \ s \in \mathbb{C} \]

\[ y_{ZS}(t) = H(-1000)f(t) + \text{linear combination of char. modes, } t \geq 0. \]
ZSR - impulse response, transfer function, and eigenresponse

- \( y_{ZS}(t) = H(s)Ae^{st+j\phi} + \text{linear combination of char. modes}, \) with \( t \geq 0 \) and \( A > 0, \phi \in \mathbb{R}, \) is the ZSR to input \( f(t) = Ae^{st+j\phi}u(t), \) with \( (D^ky_{ZS})(0-) := 0 \) for all integers \( k \geq 0. \)

- The \textit{eigenresponse} is a special case of the forced response for exponential inputs.

- If \( s = jw \) for \( w \in \mathbb{R} \) (i.e., input \( f \) is sinusoidal) and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:

\[
y(t) \sim H(jw)Ae^{j(wt+\phi)} \quad \text{as } t \to \infty.
\]

- Since \( y = h \ast f, \) we get that as \( t \to \infty \) for a causal and asymptotically stable system,

\[
y_{ZS}(t) = \int_0^t h(\tau)Ae^{j(w(t-\tau)+\phi)}d\tau = \left(\int_0^t h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j(wt+\phi)} \sim H(jw)Ae^{j(wt+\phi)},
\]

\[
\Rightarrow \int_0^\infty h(\tau)e^{-j\omega\tau}d\tau = H(jw), \quad \forall w \in \mathbb{R}.
\]

- The eigenresponse is a special case of the forced response for exponential inputs.

- If \( s = jw \) for \( w \in \mathbb{R} \) (i.e., input \( f \) is sinusoidal) and the system is asymptotically stable, then the ZSR tends to the steady-state eigenresponse of the system:

\[
y(t) \sim H(jw)Ae^{j(wt+\phi)} \quad \text{as } t \to \infty.
\]

- Since \( y = h \ast f, \) we get that as \( t \to \infty \) for a causal and asymptotically stable system,

\[
y_{ZS}(t) = \int_0^\infty h(\tau)Ae^{j(w(t-\tau)+\phi)}d\tau = \left(\int_0^\infty h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j(wt+\phi)} \sim H(jw)Ae^{j(wt+\phi)},
\]

\[
\Rightarrow \int_0^\infty h(\tau)e^{-j\omega\tau}d\tau = H(jw), \quad \forall w \in \mathbb{R}.
\]

ZSR for single-pole systems using eigenresponse

- We will see that \( y_{ZS} \) will be continuous at time 0 when \( H = P/Q \) is strictly proper (i.e., \( b_n = 0 \) so that \( m := \text{degree of } P < n := \text{degree of } Q \)); thus, \( h \) has no \( b_n \delta \) term and there is no “direct coupling” between input and output in the system.

- So given a strictly proper system with a single pole \( p \) (characteristic value), i.e., \( n = 1, \) the ZSR to a sinusoidal input \( f = Ae^{st+j\phi} \) will be of the form

\[
y_{ZS}(t) = H(s_0)f(t) + ce^{-pt} = H(s_0)Ae^{st+j\phi} + ce^{-pt}, \quad \forall t \geq 0.
\]

- So, if \( H \) is strictly proper, we can solve for the coefficient \( c \) of the system mode \( e^{-pt} \) using \( y_{ZS}(0) = 0, \) i.e.,

\[
0 = H(s_0)Ae^{j\phi} + c \quad \Rightarrow \quad c = -H(s_0)Ae^{j\phi}.
\]
Remark: time origin at $-\infty$ for an asymptotically stable system

- If we take the origin of time to be $-\infty$ for an asymptotically stable LTI system, then the system is in steady-state in “finite time” when the characteristic-modes component of the response will be negligible.

- So, for frequency $w \in \mathbb{R}$, the response to the persistent sinusoidal input $f(t) = Ae^{i(wt+\phi)}$ is, $\forall t \in \mathbb{R}$,
  \[
  \int_{-\infty}^{\infty} h(\tau) Ae^{i(w(t-\tau)+\phi)} d\tau = \left( \int_{-\infty}^{\infty} h(\tau) e^{-j\omega \tau} d\tau \right) Ae^{i(wt+\phi)} = H(jw) A e^{i(wt+\phi)},
  \]
  where the frequency $w \in \mathbb{R}$ and the integral is finite since $h$ is assumed Dirichlet.

- Thus, $H$ is the bilateral Fourier transform of the impulse response $h$,
  \[
  \int_{-\infty}^{\infty} h(\tau) e^{-j\omega \tau} d\tau = H(j\omega), \ \forall \omega \in \mathbb{R}.
  \]

- For a causal system (and as in the previous slide),
  \[
  \int_{0}^{\infty} h(\tau) e^{-j\omega \tau} d\tau = H(j\omega), \ \forall \omega \in \mathbb{R}.
  \]

- Resonance of the marginally stable series RLC circuit

  - Recall that the series RLC circuit with capacitor-voltage output is marginally stable when $R = 0$ (damping factor $\zeta = 0$) with two purely imaginary characteristic values at $\pm j/\sqrt{LC} = \pm jw_o$.

  - All sinusoidal inputs are, of course, bounded.

  - Consider an input of the form
    \[ f(t) = e^{jwt} = \cos(wt) + j\sin(wt). \]

  - When $w \neq w_o$, the steady-state total response to this complex-valued, bounded sinusoidal input, consisting of the eigenresponse $H(jw) f(t)$ plus a linear combination of the (persistent) characteristic modes $e^{\pm jw_o t}$, is bounded

  - However, if $w = w_o$ then $Q(jw_o) = 0$ and $P(jw_o) = w_o^2 \neq 0$; so $|H(jw)|$ diverges as $w \to w_o$. 
Resonance of the marginally stable series RLC circuit (cont)

• That is, though \( f(t) = e^{jw_o t} \) is a bounded input, the output (of the marginally stable circuit with resonant frequency \( w_o \)) is not bounded.

• This phenomenon is called resonance.

• When we study transient response of resonant circuits in the (complex) frequency domain, we will see that the total response of this system to \( f(t) = e^{jw_o t} \) has an unbounded output component \( te^{jw_o t} \).

• In the following, we will refer to the characteristic values of the system (roots of the characteristic polynomial \( Q \), asymptotes of transfer function \( H \)), as system poles.

Zero-state step response of causal LTI systems

• The zero-state step response of a causal LTI system with impulse response \( h \) is \( y_u = u * h \).

• That is, \( \forall t < 0, \ y_u(t) = 0 \), and \( \forall t \geq 0, \ y_u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{0}^{t} h(\tau)d\tau = H(0) \cdot 1 + \) linear combo. of char. modes

where the (steady-state, zero-frequency/DC) eigenresponse is

\[
H(0) = \int_{0}^{\infty} h(\tau)d\tau.
\]

• So, if the system is asymptotically stable, then

\[
\lim_{t \to \infty} y_u(t) = H(0).
\]

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Step response of a first-order circuit - time constant, 90% rise time

- Recall that this circuit has system equation $Dy + (RC)^{-1}y = (2RC)^{-1}f$ and impulse response $h(t) = \frac{1}{2RC}e^{-t/(RC)}u(t)$.

- Also, $H(0) = P(0)/Q(0) = 0.5$ (consistent with the infinite DC impedance of a capacitor so that the circuit is an even voltage divider in DC steady-state).

- The (ZS) step response is $y_u(t) = (h * u)(t) = \int_0^t h(\tau)d\tau = 0.5(1 - e^{-t/(RC)})u(t)$, \( t \geq 0 \).

- The 90\% rise-time of a system is the first time $t_{90}$ at which it attains 90\% of the terminal (steady-state) value of its step response, $y_u(t_{90}) = 0.9 \cdot 0.5 \Rightarrow t_{90} = RC\ln(10)$ where $RC$ is the time constant of this circuit; i.e., $t_{90}$ is $\ln(10) \approx 2.3$ time-constants.

Step response of series RLC circuit - preliminaries

- In this series RLC circuit, the capacitor voltage has been designated the output signal.

- Thus, the system equation is $D^2 y + RL^{-1}Dy + (LC)^{-1}y = (LC)^{-1}f$.

- Again, $P(s) = (LC)^{-1}$, but the characteristic polynomial remains $Q(s) = s^2 + RL^{-1}s + (LC)^{-1} = s^2 + 2w_0\zeta s + w_0^2$ where again the resonant frequency and damping factor are, respectively, $w_0 = 1/\sqrt{LC}$ and $\zeta = (R/2)\sqrt{C/L}$.

- So the characteristic values / poles again are $-w_0(\zeta \pm \sqrt{\zeta^2 - 1})$. 

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Step response of series RLC circuit - overdamped case, $\zeta > 1$

- In the overdamped case ($\zeta > 1$), there are two different, negative real poles, $r_{\pm}$.

- Again, we'll see later how the impulse response is
  \[ h(t) = \frac{w_o}{2\sqrt{\zeta^2 - 1}}(e^{r_+ t} - e^{r_- t}) u(t), \quad t \in \mathbb{R}. \]

- Thus, the ZS step response is
  \[ y_u(t) = h * u, \]
  \[ y_u(t) = \left(1 + \frac{w_o^2}{r_+(r_+ - r_-)}e^{r_+ t} + \frac{w_o^2}{r_-(r_- - r_+)}e^{r_- t}\right) u(t), \quad t \in \mathbb{R}. \]

- Again, note how the zero-state step response consists of
  - the eigenresponse $H(0) u(t) = u(t)$, with $H(s) = P(s)/Q(s) = (LC)^{-1}/(s^2 + RL^{-1}s + (LC)^{-1})$ so that $H(0) = 1$,
  - and a linear combination of characteristic modes, whose coefficients are such that the initial conditions of $y_u$ are zero consistent with a zero-state step response, $y_u(0) = 0$ and $(D y)(0) = 0$.

Step response of series RLC circuit - overdamped case, $\zeta > 1$ (cont)

- One can directly check that the step response monotonically increases:
  \[ y_u(0) = 0 \]
  \[ (D y_u)(t) \geq 0 \quad \forall t \geq 0 \]
  \[ \lim_{t \to \infty} y_u(t) = 1 = H(0) = P(0)/Q(0). \]

- To find an approximate 90% rise time in the special case when $\zeta \gg 1$ (so that $\exp(r_- t_{90}) \ll \exp(r_+ t_{90}) < 1$),
  \[ 0.9 = 0.9 y_u(\infty) = y_u(t_{90}) \approx 1 + \frac{w_o^2}{r_+(r_+ - r_-)}e^{r_+ t_{90}} \]
  \[ \Rightarrow t_{90} \approx \frac{1}{-r_+ \ln \left(\frac{10w_o^2}{-r_+(r_+ - r_-)}\right)} \]
Step response of series RLC circuit - overdamped case (cont)

- This is the step response $y_u(t)$ for the series RLC circuit example with $R = 700\Omega$, $L = 0.1\, \text{H}$, and $C = 1\, \mu\text{F}$, so that
  $$w_0 = 3162\, \text{rad/s} \quad \text{and} \quad \zeta = 1.107.$$  
- Since $\zeta > 1$, this is the overdamped case.  
- From the plot, $t_{90} \approx 0.0015\, \text{s}$.

Step response of series RLC circuit - critically damped case, $\zeta = 1$

- In the critically damped case ($\zeta = 1$), there is a repeated (double), negative real pole at $-w_0$.
- The impulse response is now
  $$h(t) = w_0^2 t e^{-w_0 t} u(t), \quad t \in \mathbb{R}.$$  
- Thus, the ZS step response $h * u$ is (after integration by parts),
  $$y_u(t) = \left(1 - w_0 t e^{-w_0 t} - e^{-w_0 t}\right) u(t), \quad t \in \mathbb{R},$$  
  noting the characteristic modes $e^{-w_0 t}$ and $t e^{-w_0 t}$.
- Again, the step response monotonically increases:
  
  \[
  \begin{align*}
  y_u(0) &= 0 \\
  (D_y u)(t) &\geq 0 \quad \forall t \geq 0 \\
  \lim_{t \to \infty} y_u(t) &= 1 = H(0).
  \end{align*}
  \]
Step response of series RLC circuit - underdamped case, \(0 < \zeta < 1\)

- In the underdamped case \((0 < \zeta < 1)\), there are two complex-conjugate poles of magnitude \(w_o\) and negative real part,

\[-wo(\zeta \pm j\sqrt{1 - \zeta^2}).\]

- Here the impulse response is

\[h(t) = \frac{wo}{\sqrt{1 - \zeta^2}} e^{-wo\zeta t} \sin(wo\sqrt{1 - \zeta^2} t) u(t), \quad \forall t \in \mathbb{R}.\]

- Thus, the ZS step response \(h \ast u\) is (after integration by parts): \(\forall t \in \mathbb{R},\)

\[y_u(t) = \left(1 - e^{-wo\zeta t} \cos(wo\sqrt{1 - \zeta^2} t) - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-wo\zeta t} \sin(wo\sqrt{1 - \zeta^2} t)\right) u(t).\]

Step response of series RLC circuit - underdamped case (cont)

- One can directly check that

\[y_u(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} y_u(t) = 1,
\]

but \(y_u\) is not monotonically increasing in the underdamped case.

- Indeed, \(y_u\) overshoots its terminal value \(y_u(\infty) = 1\).

- By solving the first-order condition, \(D y_u = 0\), we get that local extrema of \(y_u\) occur at times \(k\pi/(wo\sqrt{1 - \zeta^2})\) for integers \(k > 0\).

- In particular, \(k = 1\) corresponds to the maximum overshoot,

\[y_u \left(\frac{\pi}{wo\sqrt{1 - \zeta^2}}\right) - y_u(\infty) = e^{-\pi\zeta/\sqrt{1 - \zeta^2}} > 0,
\]

see the following figure.

- Note that the overshoot decreases with damping factor \(\zeta\).
Step response of series RLC circuit - underdamped case (cont)

- This is the previous numerical example but with the resistance reduced to $R = 100\Omega$ so that $\zeta = 0.16 < 1$, i.e., the underdamped case, and unchanged $\omega_0 = 3162\text{rad/s}$.

- Here, we see maximum overshoot of about $0.6 \approx \exp(-\pi\zeta/\sqrt{1-\zeta^2})$ at about $0.001 \approx \pi/(\omega_0\sqrt{1-\zeta^2})$ seconds.

Toward frequency domain methods

- We now begin our journey into the frequency domain.

- We first study Fourier-series representation of periodic signals in the time domain.

- Our origin of time will be $-\infty$ so that a steady-state eigenresponse of an asymptotically stable system is reached in finite time.

- Subsequently, we will study the full Zero-State Response (ZSR, including modes) using the bilateral Fourier transform on the real frequency domain, $\omega \in \mathbb{R}$.

- The Fourier transform can accommodate aperiodic input signals and not asymptotically stable and noncausal systems.

- Finally, as we have just done in the time domain, we will study the total system response (ZSR + ZIR) using the unilateral Laplace transform in the complex frequency domain, $s \in \mathbb{C}$.
Fourier-series representation of periodic signals - Outline

- Approximating Dirichlet signals $f : \mathcal{I} \rightarrow \mathbb{R}$ by minimizing square error over a finite time-interval (period) $\mathcal{I} \subset \mathbb{R}$
  - Linear vector space of signals
  - Norm, inner product, and LLSE estimation
  - Orthogonal signal-sets
  - LLSE given (projection onto the span of) orthogonal signal-sets

- Fourier series representation of periodic, Dirichlet signals $f : \mathbb{R} \rightarrow \mathbb{R}$
  - Trigonometric Fourier series: DC, fundamental, and harmonic components.
  - Compact trigonometric Fourier series.
  - Exponential Fourier series.

- Parseval’s theorem for signal power.

- Eigenresponse to a periodic input, i.e., steady-state response of an asymptotically stable LTI system.

Dirichlet signals on a finite time-interval

- We begin our study of Fourier series by reviewing basics of linear algebra
  - for continuous-time signals $f := \{ f(t) \mid t \in \mathcal{I} \}$ on a finite time-interval $\mathcal{I} \subset \mathbb{R}$ (later a period of the assumed-periodic $f$),
  - instead of vectors $\mathbf{v} \in \mathbb{R}^k$, which can be thought of as discrete-time signals on a finite time-interval, i.e., $\mathbf{v} := \{ v_l \mid l \in \{1, 2, \ldots, k\} \}$ for positive integer $k \geq 1$.

- Again, we will restrict consideration to Dirichlet signals $f : \mathcal{I} \rightarrow \mathbb{R}$, i.e., satisfying
  - the weak Dirichlet condition of absolute integrability, $\int_{\mathcal{I}} |f(t)| dt < \infty$,
  - the strong Dirichlet conditions: over any finite interval $\subset \mathcal{I}$; (i) $f$ has a finite number of jump discontinuities, and (ii) $f$ has a finite number of minima and maxima.

- The set of all such signals forms a linear vector space, e.g., it is closed under scalar multiplication and vector addition: $\forall$ such signals $f, g$ and scalars $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g : \mathcal{I} \rightarrow \mathbb{R}$ is also Dirichlet.
Square error between a signal and its estimator

- Suppose the signal \( \hat{f} \) is an approximation of \( f \) over the time-interval \( \mathcal{I} \).

- The square error between \( f \) and \( \hat{f} \) is
  \[
  ||f - \hat{f}||^2 := \int_{\mathcal{I}} |f(t) - \hat{f}(t)|^2 dt,
  \]
i.e., the squared L₂ norm of the error signal \( \varepsilon := f - \hat{f} \).

- This definition of norm has the properties of the Euclidean norm of vectors of a finite-dimensional vector space, i.e.,
  - \( \forall \) (Dirichlet) signals \( x \), \( ||x|| := \sqrt{\int_{\mathcal{I}} |x(t)|^2 dt} = 0 \Leftrightarrow x = 0 \), i.e., \( x \) is the zero signal \( (x(t) = 0 \text{ for } \text{“almost all” } t \in \mathcal{I}) \),
  - \( \forall \) signals \( x \) and scalars \( c \in \mathbb{R} \), and \( ||cx|| = |c| \cdot ||x|| \).
  - \( \forall \) signals \( x, y \), \( ||x + y|| \leq ||x|| + ||y|| \) (triangle inequality).

---

**LLSE estimator - projection and inner product**

- Consider signals \( f, g : \mathcal{I} \to \mathbb{R} \) with \( g \neq 0 \).

- We wish to approximate \( f \) "given \( g \)". i.e., using \( \hat{f} = \alpha g \) by minimizing over the scalar \( \alpha \in \mathbb{R} \) the square error
  \[
  ||f - \hat{f}||^2 = ||f - \alpha g||^2 =: E(\alpha).
  \]

- By definition of norm and linearity of the integral,
  \[
  E(\alpha) = \int_{\mathcal{I}} |f(t) - \alpha g(t)|^2 dt
  = \int_{\mathcal{I}} |f(t)|^2 dt - 2\alpha \int_{\mathcal{I}} f(t) g(t) dt + \alpha^2 \int_{\mathcal{I}} |g(t)|^2 dt,
  \]
i.e., \( E \) is quadratic.

- Since \( E \) is convex, it is minimized at \( \alpha \) satisfying the first-order optimality condition,
  \[
  0 = E'(\alpha) = -2 \int_{\mathcal{I}} f(t) g(t) dt + 2\alpha \int_{\mathcal{I}} |g(t)|^2 dt
  \Rightarrow \alpha = \frac{\int_{\mathcal{I}} f(t) g(t) dt}{\int_{\mathcal{I}} |g(t)|^2 dt}.
  \]

---
• That is, the Linear, Least Square-Error (LLSE) estimator, of \( f \) given \( g \neq 0 \) is

\[
\left( \frac{\int_I f(t)g(t)\,dt}{\int_I |g(t)|^2\,dt} \right) g = \frac{\langle f, g \rangle}{\|g\|^2} g,
\]

where we have identified the inner product operator mapping two signals to a scalar,

\[
\langle f, g \rangle := \int_I f(t)g(t)\,dt \in \mathbb{R}.
\]

• The LLSE estimator of the signal \( f \) given the signal \( g \neq 0 \) is the projection of \( f \) onto the one-dimensional vector space (of signals)

\[
\{ \alpha g | \alpha \in \mathbb{R} \} = \text{span}(g).
\]

LLSE estimator - LLSE signal and given signals are orthogonal

• The error signal for this LLSE estimator is

\[
\varepsilon := f - \frac{\langle f, g \rangle}{\|g\|^2} g = f - \hat{f}.
\]

• Note that the inner product of real-valued signals is obviously linear in both arguments and commutative.

• Also, for all signals \( g \),

\[
\langle g, g \rangle = \| g \|^2.
\]

• Thus, if \( g \neq 0 \) then

\[
\langle \varepsilon, g \rangle = \langle f - \frac{\langle f, g \rangle}{\|g\|^2} g, g \rangle = \langle f, g \rangle - \frac{\langle f, g \rangle}{\|g\|^2} \langle g, g \rangle = 0,
\]

i.e., the error \( \varepsilon = f - \alpha g \) of the LLSE estimator of \( f \) given \( g \) is orthogonal to \( g \).
LLSE estimator given orthogonal signal-sets

- Consider a group of \( k \geq 1 \) non-zero signals
  \[ g_1, g_2, \ldots, g_k : \mathcal{I} \to \mathbb{R} \]
  that are orthogonal to each other, i.e.,
  \[ \forall p \neq q \in \{1, 2, \ldots, k\}, \quad \int_{\mathcal{I}} g_p(\tau)g_q(\tau)d\tau = \langle g_p, g_q \rangle = 0. \]

- For any signal \( f : \mathcal{I} \to \mathbb{R} \), we can easily generalize the LLSE estimator given one signal to find the LLSE that is a linear combination of these \( k \) given signals:
  \[ \hat{f} = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_k g_k, \]
  i.e., the LLSE estimator.

- To this end, define the square-error function,
  \[ E(\alpha_1, \alpha_2, \ldots, \alpha_k) = ||f - \hat{f}||^2 = ||\varepsilon||^2 = \langle \varepsilon, \varepsilon \rangle. \]

LLSE estimator given orthogonal signal-sets (cont)

- By the linearity and commutativity property of inner product,
  \[ E(\alpha_1, \alpha_2, \ldots, \alpha_k) = \int_{\mathcal{I}} |f(\tau) - \alpha_1 g_1(\tau) - \alpha_2 g_2(\tau) - \ldots - \alpha_k g_k(\tau)|^2 d\tau \]
  \[ = \langle f - \sum_{i=1}^{k} \alpha_i g_i, f - \sum_{i=1}^{k} \alpha_i g_i \rangle \]
  \[ = ||f||^2 + 2 \sum_{p<q} \alpha_p \alpha_q \langle g_p, g_q \rangle + \sum_{l} (-2\alpha_l \langle f, g_l \rangle + \alpha_l^2 ||g_l||^2) \]

- By the orthogonality assumption, i.e., \( \forall p \neq q, \langle g_p, g_q \rangle = 0 \),
  \[ E(\alpha_1, \alpha_2, \ldots, \alpha_k) = ||f||^2 + \sum_{l} (-2\alpha_l \langle f, g_l \rangle + \alpha_l^2 ||g_l||^2) \]

- Again, the coefficients \( \alpha_l \) that minimize \( E \) are obtained by the \( k \) first-order conditions
  \[ 0 = \frac{\partial E}{\partial \alpha_l} = -2\alpha_l \langle f, g_l \rangle + 2\alpha_l ||g_l||^2, \quad l \in \{1, 2, \ldots, k\}. \]
Thus, the LLSE coefficients are
\[ \alpha_l = \frac{\langle f, g_l \rangle}{||g_l||^2} = \frac{\int_{-\infty}^{\infty} f(\tau) g_l(\tau) d\tau}{\int_{-\infty}^{\infty} |g_l(\tau)|^2 d\tau} \]

So for an orthogonal signal-set, the LLSE coefficients are the same as those of LLSE estimators given just the individual signals.

Note: If the given signal-set \( \{g_1, g_2, \ldots, g_k\} \) is not orthogonal, the Gram-Schmidt procedure can be used to find an orthogonal signal-set that spans the linear vector space of the given one, where
\[ \text{span}\{g_1, g_2, \ldots, g_k\} = \{\alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_k g_k | \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\} \]

Actually, Gram-Schmidt will work on a given signal-set that is not even linearly independent to produce an orthonormal basis of its span.
• Consider the exponential signal \( f = 5e^{2t} \) for \( t \in \mathbb{I} = [-2, 2] \).

• To find the LLSE estimator of \( f \) given the signals \( g_1(t) = 1 \) and \( g_2(t) = t, t \in \mathbb{I} \),

\[
\hat{f} = \alpha_1 g_1 + \alpha_2 g_2,
\]

first note that

\[
\langle g_1, g_2 \rangle = \int_{-2}^{2} g_1(t)g_2(t)dt = \int_{-2}^{2} 1 \cdot t dt = \left. \frac{1}{2} t^2 \right|_{-2}^{2} = 0.
\]

• Thus, \( g_1 \) and \( g_2 \) are orthogonal.

• Recall that the product of an even and odd signal is odd, and that zero is the integral (or average value) of an odd signal over an interval of time \( \mathbb{I} \) that equally straddles the origin.

\[
\alpha_1 = \frac{\langle f, g_1 \rangle}{||g_1||^2} = \frac{\int_{-2}^{2} f(t)g_1(t)dt}{\int_{-2}^{2} |g_1(t)|^2 dt} = \frac{\int_{-2}^{2} 5e^{2t} \cdot 1 dt}{\int_{-2}^{2} |g_1(t)|^2 dt} = \frac{(5/2)e^{4t}}{4}
\]

\[
\alpha_2 = \frac{\langle f, g_2 \rangle}{||g_2||^2} = \frac{\int_{-2}^{2} f(t)g_2(t)dt}{\int_{-2}^{2} |g_2(t)|^2 dt} = \frac{\int_{-2}^{2} 5e^{2t} \cdot t dt}{\int_{-2}^{2} |g_2(t)|^2 dt} = \frac{(5/2)te^{2t} - (5/4)e^{4t}}{(1/3)t^3}
\]

• So, the LLSE of \( f \) given \( \{g_1, g_2\} \) is

\[
\hat{f} = \frac{(5/4) \cosh(4)}{g_1} + \left((45/64)e^4 + (75/64)e^{-4}\right)g_2.
\]
• Now consider periodic signals on $\mathbb{R}$.

• Recall that if the ratio of the periods of $g_1$ and $g_2$ is rational, or if one is periodic and the other is constant, then all linear combinations $\alpha_1 g_1 + \alpha_2 g_2$ are periodic, i.e., $\forall$ scalars $\alpha_1, \alpha_2 \in \mathbb{R}$.

• The following theorem is the basic to representation periodic signals by Fourier series.

• Theorem: The span of the set of signals
  \[ \{ \cos(kw_0 t), \ t \in \mathbb{R} \}, \ k \in \{ 0, 1, 2, 3, \ldots \} \]
  and
  \[ \{ \sin(kw_0 t), \ t \in \mathbb{R} \}, \ k \in \{ 1, 2, 3, \ldots \} \]
  includes all piecewise-continuous, periodic Dirichlet signals with period $T := 2\pi/w_0 > 0$ that are valued at the midpoint of their jump discontinuities.

• Note that the constant signal is included in this signal set: if $k = 0$ then $\forall t \in \mathbb{R}, \ 1 = \cos(kw_0 t)$.

• Also for $k > 0$, the period of $\cos(kw_0 t)$ and $\sin(kw_0 t)$ is $2\pi/(kw_0) = T/k$.

• So, linear combinations of these signals will be periodic with period $T/k$ for some $k \in \{ 1, 2, \ldots \}$ or constant (corresponding to $k = 0$).

### Fourier series coefficients (coordinates w.r.t. trigonometric basis)

• Suppose $f : \mathbb{R} \to \mathbb{R}$ is Dirichlet and periodic with period
  \[ T := \frac{2\pi}{w_0} > 0, \]
  where $w_0$ is the fundamental frequency of $f$.

• Define the infinitely large, trigonometric signal-set:
  \[ g_{c,k} := \{ \cos(kw_0 t), \ t \in \mathbb{R} \}, \ k \in \mathbb{Z}^{\geq 0} \]
  and
  \[ g_{s,k} := \{ \sin(kw_0 t), \ t \in \mathbb{R} \}, \ k \in \mathbb{Z}^{\geq 1} \]

• Note that all of these $g$ have the property that $\forall t \in \mathbb{R}, \ g(t) = g(t + T)$, consistent with $g$ either being constant ($k = 0$), or having period $T/k$ for some integer $k > 0$ so that
  \[ g(t) = g(t + T/k) \]
  \[ = g(t + T/k + T/k) = g(t + 2T/k) \ldots \]
  \[ = g(t + (k - 1)T/k + T/k) \]
  \[ = g(t + T). \]

• The LLSE of $f$ given this signal set over an arbitrary interval $\mathcal{I}$ of length $T$ is
  \[ \hat{f} = \sum_{k=0}^{\infty} \alpha_k g_{c,k} + \sum_{k=1}^{\infty} \beta_k g_{s,k} = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k g_{c,k} + \beta_k g_{s,k}), \]
  where by the previous theorem, the $\{\alpha_k, \beta_k\}$ are the coordinates of $f$ w.r.t. the trig basis, i.e., the LLSE has zero square error, $||\varepsilon||^2 = ||f - \hat{f}||^2 = 0$. 

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Orthogonality of the given trigonometric signals - different cosines

- Again, \( w_o := 2\pi/T \) is the fundamental frequency of \( T \)-periodic \( f \).
- Let \( \int_T \ldots dt \) represents integration over any interval of length \( T \) (i.e., any period of \( f \)).
- \( \forall \) integers \( k \neq l \geq 0 \),
  \[
  \langle g_{c,k}, g_{c,l} \rangle = \int_{T} \cos(kw_o t) \cos(lw_o t) dt \\
  = \int_0^T 0.5 \cos((k + l)w_o t) + 0.5 \cos((k - l)w_o t) \ dt \\
  = \left. \frac{0.5}{(k + l)w_o} \sin((k + l)w_o t) + \frac{0.5}{(k - l)w_o} \sin((k - l)w_o t) \right|_{0}^{T} \\
  = \frac{0.5}{(k + l)w_o} \sin((k + l)2\pi) + \frac{0.5}{(k - l)w_o} \sin((k - l)2\pi) \\
  = 0.
  \]

Orthogonality of the given trigonometric signals - different sines

- \( \forall \) integers \( k \neq l \geq 1 \),
  \[
  \langle g_{s,k}, g_{s,l} \rangle = \int_{T} \sin(kw_o t) \sin(lw_o t) dt \\
  = \int_0^T 0.5 \cos((k - l)w_o t) - 0.5 \cos((k + l)w_o t) \ dt \\
  = \left. \frac{0.5}{(k - l)w_o} \sin((k - l)w_o t) - \frac{0.5}{(k + l)w_o} \sin((k + l)w_o t) \right|_{0}^{T} \\
  = 0.
  \]
Orthogonality of the given trigonometric signals - a sine and cosine

- \( \forall \) integers \( k \geq 0, l \geq 1, \)
  \[
  \langle g_{c,k}, g_{s,l} \rangle = \int_{T}^{T} \cos(kw_o t) \sin(lw_o t) \, dt \\
  = \int_{0}^{T} 0.5 \sin((l + k)w_o t) + 0.5 \sin((l - k)w_o t) \, dt \\
  = -\frac{0.5}{(l + k)w_o} \cos((l + k)w_o t) - \frac{0.5}{(l - k)w_o} \cos((l - k)w_o t) \bigg|_{0}^{T} \\
  = -\frac{0.5}{(l + k)w_o} (\cos((l + k)2\pi) - 1) - \frac{0.5}{(l - k)w_o} (\cos((l - k)2\pi) - 1) \\
  = 0
  \]

Fourier coefficients/coordinates

- By the orthogonality of the given trigonometric signal-set, the LLSE of \( f \) over a period \( T = 2\pi/w_o \) has coefficients
  \[
  \forall k \geq 0, \alpha_k = \frac{\langle f, g_{c,k} \rangle}{||g_{c,k}||^2} = \frac{\int_{T} f(t) \cos(kw_o t) \, dt}{\int_{T} |\cos(kw_o t)|^2 \, dt} \\
  = \begin{cases} 
  \frac{2}{T} \int_{T} f(t) \cos(kw_o t) \, dt & \text{if } k > 0 \\
  \frac{1}{T} \int_{T} f(t) \, dt & \text{if } k = 0
  \end{cases}
  \]
  \[
  \forall k \geq 1, \beta_k = \frac{\langle f, g_{s,k} \rangle}{||g_{s,k}||^2} = \frac{\int_{T} f(t) \sin(kw_o t) \, dt}{\int_{T} |\sin(kw_o t)|^2 \, dt} \\
  = \frac{2}{T} \int_{T} f(t) \sin(kw_o t) \, dt.
  \]
- Note that \( \forall k > 0, T^{-1}||g_{c,k}||^2 = T^{-1}||g_{s,k}||^2 = 1/2 \), consistent with our previous calculation of the power of a sinusoid with unit amplitude.
Fourier coefficients/coordinates (cont)

• So, at all points \( t \) of continuity of \( f \) (when \( f = \hat{f} \)),

\[
f(t) = \alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos(kw_0 t) + \beta_k \sin(kw_0 t)) =: \hat{f}(t).
\]

• Note that \( \alpha_0 = T^{-1} \int_T f(t) \, dt \) is just the mean/average value (DC component) of \( f \).

• The \( T \)-periodic \( (k = 1) \) component is the fundamental of \( T \)-periodic \( f \).

• For \( k \geq 2 \), the \( T/k \)-periodic component is the \( (k - 1)^{\text{th}} \) harmonic of \( f \).

Trigonometric Fourier series - square-wave example

• The Fourier coefficients of this square wave \( f \) of period 7 and duty cycle 4 are

\[
\begin{align*}
\alpha_0 &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, dt = \frac{1}{7} \int_{-2}^{2} 5 \, dt = \frac{5}{7} \bigg|_{-2}^{2} = \frac{20}{7}. \\
\forall k \geq 1, \quad \alpha_k &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(kw_0 t) \, dt = \frac{2}{7} \int_{-2}^{2} 5 \cos(2\pi/7) t \, dt \\
&= \frac{5}{k\pi} \sin(k(2\pi/7) \bigg|_{-2}^{2} = \frac{10}{k\pi} \sin(4\pi/7). \\
\forall k \geq 1, \quad \beta_k &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(kw_0 t) \, dt = 0.
\end{align*}
\]
Trigonometric Fourier series - square-wave example (cont)

• So,
\[
f(t) = \frac{20}{7} + \sum_{k=1}^{\infty} \frac{10}{k \pi} \sin\left(\frac{4k\pi}{7}\right) \cos\left(\frac{2k\pi t}{7}\right) =: \hat{f}(t), \quad \forall t \in \mathbb{R},
\]
except possibly at points \(t\) of jump discontinuity of \(f\) where \(\hat{f}(t) = 5/2\), e.g., at \(t = \pm 2\).

• Note \(\beta_k = 0\ \forall k \geq 1\) simply because the integrand is odd on the interval of integration \([-3.5, 3.5]\) - recall that the product of an even \((f)\) and odd \((\text{sine})\) function is odd.

• Also note the interesting identity that ensues at time \(t = 0\) (a point of continuity of \(f\)):
\[
-\frac{2\pi}{7} = \sum_{k=1}^{\infty} \frac{\sin\left(\frac{4k\pi}{7}\right)}{k}.
\]

Trigonometric Fourier series - triangle-wave example

\[
\begin{align*}
\alpha_0 &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{1}{10} \int_{-5}^{5} 3t dt = \frac{3}{20} t^2 \bigg|_{-5}^{5} = 0. \\
\forall k \geq 1, \quad \alpha_k &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(k\omega_0 t) dt = 0. \\
\forall k \geq 1, \quad \beta_k &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(k\omega_0 t) dt = \frac{1}{5} \int_{-5}^{5} 3t \sin\left(k\left(\frac{2\pi}{10}\right)t\right) dt \\
&= \frac{3}{5} \left(-\frac{5}{k\pi} t \cos\left(k\pi t/5\right) + \frac{5^2}{(k\pi)^2} \sin\left(k\pi t/5\right)\right) \bigg|_{-5}^{5} \text{ after integration by parts} \\
&= -\frac{30}{k\pi} \cos(k\pi) = \frac{30}{k\pi} (-1)^{k+1}
\end{align*}
\]
Trigonometric Fourier series - even and odd signals

- Thus, for this triangle wave \( f \),
  \[
  f(t) = \sum_{k=1}^{\infty} \frac{30}{k\pi} (-1)^{k+1} \sin \left( \frac{k\pi t}{5} \right)
  \]
  \( \forall t \in \mathbb{R} \) except possibly at \( t = 5k \) for odd \( k \in \mathbb{Z} \) (i.e., points of jump discontinuity in \( f \) where the RHS \( \hat{f}(5k) = 0 \)).

- Note \( \alpha_k = 0 \ \forall k \geq 0 \) in the previous triangle-wave example simply because the integrand is odd on the interval of integration \([-5, 5]\) - recall that the product of an odd \((f)\) and even \((\cos)\) function is odd.

- In general, if \( T \)-periodic \( f(t) \) is even on \([-T/2, T/2]\), then \( \forall k \geq 1, \beta_k = 0 \), i.e., \( f \) has a cosine (even) trigonometric Fourier series.

- Alternatively, if \( f(t) \) is odd on \([-T/2, T/2]\) then \( \forall k \geq 0, \alpha_k = 0 \), i.e., \( f \) has a sine (odd) trigonometric Fourier series.

Partial Fourier series approximation

- Consider the partial Fourier series \( \hat{f}_K \) of \( T \)-periodic, Dirichlet \( f \) up to the \((K - 1)\)th harmonic, i.e., \( \forall K \geq 0, t \in \mathbb{R}, \)
  \[
  \hat{f}_K(t) := \alpha_0 + \sum_{k=1}^{K} \alpha_k \cos(k\omega_0 t) + \beta_k \sin(k\omega_0 t),
  \]

- For Dirichlet \( f \) (in \( L^2 \)),
  \[
  \lim_{K \to \infty} \int_T |\hat{f}_K(t) - f(t)|^2 dt = \lim_{K \to \infty} ||\hat{f}_K(t) - f(t)||^2 = 0.
  \]

- So, at points \( t \) of continuity of \( f \),
  \[
  \lim_{K \to \infty} \hat{f}_K(t) := \hat{f}(t) = f(t).
  \]

- In the following we will no longer point out possible differences between \( f \) and its (complete) Fourier series \( \hat{f} \) at points of jump discontinuity of \( f \).
Partial Fourier series approximation - square-wave example

• In the following, we plot $\hat{f}_0 = 20/7$, $\hat{f}_1$, $\hat{f}_{10}$, and $\hat{f}_{40}$ for the previous square-wave example.

![Graph showing partial Fourier series approximations]

• The non-uniform convergence near the jump discontinuity is called the Gibbs phenomenon.

Parseval’s theorem for signal power

• Recall that the average power of a sinusoid $A \cos(wt + \phi)$, $t \in \mathbb{R}$, is $A^2/2$.

• Discrepancies only at points of jump discontinuity will not affect mean power calculations.

• So, the Fourier series representation of a $T$-periodic, Dirichlet signal $f$ can be used to compute its mean power, i.e.,

$$P_f = T^{-1} \int_T |f(t)|^2 dt$$

$$= T^{-1} \int_T \left(\alpha_0 + \sum_{k=1}^{\infty} (\alpha_k \cos(kw_0t) + \beta_k \sin(kw_0t))\right)^2 dt$$

$$= T^{-1} \int_T \alpha_0^2 + \sum_{k=1}^{\infty} \alpha_k^2 \cos^2(kw_0t) + \beta_k^2 \sin^2(kw_0t) dt \text{ by orthogonality}$$

$$= \alpha_0^2 + \sum_{k=1}^{\infty} \left(\frac{\alpha_k^2}{2} + \frac{\beta_k^2}{2}\right)$$

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Parseval's theorem for signal power (cont)

- Suppose we want to compute the fraction $\varphi$ of $T$-periodic $f$'s power in the frequency band $[w_1, w_2]$ where $w_2 > w_1 \geq 0$ are frequencies in radians/s and $w_o = 2\pi/T$ is the fundamental of $f$.

- First identify which harmonics of $f$ lie in the band, i.e., for what integers $k \geq 0$ is $kw_o \in [w_1, w_2]$. equivalently
  
  $k_1 := \left\lceil \frac{w_1}{w_o} \right\rceil = \frac{w_1 T}{(2\pi)} \leq k \leq k_2 := \left\lfloor \frac{w_2}{w_o} \right\rfloor = \frac{w_2 T}{(2\pi)}$, 

  where for $z \in \mathbb{R}$, $\lceil z \rceil$ is the smallest integer $\geq z$ and $\lfloor z \rfloor$ is the greatest integer $\leq z$.

- So,
  
  $\varphi = \frac{1}{P_f} \sum_{k=k_1}^{k_2} \frac{\alpha_k^2 + \beta_k^2}{2}$,

  where we’ve defined $\beta_0 = \alpha_0$ to simplify this expression when $k_1 = 0$, and $\sum_{k=k_1}^{k_2} \ldots := 0$ when $k_1 > k_2$.

- A complementary problem is to find the smallest frequency interval $[w_1, w_2]$ for which $\varphi \geq 0.95$.

Parseval’s theorem for signal power - example

- To find the fraction of the power of the previous square wave (with period $T = 7$) in the frequency band $[20.5\text{Hz}, 100.5\text{Hz}]$, recall its Fourier series:

  $\frac{20}{7} + \sum_{k=1}^{\infty} \frac{10}{k\pi} \sin(4k\pi/7) \cos(2k\pi t/7)$.

- Thus, its total mean power is

  $P_f := \left( \frac{20}{7} \right)^2 + \sum_{k=1}^{\infty} \frac{10}{k\pi} \sin(4k\pi/7)^2$.

- Since the frequency band in radians/s is $[41\pi, 201\pi]$ and the signal’s period is 7,
  
  - the lower harmonic index in the band is $k_1 = \left\lfloor 41\pi/(2\pi/7) \right\rfloor = 143.5 = 144$,
  
  - the upper one is $k_2 = \left\lceil 201\pi/(2\pi/7) \right\rceil = 703.5 = 703$.

- So, the fraction of $f$’s mean power in $[20.5\text{Hz}, 100.5\text{Hz}]$ is

  $P_f^{-1} \sum_{k=144}^{703} \frac{1}{2} \left( \frac{10}{k\pi} \sin(4k\pi/7) \right)^2$.
Parseval’s theorem for signal power - example (cont)

- Note that $P_f$ is easily computed directly as
  $$P_f = \frac{1}{T} \int_T |f(t)|^2 dt = \frac{1}{7} 5^2 \cdot 4 = \frac{100}{7} \approx 14.$$

- Also note that the $f$’s DC power is
  $$\left( \frac{1}{T} \int_T f(t) dt \right)^2 = \alpha_0^2 = \left( \frac{20}{7} \right)^2 \approx 8.$$

The compact trigonometric Fourier series

- Using the trigonometric identity
  $$\forall \alpha, \beta, w \in \mathbb{R}, \quad \alpha \cos(wt) + \beta \sin(wt) = \sqrt{\alpha^2 + \beta^2} \cos(wt - \tan^{-1}(\beta/\alpha)).$$
  we obtain the compact trigonometric Fourier series (referenced to cosine) of any $T$-periodic, Dirichlet $f$ from its trigonometric one:
  $$\hat{f}(t) = A_0 \cos(\phi_0) + \sum_{k=1}^{\infty} A_k \cos(k \omega_0 t + \phi_k), \quad t \in \mathbb{R}, \text{ where}$$
  $$A_k = \sqrt{\alpha_k^2 + \beta_k^2}$$
  $$\phi_k = -\tan^{-1}(\beta_k/\alpha_k)$$
  with $\beta_0 := 0$.

- Note that the amplitudes $A_k \geq 0$, $\forall k \geq 0$.

- Also, if $\alpha_0 < 0$ then $A_0 = |\alpha_0|$ and $\phi_0 = \pi$. 
• Note that \( f(t) = 5 \cos(6t + 3) + 7 \sin(13t + 5) \) is periodic since the ratio, \( 13/6 \) or \( 6/13 \), of the periods, \( 2\pi/6 \) and \( 2\pi/13 \), of its components is rational.

• The period of \( f \) is the least common integer multiple of its components’ periods, \( i.e. \),
  \[ T = 2\pi = 6(2\pi/6) = 13(2\pi/13) \text{ seconds} \]

• So \( f \)'s fundamental is \( w_0 = 2\pi/T = 1 \text{ radian/s} = \text{the greatest common divisor of } 6 \text{ and } 13. \)

• The above \( f \) is given in compact trigonometric form and has only \( 5^{th} \) and \( 12^{th} \) harmonics (no DC or fundamental components in particular):
  \[ f(t) = 5 \cos(6t + 3) + 7 \cos(13t + 5 - \pi/2) \]

• We can write the above \( f \) as a trigonometric Fourier series,
  \[ f(t) = [5 \cos(3) \cos(6t) - 5 \sin(3) \sin(6t)] + [7 \sin(5) \cos(13t) + 7 \cos(5) \sin(13t)] \]

• The derived trigonometric Fourier series of the previous: (a) even square-wave happened to be in compact form, and (b) odd triangle-wave can be transformed to compact form again by using the identity \( \forall v \in \mathbb{R} \), \( \sin(v) = \cos(v - \pi/2) \).

---

Compact trigonometric Fourier series - square-wave example

• For this (not even) square-wave example with period \( T = 7 \), the DC trigonometric Fourier coefficient is
  \[ a_0 = \frac{1}{T} \int_{-2.5}^{-2.5+T} f(t)dt = \frac{1}{7} \int_{-1}^{3} 5dt = \frac{5}{7} t \bigg|_{-1}^{3} = \frac{20}{7}. \]
The other trigonometric Fourier coefficients are, ∀k ≥ 1:

\[ \alpha_k = \frac{2}{T} \int_{-2.5}^{-T} f(t) \cos(kt) \, dt = \frac{2}{7} \int_{-1}^{3} 5 \cos(2\pi t/7) \, dt \]
\[ = \frac{5}{k\pi} \sin(2\pi t/7) \bigg|_{-1}^{3} = \frac{5}{k\pi} (\sin(6\pi/7) - \sin(-2\pi/7)) \]
\[ = \frac{10}{k\pi} \cos(2\pi/7) \sin(4\pi/7). \]

\[ \beta_k = \frac{2}{T} \int_{-2.5}^{-T} f(t) \sin(kt) \, dt = \frac{2}{7} \int_{-1}^{3} 5 \sin(2\pi t/7) \, dt \]
\[ = -\frac{5}{k\pi} \cos(2\pi t/7) \bigg|_{-1}^{3} = -\frac{5}{k\pi} (\cos(6\pi/7) - \cos(-2\pi/7)) \]
\[ = \frac{10}{k\pi} \sin(2\pi/7) \sin(4\pi/7). \]

So, the compact trigonometric Fourier coefficients are \( A_0 = 20/7, \phi_0 = 0, \) and ∀k ≥ 1,

\[ A_k = \sqrt{\alpha_k^2 + \beta_k^2} = \frac{10}{k\pi} \left| \sin\left(\frac{4k\pi}{7}\right)\right| \sqrt{\cos^2\left(\frac{2k\pi}{7}\right) + \sin^2\left(\frac{2k\pi}{7}\right)} = \frac{10}{k\pi} \left| \sin\left(\frac{4k\pi}{7}\right)\right| \]

\[ \phi_k = -\tan^{-1}\left(\beta_k/\alpha_k\right) = -\tan^{-1}\left(\frac{\sin(4k\pi/7) \sin(2k\pi/7)}{\sin(4k\pi/7) \cos(2k\pi/7)}\right) \]
\[ = \left\{ \begin{array}{ll}
-\tan^{-1}(\tan(2k\pi/7)) & = -2k\pi/7 \quad \text{if } \sin(4k\pi/7) > 0 \\
-\tan^{-1}(\tan(2k\pi/7)) + \pi & = -2k\pi/7 + \pi \quad \text{if } \sin(4k\pi/7) < 0
\end{array} \right. \]
\[ \Rightarrow \hat{f}(t) = \frac{20}{7} + \sum_{k=1}^{\infty} \frac{10}{k\pi} \left| \sin(4k\pi/7)\right| \cos(2k\pi t/7 - 2k\pi/7 + \pi \left\{ \begin{array}{ll}
1 & \text{if } \sin(4k\pi/7) < 0 \\
0 & \text{else}
\end{array} \right. \]
\[ = \frac{20}{7} + \sum_{k=1}^{\infty} \frac{10}{k\pi} \sin(4k\pi/7) \cos(2k\pi t/7 - 2k\pi/7 + \pi)\{\sin(4k\pi/7) < 0\} \]
consistent with this square wave being the previous example even square-wave delayed by one second, where

- the indicator function \( 1_B = 1 \) if \( B \) is true, else \( 1_B = 0. \)
• Recall the Euler-De Moivre identities:

\[
\cos(wt) = \frac{e^{jwt} + e^{-jwt}}{2}, \quad \sin(wt) = \frac{e^{jwt} - e^{-jwt}}{2j}
\]

\[\Rightarrow \alpha \cos(wt) + \beta \sin(wt) = \frac{\alpha_k - j\beta_k}{2} e^{jwt} + \frac{\alpha_k + j\beta_k}{2} e^{-jwt}\]

• We obtain the complex-exponential Fourier series of any \(T\)-periodic, Dirichlet signal \(f\) by substituting this identity into its trigonometric one:

\[
\hat{f}(t) = \sum_{k=-\infty}^{\infty} D_k e^{jkw_o t}
\]

where, again, \(w_o = 2\pi/T\),

\[\alpha_0 = D_0\]

and \(\forall k \geq 1,\)

\[D_k = \frac{\alpha_k - j\beta_k}{2} \quad \text{and} \quad D_{-k} = \frac{\alpha_k + j\beta_k}{2}.
\]

The complex-exponential Fourier series (cont)

• By substituting the expressions for the trigonometric Fourier coefficients, we get that

\[\forall k \in \mathbb{Z}, \quad D_k = \frac{1}{T} \int_T f(t) e^{-jkw_o t} dt.
\]

• Note that if \(f\) is real valued, then \(\forall k \geq 0, \alpha_k, \beta_k \in \mathbb{R}\)

\[\Rightarrow \forall k \in \mathbb{Z}, \quad D_{-k} = \overline{D_k}, \quad \text{the complex conjugate of} \ D_k \in \mathbb{C}.
\]

• In turn, \(\forall k \geq 1, \overline{D_k} e^{jkw_o t} = D_{-k} e^{-jkw_o t}, \) consistent with \(f\) being real valued.

• Similarly, using Euler’s identity \(e^{jwt} = \cos(wt) + j\sin(wt), \) we can transform the complex-exponential to the trigonometric Fourier coefficients: \(\forall k \geq 1,\)

\[\alpha_k = D_k + D_{-k} \quad \text{and} \quad \beta_k = j(D_k - D_{-k}).
\]
Complex-exponential Fourier series - example

- Define \( \forall t \in \mathbb{R} \),
  \[
  f(t) = \sum_{k=-\infty}^{\infty} 7e^{-2(t-5k)}(u(t - 5k) - u(t - 3 - 5k)),
  \]
  i.e., a Dirichlet signal with period \( T = 5 \) and duty cycle 3 (within each period).

- The complex-exponential Fourier coefficients are, \( \forall k \in \mathbb{Z} \),
  \[
  D_k = \frac{1}{T} \int_T f(t)e^{-jk\omega_0 t}dt = \frac{1}{T} \int_0^T f(t)e^{-jk\omega_0 t}dt
  = \frac{1}{5} \int_0^3 7e^{-2t}e^{-jk2\pi t/5}dt = \frac{1}{5} \cdot \frac{7}{(2+jk2\pi/5)} e^{-(2+jk2\pi/5)\frac{3}{5}} \\
  = \frac{7}{10+jk2\pi} \left( 1 - e^{-3(2+jk2\pi/5)} \right). 
  \]

Complex-exponential Fourier series - orthogonality

- For any period \( T > 0 \), the \( \mathbb{C} \)-valued signal-set
  \[
  \left\{ g_{e,k}(t) := e^{jk2\pi t/T} = e^{jk\omega_0 t}, \quad t \in \mathbb{R} \right\}_{k \in \mathbb{Z}} 
  \]
  is orthogonal (and a basis for the set of all periodic Dirichlet functions with fundamental frequency \( \omega_0 \) or some whole multiple, \( kw_0 \)).

- Orthogonality is easily verified: \( \forall k \neq l \in \mathbb{Z} \),
  \[
  < g_{e,k}, g_{e,l} > := \int_T g_{e,k}(t)\overline{g_{e,l}(t)}dt = \int_T e^{jk\omega_0 t}e^{-jl\omega_0 t}dt
  = \frac{1}{j(k-l)\omega_0} \left| e^{j(k-l)\omega_0 T} \right|_0^T = 0
  \]
  \[
  \| g_{e,k} \|^2 = < g_{e,k}, g_{e,k} > = \int_T g_{e,k}(t)\overline{g_{e,k}(t)}dt = \int_T |g_{e,k}(t)|^2 dt = \int_T 1 dt = T.
  \]

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Complex-exponential Fourier series - orthogonality (cont)

• Note that inner product for \(C\)-valued signals is linear in the first argument and conjugate linear in the second argument.

• So, for inner product thus defined, we have again that the complex-exponential Fourier coefficients for \(T\)-periodic, Dirichlet \(f\) are, \(\forall k \in \mathbb{Z}\),

\[
D_k = \frac{\langle f, g_{e,k} \rangle}{||g_{e,k}||^2} = \frac{\int_T f(t)g_{e,k}(t)dt}{T} = \frac{1}{T} \int_T f(t)e^{-jk\omega_0 t}dt.
\]

Spectrum (frequency-domain representation) of a signal

• Recall the previous 10-periodic, odd, triangle-wave example signal, \(f\).

• The trigonometric Fourier coefficients of \(f\) are \(\alpha_k = 0\) for \(k \geq 0\) and \(\beta_k = (-1)^{k+1}30/(k\pi)\) for \(k > 1\).

• Thus, the exponential Fourier coefficients are

\[
D_k = \begin{cases} 
0 & \text{if } k = 0 \\
(-1)^{k+1}j15/(k\pi) & \text{if } k > 0 \\
(-1)^k j15/(k\pi) & \text{if } k < 0
\end{cases}
\]

• We can plot magnitude and phase spectrum of \(f\), respectively:

\[
|D_k| = \frac{15}{k\pi} = \frac{3}{k\omega_0} \quad \text{for } k \neq 0, \text{ and}
\]

\[
\angle D_k = \begin{cases} 
\text{don’t care} & \text{if } k = 0 \\
\pi/2 & \text{if } k > 0 \text{ and odd or } k < 0 \text{ and even} \\
-\pi/2 & \text{if } k < 0 \text{ and odd or } k > 0 \text{ and even}
\end{cases}
\]

versus \(k\omega_0 = k2\pi/10 = k\pi/5\) for \(k \in \mathbb{Z}\).
Complex-exponential Fourier series - Parseval's theorem

- We can argue as before that the mean power of $T$-periodic, Dirichlet $f$ is

$$P_f = T^{-1} \int_T |f(t)|^2 dt = T^{-1} \int_T f(t)\overline{f(t)} dt$$

$$= T^{-1} \int_T \sum_{k=-\infty}^{\infty} D_k e^{jk\omega_0 t} \sum_{l=-\infty}^{\infty} D_l e^{jkw_0 t} dt$$

$$= T^{-1} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} D_k \overline{D_l} \int_T e^{jk\omega_0 t} e^{jkw_0 t} dt$$

$$= T^{-1} \sum_{k=-\infty}^{\infty} D_k \overline{D_k} \int_T e^{jk\omega_0 t} e^{jkw_0 t} dt \quad \text{by orthogonality}$$

$$= \sum_{k=-\infty}^{\infty} |D_k|^2 T^{-1} \int_T 1 dt$$

$$= \sum_{k=-\infty}^{\infty} |D_k|^2 = |D_0|^2 + 2 \sum_{k=1}^{\infty} |D_k|^2.$$  

where the last equality is because $f$ is $\mathbb{R}$-valued $\Rightarrow \forall k \in \mathbb{Z}, D_{-k} = \overline{D_k} \Rightarrow |D_k| = |D_{-k}|.$
Complex-exponential Fourier series - Parseval’s theorem - example

- Recall that 5-periodic \( f(t) = \sum_{k=-\infty}^{\infty} 7e^{-2(t-5k)}(u(t-5k) - u(t-3-5k)), t \in \mathbb{R}, \) had complex-exponential Fourier series coefficients: \( \forall k \in \mathbb{Z}, \)
  \[
  D_k = \frac{7}{10 + jk2\pi} \left( 1 - e^{-3(2+jk2\pi/5)} \right)
  = \frac{7}{10 + jk2\pi} \left( 1 - e^{-6\cos(6k\pi/5)} + je^{-6\sin(6k\pi/5)} \right).
  \]
- By Parseval’s theorem, the mean power of \( f \) is
  \[
  P_f = \sum_{k=-\infty}^{\infty} \left| \frac{7}{10 + jk2\pi} \left( 1 - e^{-3(2+jk2\pi/5)} \right) \right|^2
  = \sum_{k=-\infty}^{\infty} \frac{7}{\sqrt{100 + (k2\pi)^2}^2} \sqrt{(1 - e^{-6\cos(6k\pi/5)})^2 + (e^{-6\sin(6k\pi/5)})^2}
  = \frac{1}{T} \int_T |f(t)|^2 dt = \frac{1}{5} \int_0^3 49e^{-4t} dt = \frac{49}{20} (1 - e^{-12}).
  \]

Complex-exponential Fourier series - power in a frequency band

- Consider the frequency band \([w_1, w_2]\) with \( w_2 > w_1 \geq 0 \) radians/s.
- To compute the amount of \( T \)-periodic \( f \)'s mean power in this band, recall that we need to identify which of its harmonics \( k \) lie in it:
  \[ k_1 := \left\lfloor \frac{w_1}{w_0} \right\rfloor \leq k \leq \left\lfloor \frac{w_2}{w_0} \right\rfloor =: k_2, \]
  where again the fundamental frequency of \( f \) is \( w_o = 2\pi/T \).
- Using the complex-exponential Fourier series of \( f \), note that we need to account for both positive \((kw_o)\) and negative \((-kw_o)\) frequencies for this index set \( k \).
- That is, if \( \{D_k\}_{k \in \mathbb{Z}} \) are the complex-exponential Fourier coefficients of \( f \) and \( k_1 > 0 \), then the amount of \( f \)'s power in this band is
  \[
  \sum_{k=k_1}^{k_2} (|D_{-k}|^2 + |D_k|^2) = \sum_{k=-k_2}^{-k_1} |D_k|^2 + \sum_{k=k_1}^{k_2} |D_k|^2 = 2 \sum_{k=k_1}^{k_2} |D_k|^2.
  \]
- Note that if \( k_1 = 0 \) \( (i.e., w_1 = 0) \), then we should include the term \( |D_0|^2 \) (representing \( f \)'s DC power) only once in the expression above.
Eigenresponse

• Recall that for an asymptotically stable system with transfer function \( H \), the steady-state response to the (persistent) sinusoidal input \( f(t) = Ae^{j(wt+\phi)}, \ t \in \mathbb{R} \), with amplitude \( A > 0 \), frequency \( w \), and phase \( \phi \), is the eigenresponse,
  \[
y(t) = H(jw)Ae^{j(wt+\phi)} = |H(jw)|Ae^{j(wt+\phi+\angle H(jw))}, \ t \in \mathbb{R}.
  \]
  
• That is,
  – the magnitude response or system gain of the system is \( |H(jw)| \), or \( 20 \log_{10} |H(jw)| \) in decibels (dB):
    
    \[
    \text{input magn.} \quad 20 \log_{10} A \ \text{dB} \quad \rightarrow \quad \text{output magn.} \quad 20 \log_{10} A + 20 \log_{10} |H(jw)| \ \text{dB}
    \]
  – the phase response of the system is \( \angle H(jw) \):
    
    \[
    \text{input phase} \ \ \phi \ \text{radians} \quad \rightarrow \quad \text{output phase} \ \ \phi + \angle H(jw) \ \text{radians}
    \]

• So, the eigenresponse to real-valued \( f(t) = A \cos(wt+\phi), \ t \in \mathbb{R} \), is real-valued
  \[
y(t) = A|H(jw)| \cos(wt+\phi+\angle H(jw)), \ t \in \mathbb{R}.
  \]

Eigenresponse and Fourier series

• Given a complex-exponential Fourier series of an arbitrary (Dirichlet) periodic input,
  \[
f(t) = \sum_{k=-\infty}^{\infty} D_k e^{jkwt}
  \]
  the eigenresponse of an asymptotically stable, LTI system \( H \) is, by linearity,
  \[
y(t) = \sum_{k=-\infty}^{\infty} D_k H(jkw_o) e^{jkwt}, \ t \in \mathbb{R}.
  \]

• Equivalently, the eigenresponse to \( f \) with the compact Fourier-series representation
  \[
f(t) = \sum_{k=0}^{\infty} A_k \cos(kw_o t + \phi_k)
  \]
  is
  \[
y(t) = \sum_{k=0}^{\infty} |H(jkw_o)|A_k \cos(kw_o t + \phi_k + \angle H(jkw_o)), \ t \in \mathbb{R}.
  \]
For this example LTIC circuit, the transfer function is

\[ H(jw) = \frac{R}{R + (jwL||R)} = \frac{R}{R + ((jwL)^{-1} + R^{-1})^{-1}} \]

\[ = \frac{jw/2 + R/(2L)}{jw + R/(2L)} \]

\[ = \frac{P(jw)}{Q(jw)} \]

The transfer function can be verified by the system equation \( Q(D)y = P(D)f \) and \( H = P/Q \), where by KCL

\[ 0 = \frac{f - y}{R} + \frac{0 - y}{R} + i \text{ where } LD i = f - y \]

\[ \Rightarrow 0 = Df - Dy + 0 - Di = \frac{Df}{D} + \frac{0 - Dy}{D} + \frac{f - y}{L} \]

\[ \Rightarrow Q(D)y := Dy + \frac{R}{2L}y = \frac{1}{2}Df + \frac{R}{2L}f =: P(D)f. \]
• So, the magnitude response in dB at frequency \( w \) is
\[
20 \log_{10} |H(jw)| = 20 \log_{10} \frac{\sqrt{(w/2)^2 + (R/(2L))^2}}{w^2 + (R/(2L))^2} = 10 \log_{10} \frac{w^2 + (R/(2L))^2}{w^2 + (R/(2L))^2} \]
and the phase response is
\[
\angle H(jw) = \tan^{-1} \left( \frac{w/2}{R/(2L)} \right) - \tan^{-1} \left( \frac{w}{R/(2L)} \right).
\]

• Note that for very low frequencies \( \lim_{w \to 0} |H(jw)| = H(0) = 1 \) (= 0 dB), and for high frequencies \( \lim_{w \to \infty} |H(jw)| = 0.5 \) (\( \approx -6 \) dB).

• So, higher frequencies are attenuated by about 6 dB, while low frequencies are not significantly attenuated, i.e., they pass through the system (because the DC impedance of the inductor is zero).

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Eigenresponse and Fourier series - circuit example (cont)

• Again recall that \( f(t) = \sum_{k=-\infty}^{\infty} 7e^{-2(t-5k)}(u(t-5k) - u(t-3-5k)), t \in \mathbb{R}, \) had complex-exponential Fourier series coefficients,
\[
D_k = \frac{7}{10 + jk2\pi} \left( 1 - e^{-3(2+jk2\pi/5)} \right)
\]
\[
f(t) = \sum_{k=-\infty}^{\infty} D_k e^{jk2\pi t/5}
\]

• So, if \( R = 10k\Omega \) and \( L = 0.1H \) (i.e., \( R/L = 10^5 \)), then the eigenresponse of this circuit to this input is, \( \forall t \in \mathbb{R}, \)
\[
y(t) = \sum_{k=-\infty}^{\infty} H(jk2\pi/5)D_k e^{jk2\pi t/5}
= \sum_{k=-\infty}^{\infty} \frac{jk\pi/5 + 10^5/2}{jk2\pi/5 + 10^5/2} \frac{7}{10 + jk2\pi} \left( 1 - e^{-3(2+jk2\pi/5)} \right) e^{jk2\pi t/5}
\]

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Recall that a SISO LTIC system relating input signal $f$ to output $y$ by $P(D)f = Q(D)y$ has (zero state) transfer function $H(s) = P(s)/Q(s)$.

The roots of $Q$ are called system characteristic values or poles, while the roots of $P$ are system zeros.

Recall that for an asymptotically stable system with transfer function $H$, the steady-state (eigen-) response to real-valued, persistent, sinusoidal input

$$f(t) = A\cos(\omega t + \phi), \quad t \in \mathbb{R}$$

is real-valued sinusoidal output at the same frequency

$$y(t) = A|H(j\omega)|\cos(\omega t + \phi + \angle H(j\omega)), \quad t \in \mathbb{R},$$

with input parameters: amplitude $A > 0$, frequency $\omega \geq 0$, and phase $\phi$.

The magnitude response or system gain of the system is $|H(j\omega)|$, or $20\log_{10} |H(j\omega)|$ in decibels (dB):

input magnitude $20\log_{10} A$ dB $\rightarrow$ output magnitude $20\log_{10} A + 20\log_{10} |H(j\omega)|$ dB

the phase response of the system is $\angle H(j\omega)$:

input phase $\phi$ radians $\rightarrow$ output phase $\phi + \angle H(j\omega)$ radians

Bode plots of the system

The magnitude Bode plot of the system depicts the amplitude gain in decibels $20\log_{10} |H(j\omega)|$ as a function of the log-frequency $\log_{10} \omega$ for $\omega \geq 0$, i.e., a “log-log” plot.

The phase Bode plot of the system depicts the phase change $\angle H(j\omega)$ as a function of $\log_{10} \omega$ for $\omega \geq 0$, i.e., a semi-log plot.

In the following discussion of Bode plots, logarithm is assumed base 10.

Note that DC ($\omega = 0$) gain/phase-change is depicted at $\log \omega = -\infty$, i.e., the origin on a Bode plot is a low frequency $\log \omega = 0 \Rightarrow \omega = 1$ radians/s, but not DC.

A change in frequency by a factor of 10 is called a decade (by a factor of 8 is called an octave). 

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A change in frequency by a factor of 10 is called a decade (by a factor of 8 is called an octave).
Example of a second-order transfer function

- Recall that for this second-order, series RLC circuit, the transfer function is
  \[ H(s) = \frac{LC^{-1}}{s^2 + RL^{-1}s + (LC)^{-1}}. \]
- Consider the overdamped case where there are two real poles \( r_{\pm} < 0 \), and define the positive frequencies \( w_1 := -r_+ < w_2 := -r_- \).

Example of a second-order transfer function (cont)

- Thus, the transfer function can be written as
  \[ H(jw) = \frac{H(0)}{(1 + jw/w_1)(1 + jw/w_2)} = \frac{H(0)}{(1 - jw/r_+)(1 - jw/r_-)}, \]
  where here the DC gain is
  \[ H(0) = \frac{LC^{-1}}{w_1w_2} = \frac{LC^{-1}}{r_+r_-} = 1. \]
- For example, if \( R = 1010\Omega, L = 10\text{mH} \) and \( C = 1\mu\text{F} \), then \( w_1 = 10^3 \text{ radians/s} \) and \( w_2 = 10^5 \text{ radians/s} \).
"Standard form" transfer function to explain Bode plots

- Consider a transfer function that is a proper, rational polynomial
  \[ H(jw) = \frac{P(jw)}{Q(jw)}, \tag{1} \]
  for frequencies \( w \geq 0 \),
  where by “proper” we mean that the degree of \( P \leq \) that of \( Q \).

- In polar form (not indicating dependence on real frequency \( w \geq 0 \)),
  \[ |H| e^{j\angle H} = \frac{|P| e^{j\angle P}}{|Q| e^{j\angle Q}}. \]

- Thus,
  \[ |H| = \frac{|P|}{|Q|} \quad \text{and} \quad \angle H = \angle P - \angle Q. \]

---

So, for the above second-order RLC circuit, we can write

\[
\begin{align*}
20 \log |H(jw)| &= 20 \log |H(0)| - 20 \log |1 + jw/w_1| - 20 \log |1 + jw/w_2| \\
\angle H(jw) &= \angle H(0) - \angle(1 + jw/w_1) - \angle(1 + jw/w_2)
\end{align*}
\]

- More generally, assume that \( P \) and \( Q \) have only real roots and no common roots, \( i.e., \forall k, l, w_{z,k} \neq w_{p,l} \).

- Factoring \( P \) and \( Q \) we get for the case of no DC poles or zeroes \((i.e., \forall k, -w_{z,k}, -w_{p,k} \neq 0, \) so that DC gain in decibels is finite\):
  \[ H(jw) = H(0) \frac{\prod_{k=1}^{n} (1 + jw/w_{z,k})}{\prod_{k=1}^{m} (1 + jw/w_{p,k})} \tag{i} \]

- If \( H \) has \( r \leq m \) DC zeroes \((i.e., 0 \text{ DC gain or } -\infty \text{ in decibels}) \) or \( r \leq n \) DC poles \((i.e., \infty \text{ DC gain}) \), then for some scalar \( \eta \in \mathbb{R} \), respectively:
  \[
  \begin{align*}
  H(jw) &= \eta (jw)^r \frac{\prod_{k=1}^{n-r} (1 + jw/w_{z,k})}{\prod_{k=1}^{m-r} (1 + jw/w_{p,k})} \tag{ii} \\
  \text{or} \quad H(jw) &= \eta (jw)^r \frac{\prod_{k=1}^{n-r} (1 + jw/w_{z,k})}{(jw)^r \prod_{k=1}^{m-r} (1 + jw/w_{p,k})} \tag{iii}
  \end{align*}
  \]

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Since \( \forall w > 0, |jw| = w \) and \( \angle jw = \pi/2 \), the phase and magnitude for the above three cases (i)-(iii) are, for \( w \geq 0 \):

\[
\angle H(jw) = \angle H(0) + \sum_{k=1}^{m} \angle(1 + jw/w_{z,k}) - \sum_{k=1}^{n} \angle(1 + jw/w_{p,k}) \quad (i)
\]

\[
20 \log |H(jw)| = 20 \log |H(0)| + \sum_{k=1}^{m} 20 \log |1 + jw/w_{z,k}| - \sum_{k=1}^{n} 20 \log |1 + jw/w_{p,k}| \quad (i)
\]

\[
\angle H(jw) = \angle H(0) + \sum_{k=1}^{m} \angle(1 + jw/w_{z,k}) - \sum_{k=1}^{n} \angle(1 + jw/w_{p,k}) \quad (ii)
\]

\[
20 \log |H(jw)| = 20 \log |\eta| + \sum_{k=1}^{m-\rho} 20 \log |1 + jw/w_{z,k}| - \sum_{k=1}^{n} 20 \log |1 + jw/w_{p,k}| \quad (ii)
\]

\[
\angle H(jw) = \angle H(0) + \sum_{k=1}^{m-\rho} \angle(1 + jw/w_{z,k}) - \sum_{k=1}^{n} \angle(1 + jw/w_{p,k}) \quad (ii)
\]

\[
\angle H(jw) = \angle H(0) + \sum_{k=1}^{m-\rho} \angle(1 + jw/w_{z,k}) - \sum_{k=1}^{n} \angle(1 + jw/w_{p,k}) \quad (ii)
\]

\[
20 \log |H(jw)| = 20 \log |\eta| - \sum_{k=1}^{m} 20 \log |1 + jw/w_{z,k}| - \sum_{k=1}^{n} 20 \log |1 + jw/w_{p,k}| \quad (iii)
\]

\[
\angle H(jw) = \angle H(0) + \sum_{k=1}^{m-\rho} \angle(1 + jw/w_{z,k}) - \sum_{k=1}^{n} \angle(1 + jw/w_{p,k}) \quad (iii)
\]

Magnitude Bode plots with DC zeros or poles

- If there is a DC pole or zero, the magnitude Bode plot in decibels is obviously not finite at DC.
- Note the terms
  \[
  20 \log |\eta| \pm r 20 \log w
  \]
  for the above magnitude Bode plots with DC poles or zeros (of degree \( r \)).
- As a function of \( \log w \), these terms have
  - slope \( \pm 20r \) and
  - \( y \)-intercept \( 20 \log |\eta| \), where
  - the \( y \)-axis is \( \log w = 0 \), i.e., \( w = 1 \) radians/s (not DC where \( \log w = -\infty \) or \( w = 0 \)).
Magnitude Bode plot: \( \log |1 + jw/w_1| \) vs. \( \log w \) with \( w_1 > 0 \)

- So, to construct the magnitude Bode plot, we first need to construct that of its component terms of the form
  \[
  20 \log |1 + jw/w_1| = 10 \log(1 + (w/w_1)^2) \quad \text{versus} \quad \log w, \; w \geq 0,
  \]
  for an arbitrary frequency \( w_1 > 0 \).

- Case 1: If \( w < w_1/10 \) (i.e., \( \log w < -1 + \log w_1 \)), then \( 1.1 > |1 + jw/w_1| > 1 \) so that \( |1 + jw/w_1| \approx 1 \) and
  \[
  20 \log |1 + jw/w_1| \approx 0.
  \]

- Case 2: If \( w = w_1 \), then
  \[
  20 \log |1 + jw/w_1| = 20 \log |1 + j| = 20 \log \sqrt{2} \approx 3.
  \]

- Case 3: If \( w > 10w_1 \) (i.e., \( \log w > 1 + \log w_1 \)), then \( |1 + jw/w_1| \approx w/w_1 \) and
  \[
  20 \log |1 + jw/w_1| \approx 20 \log w - 20 \log w_1;
  \]
as a function of \( \log w \), this line is of slope 20dB/decade with \( x \)-intercept \( \log w_1 \).

These three cases are summarized in the following figure.

Note that if \( w_1 \) is real but negative, the magnitude Bode plot above does not change; the above analysis applies to \( -w_1 = |w_1| \).

If \( w_1 \) is complex-valued,
- the same three cases above apply using its modulus \( |w_1| \) (instead of \( w_1 \)) leading to the same asymptotic magnitude Bode plot,
- however at \( \log |w_1| \) in Case 2, the true magnitude Bode plot will not differ by 3dB from (the corner of) the asymptotic plot, cf. the resonant circuit example.
Magnitude Bode plots - rules of thumb

- So, to sketch the asymptotic magnitude Bode plot of a transfer function $H$:
  1. Mark off on the x-axis the modulus of the non-zero zeroes and poles of $H$.
  2. Each zero, respectively pole, contributes an additional 20 dB/decade slope only for frequencies larger than (i.e., to the right of) its modulus.
  3. If the DC (from $-\infty = \log 0$) gain is finite, then mark it off on the y-axis; else
  4. If there are $r$ DC zeroes (Case (ii)), then the initial (again from $-\infty = \log 0$) slope is $20r$ dB/decade.
  5. If there are $r$ DC poles (Case (iii)) then the initial slope is $-20r$ dB/decade.

- If the $k^{th}$ repeated zero (resp., pole) is real, then the true Bode plot will be $3k$ dB larger (resp., smaller) than the asymptotic Bode plot at the zero (resp., pole).
Suppose two poles at $w_1 = -10^3$ radians/s and $-w_2 = -10^5$ radians/s and DC gain $H(0) = 1$ (20 log 1 = 0 dB), consider again the example second-order transfer function

$$H(jw) = \frac{H(0)}{(1 + jw/w_1)(1 + jw/w_2)}$$

$$\Rightarrow \quad 20 \log |H(jw)| = 0 - 20 \log |1 + jw/10^3| - 20 \log |1 + jw/10^5|.$$

Phase Bode plot - $\angle(1 + jw/w_1)$ vs. log $w$ for $w_1 > 0$

- To plot $\angle(1 + jw/w_1) = \tan^{-1}(w/w_1)$, for $w_1 > 0$, we again take three cases.
- Case 1: If $w < w_1/10$ (i.e., log $w < -1 + \log w_1$), then
  $$0 \leq \tan^{-1}(w/w_1) < \tan^{-1}(1/10) \approx 0.$$
- Case 2: If $w = w_1$, then $\tan^{-1}(w/w_1) = \tan^{-1}(1) = \pi/4$ radians (=45 degrees).
- Case 3: If $w > 10w_1$ (i.e., log $w > 1 + \log w_1$), then
  $$\pi/2 \geq \tan^{-1}(w/w_1) > \tan^{-1}(10) \approx \pi/2 \text{ radians (=90 degrees).}$$
- These three cases are summarized in the following figure.
Phase Bode plot - $\angle(1 + jw/w_1)$ vs. $\log w$ for $w_1 > 0$

So, the change in phase due to the term

$$\angle(1 + jw/w_1) = \tan^{-1}(w/w_1)$$

is $\pi/2$ radians over a two decade interval $[w_1/10, 10w_1]$, i.e., a slope of 45 degrees/decade.

Phase Bode plot - discussion

- **Unlike** the magnitude response, for the phase response:
  - the pole or zero has an effect on the one decade before $w_1$,
  - the effect of the pole or zero “saturates” to a total change in phase of $\pm \pi/2$ one decade after $w_1$, and
  - we do not annotate deviations between the true and asymptotic plots (as 3 dB gaps).

- If $-\infty < H(0) < 0$, then the initial phase $\angle H(0) = \pm \pi$.

- Note that if $w_1$ is real but negative, the phase change is instead $-\pi/2$ over two decades at $\log(-w_1) = \log |w_1|$, i.e., -45 degrees/decade.

- Also, if $w_1$ is complex-valued, the phase change remains $\pi/2$ over two decades, but the phase Bode plot departs from the asymptotic and is steeper at $\log |w_1|$, cf. the resonant circuit example.
Phase Bode plot - second-order system example

- For two poles at $w_1 = 10^3$ rad/s and $w_2 = 10^5$ rad/s and DC gain $H(0) = 1$, consider again the example second-order transfer function

$$H(jw) = \frac{H(0)}{(1 + jw/w_1)(1 + jw/w_2)}$$

$$\angle H(jw) = 0 - \tan^{-1}(w/10^3) - \tan^{-1}(w/10^5)$$

Bode plots - second-order system example 2

$$H(s) = \frac{10s + 10^7}{s^2 + 10100s + 10^6} = \frac{10(s + 10^6)}{(s + 10^4)(s + 10^2)} = \frac{10(1 + s/10^6)}{(1 + s/10^4)(1 + s/10^2)}$$
Bode plots - example with DC zero

\[ H(s) = \frac{10s(s + 100)}{s + 10^4} \Rightarrow |H(j1)| \approx \frac{10 \cdot 100}{10^4} = 10^{-1} = -20\text{dB}. \]

Op-amp circuit example - noninverting configuration

- KCL at the negative input terminal (with node voltage \( f \)) gives

\[ 0 = \frac{0 - f}{R_1} + \frac{y - f}{R_F} + CD(y - f) \]
• Thus, the system equation and transfer function are:

\[
Q(D)y = D \cdot y + \frac{1}{CR_F} y = D \cdot f + \frac{1}{C} \left( \frac{1}{R_F} + \frac{1}{R_1} \right) f = P(D) f \\
\Rightarrow H(s) = \frac{P(s)}{Q(s)} = \frac{s + C^{-1}(R_1^{-1} + R_F^{-1})}{s + (CR_F)^{-1}}
\]

\[
= \frac{s + 1010}{s + 10},
\]

the last equality when \( C = 0.01 \mu F, R_1 = 100k\Omega, R_F = 10M\Omega. \)

• So, this system has
  - a pole at \(-10\),
  - a zero at \(-1010 \approx -10^3\), and
  - DC gain \( H(0) \approx 10^2 = 40\) dB.

• The zero being larger than the pole will restore the slope to 0 dB/decade in the magnitude plot, and the phase to 0 degrees in the phase plot.

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Op-amp circuit example - asymptotic Bode plots

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Op-amp circuit example 2 - inverting configuration

Here, \( y = v_c \) so KCL at the negative input terminal (with node voltage 0) gives

\[
0 = \frac{f - 0}{R_1} + \frac{y - 0}{R_F} + C \frac{d}{dt}(y - 0) \Rightarrow H(s) = \frac{-(CR_1)^{-1}}{s + (CR_F)^{-1}}
\]

Op-amp circuit example 2 - inverting configuration (cont)

So when \( C = 0.01 \mu F, R_1 = 100k\Omega, \) and \( R_F = 10M\Omega, \)

\[
H(s) = \frac{-10^3}{s + 10}
\]

The DC gain is again 20dB, but the DC phase is \( \pm \pi \), i.e., \( H(0) = -100 < 0 \).

At high-frequencies, \( |H(jw)| \sim 10^3/w \rightarrow 0(\sim -\infty \text{ dB}) \) and \( \angle H(jw) \sim \angle(-1/j) = \pi/2 \).

Note that the previous noninverting op-amp at high frequencies, \( |H(jw)| \sim 1(\equiv 0 \text{ dB}) \) and \( \angle H(jw) \sim \angle 1 \equiv 0 \).
Recall the series RLC circuit transfer function with output signal the capacitor voltage,

\[ H(s) = \frac{(LC)^{-1}}{s^2 + RL^{-1}s + (LC)^{-1}} = \frac{w_o^2}{s^2 + 2\zeta w_o s + w_o^2} \]

- \( H \) has a complex-conjugate pair of poles when damping factor \( \zeta < 1 \) each with magnitude \( w_o \):

\[-w_o(\zeta \pm j\sqrt{1-\zeta^2})\]

- \( H \) is marginally stable (divergent eigenresponse at resonant frequencies \( \pm w_o \), i.e., poles at \( s = \pm jw_o \)) when \( \zeta = 0 \) (\( R = 0 \)).
Consider $H(jw_o) = -j/(2\zeta)$ as a function of $\zeta \leq 1$.

For the magnitude response:
- If $\zeta = 1$, then $|H(jw_o)| = 1/2 = -6$ dB consistent with a double pole at $-w_o$ and a DC gain $|H(0)| = 1(= 0$ dB).
- If $\zeta = 0.5$, then $|H(jw_o)| = 1(= 0$ dB), i.e., the 6dB gap disappears and the true Bode plot meets the asymptotic one at its corner.
- $\lim_{\zeta \to 0} |H(jw_o)| = \infty$; this resonance effect is why this system is called a “tuned circuit”: it dramatically amplifies only in a narrow band around $w_o$.

The phase

$$\angle H(jw) = -\tan^{-1}\left(\frac{2\zeta}{w_0/w - w/w_o}\right).$$

The change in phase due to the pair of poles (each with magnitude $w_o$) is $-\pi$ as frequency $w$ increases past $w_o$; note that the denominator is positive for $w < w_o$ and negative for $w > w_o$.

As $\zeta \to 0$, this change in phase becomes more abrupt, not gradual at $-90$ degrees/decade over two decades.

Recall that for $L = 0.1$H, and $C = 1\mu$F, the resonant frequency is $w_o = (LC)^{-0.5} = 3162$ radians/s.

Furthermore, if $R = 100\Omega$ then $\zeta = (R/2)\sqrt{C/L} = 0.16 < 1$ (the underdamped case).
The following Bode plots are for the underdamped cases corresponding to \( R = 100\Omega \) (\( \zeta = 0.16 \)) and \( R = 10\Omega \) (\( \zeta = 0.016 \)), as well as the critically damped case (\( \zeta = 1 \)) with a repeated (double) pole at \(-\omega_o\) (note the -6dB gap and more gradual -90 degrees/decade slope at \( \log \omega_o \)).

The overdamped case (\( \zeta > 1 \)) would have two real poles straddling \( \log \omega_o \) and Bode plots similar to the previous two-pole example of a series RLC transfer function.
Recall that if the output of the RLC circuit $y$ is instead the voltage across the resistor, then $H$ has a DC zero:

$$H(s) = \frac{RL^{-1}s}{s^2 + RL^{-1}s + (LC)^{-1}} = \frac{2\zeta \omega_0 s}{s^2 + 2\zeta \omega_0 s + \omega_0^2}.$$

The following asymptotic Bode plots are for the underdamped case case when $\omega_0 = 3162$ radians/s.

- The DC gain is $0(= -\infty \text{ dB})$ and that the gain increases from DC at 20 dB/decade, consistent with a DC zero.
- The DC phase is $\pi/2$ radians since $\angle H(jw) \approx \angle j$ for very small frequencies $w \downarrow 0$.
- At frequency $\omega_0$ the system response is $H(j\omega_0) = 1$, i.e., 0 dB and 0 phase.
Bilateral Fourier transform

- We now study in the frequency domain the ZSR of a LTI system to a not necessarily periodic input.
- We will employ the bilateral Fourier transform spanning positive and negative frequencies (as the complex-exponential Fourier series for periodic signals).

Outline of topics to be covered:
- Derivation of bilateral Fourier transform (FT) from Fourier series of periodically extended pulse of finite support.
- Basic FT pairs and properties.
- Parseval’s theorem for signal energy.
- Ideal filters and causality issues.
- Modulation and demodulation.
- Poisson’s identity and Nyquist sampling.

Periodic extensions of aperiodic signals of finite support

- Recall periodic extensions of pulses \( x \) of finite support.

- In this example, the pulse \( x \) has support of duration 4 and for \( T \geq 4 \), the \( T \)-periodic extension of \( x \) depicted in this figure is

\[
x_T(t) := \sum_{k=-\infty}^{\infty} x(t - kT) = \sum_{k=-\infty}^{\infty} (\Delta_{kT} x)(t), \quad t \in \mathbb{R},
\]

i.e., superposition of time-shifted pulses \( x \) by all integer \( (n) \) multiples of \( T \).

- Consider the Fourier series of \( x_T \): \( \forall t \in \mathbb{R} \),

\[
x_T(t) = \sum_{k=-\infty}^{\infty} D_{k,T} e^{jk2\pi t/T}, \quad \text{where} \quad D_{k,T} = \frac{1}{T} \int_{-T/2}^{T/2} x(\tau) e^{-jk2\pi \tau/T} d\tau.
\]
• Define the function,

\[ X(k2\pi/T) := \int_{-\infty}^{\infty} x(\tau) e^{-jk2\pi\tau/T} d\tau. \]

• Thus, for large \( T \), \( D_{k,T} \approx T^{-1}X(k2\pi/T) \).

• Also, \( \forall t \in \mathbb{R}, x(t) = \lim_{T \to \infty} x_T(t) \).

• So, substituting this approximation of the Fourier coefficients \( D \) into the complex-exponential Fourier series of \( x_T \) gives, by Riemann integration, \( \forall t \in \mathbb{R} \),

\[
x(t) = \lim_{T \to \infty} \sum_{k=\infty}^{\infty} X(k2\pi/T) e^{jk2\pi t/T} T^{-1}
\]

\[
= \int_{-\infty}^{\infty} X(w) e^{jw t/(2\pi)} dw/(2\pi)
\]

where \( w = 2\pi/T \) so that \( T^{-1} \) converges to the differential \( dw/(2\pi) \) as \( T \to \infty \).

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**Fourier transform and inverse Fourier transform**

• In summary, we defined the *Fourier transform* of time-domain signal \( x : \mathbb{R} \to \mathbb{R} \) as

\[
X(w) := \int_{-\infty}^{\infty} x(t) e^{-jwt} dt \quad \forall \text{ frequencies } w \in \mathbb{R}.
\]

• Using the complex-exponential Fourier series of the periodic extension of \( x \), we heuristically derived the *inverse Fourier transform* of the frequency-domain signal \( X : \mathbb{R} \to \mathbb{R} \) as

\[
x(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jwt} dw \quad \forall \text{ time } t \in \mathbb{R}.
\]

• We write \( X = \mathcal{F}x \) (or \( \mathcal{F}\{x\} \)) and \( x = \mathcal{F}^{-1}X \), again emphasizing functional transformation.

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Fourier transform properties - even magnitude and odd phase

- If \( x : \mathbb{R} \to \mathbb{R} \), i.e., \( x \) is a real-valued signal, then \( X = \mathcal{F}x \) has
  - even magnitude, i.e., \( |X(-w)| = |X(w)| \forall w \in \mathbb{R} \), and
  - odd phase, i.e., \( \angle X(w) = -\angle X(-w) \forall w \in \mathbb{R} \),

- That is, \( X(-w) = \overline{X(w)} \forall w \in \mathbb{R} \):
  \[
  X(-w) = \int_{-\infty}^{\infty} x(t)e^{-j(-w)t}dt = \int_{-\infty}^{\infty} x(t)e^{jwt}dt = \int_{-\infty}^{\infty} x(t)e^{-jwdx}dt \quad \text{(since } x \text{ is } \mathbb{R}-\text{valued)}
  \]
  \[
  \overline{X(w)} \forall w \in \mathbb{R}.
  \]

Fourier transform properties - linearity & differentiation

- The Fourier transform is a linear mapping simply because of the linearity of integration.
- For all differentiable (Dirichlet) signals \( x \), the Fourier transform of a time derivative is
  \[
  \forall w \in \mathbb{R}, \quad (\mathcal{F}\mathcal{D}x)(w) = jw(\mathcal{F}x)(w).
  \]
- Recalling \( \mathcal{D} := \frac{d}{dt} \), this property is proved by simple integration by parts: \( \forall w \in \mathbb{R} \),
  \[
  (\mathcal{F}\mathcal{D}x)(w) = \int_{-\infty}^{\infty} (\mathcal{D}x)(t)e^{-jwt}dt
  \]
  \[
  = \int_{-\infty}^{\infty} e^{-jwt}dx(t)
  \]
  \[
  = x(t)e^{-jwt}\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)de^{-jwt}
  \]
  \[
  = 0 + jw\int_{-\infty}^{\infty} x(t)e^{-jwt}dt \quad \text{(*)}
  \]
  \[
  = jwX(w),
  \]
  where (*) is by the (weak) Dirichlet condition of absolute integrability which implies that
  \( \lim_{t \to \pm \infty} x(t) = 0 \).
We can use the Fourier transform (FT) to analyze the zero-state response of the system

\[ P(D)f = Q(D)y. \]

Let the FT of the input and output are, respectively, \( F = \mathcal{F}f \) and \( Y = \mathcal{F}y. \)

By the linearity and differentiation properties of the FT: \( \forall w \in \mathbb{R}, \)

\[ P(jw)F(w) = Q(jw)Y(w). \]

Thus, the transfer function,

\[ H(jw) = \frac{P(jw)}{Q(jw)} = \frac{Y(w)}{F(w)}, \quad \forall w \in \mathbb{R}. \]

**Change of notation:** In the following discussion of Fourier transforms, we will use "\( H(w) \)" instead of "\( H(jw) \)" for the transfer function, i.e.,

\[ H(w) = \frac{P(jw)}{Q(jw)} = \frac{Y(w)}{F(w)}, \quad \forall w \in \mathbb{R}. \]

---

**Fourier transform of the unit impulse, \( \delta \), and \( H = \mathcal{F}h \)**

By the ideal sampling property, the Fourier transform of the impulse is, \( \forall w \in \mathbb{R}, \)

\[ (\mathcal{F}\delta)(w) = \int_{-\infty}^{\infty} \delta(t)e^{-jwt}dt = e^{-jwt}|_{t=0} = 1. \]

So, \( \mathcal{F}\delta \equiv 1. \)

Note that by inverse Fourier transform, this means

\[ 2\pi\delta(t) = \int_{-\infty}^{\infty} e^{jwt}dw \quad \forall t \in \mathbb{R}. \]

Thus, the LTI SISO system with input \( f = \delta \) has ZSR \( y = h \) (impulse response), so that in the frequency domain

\[ Y = HF = H1 = H, \]

and so,

\[ \mathcal{F}h = H, \]

i.e., the Fourier transform of the impulse response is the transfer function.
Transfer function is Fourier transform of (ZS) impulse response

- In summary, given the input \( f \), to solve for the ZS \( y \) the LTl SISO system described as \( P(D)f = Q(D)y \) (or as \( y = f * h \)), we will employ the Fourier transform:
  \[
y = \mathcal{F}^{-1}\{HF\},
\]
where \( H = \mathcal{F}h \) and \( F = \mathcal{F}f \).

- From this, we can conclude the convolution property:
  \[
  \mathcal{F}\{f * h\} = \mathcal{F}\{f\}\mathcal{F}\{h\}.
  \]

- Sometimes the inverse Fourier transform is easily computed by integration.

- Otherwise, certain Fourier-transform properties and known Fourier-transform pairs can simplify computing \( \mathcal{F}^{-1} \).

Direct proof of the convolution property, \( \mathcal{F}\{f * h\} = \mathcal{F}\{f\}\mathcal{F}\{h\} \).

\[
\mathcal{F}\{f * h\}(w) = \int_{-\infty}^{\infty} (f * h)(t)e^{-jwt}dt
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)h(t - \tau)e^{-jwt}d\tau dt
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)h(t - \tau)e^{-jw(t - \tau)}e^{-j\omega\tau}d\tau dt
\]
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)h(t')e^{-jw(t' - \tau)}e^{-j\omega\tau}d\tau dt' \quad (t' = t - \tau)
\]
\[
= \int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{\infty} h(t')e^{-jwt'}dt'
\]
\[
= \mathcal{F}\{f\}(w)\mathcal{F}\{h\}(w)
\]
\[
= F(w)H(w) \quad \forall w \in \mathbb{R}.
\]
Fourier transform pairs - exponential functions of time

- Again, let $X = \mathcal{F}x$.
- If the causal signal $x(t) = e^{so}u(t)$, $t \in \mathbb{R}$, with $s_0 \in \mathbb{C}$ and $\text{Re}\{s_0\} < 0$, then
  $$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt = \int_{0}^{\infty} e^{so}e^{-jwt}dt = \frac{1}{jw - s_0}.$$  
- Note that if $s_0 \in \mathbb{R}$ (so that $x$ is $\mathbb{R}$-valued), then $|X(w)| = (w^2 + s_0^2)^{-0.5}$ is even, $\angle X(w) = -\tan^{-1}(w/s_0)$ is odd, and, again, $X$ is the frequency domain representation of an aperiodic signal $x$.
- Also note that the “anti-causal” signal $\{-e^{so}u(-t), t \in \mathbb{R}\}$ has the same Fourier transform expression but requires $\text{Re}\{s_0\} > 0$ for convergence.
- For example, $\forall w \in \mathbb{R}$,
  $$\mathcal{F}\{e^{3t}u(t)\}(w) = \frac{1}{jw + 3} \quad \text{and} \quad \mathcal{F}\{e^{2t}u(-t)\}(w) = \frac{1}{2 - jw}.$$  
- Neither the causal signal $e^{3t}u(t)$ nor the anti-causal signal $e^{-2t}u(-t)$ are Dirichlet because they do not have a finite integral.

---

Fourier analysis of RL circuit with exponential input

- Recall that for this circuit, the transfer function is
  $$H(w) = \frac{jw/2 + R/(2L)}{jw + R/(2L)} = \frac{1}{2} + \frac{R/(4L)}{jw + R/(2L)} \quad \forall w \in \mathbb{R}.$$  
- So, by linearity and the Fourier-transform pairs we have already discussed, the impulse response of this system is
  $$h(t) = \frac{1}{2}\delta(t) + \frac{R}{4L}e^{-Rt/(2L)}u(t) \quad \forall t \in \mathbb{R}.$$  

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Inverse Fourier transforms by known pairs

- Again, note that if we try to use the integral definition of $F^{-1}$ on the frequency domain signal $c$ (constant) or $b/(jw + a)$ we do not get an integrand corresponding to an indefinite integral, i.e., we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} ce^{jwt} dw$$

or

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{jw + a} e^{jwt} dw.$$

- We found the inverse Fourier transforms by remembering the easily computed Fourier transforms $F\{c\delta\}$ for constant $c \in \mathbb{R}$ and $F\{e^{-at}u(t)\}$ for constant $a > 0$.

Fourier analysis of RL circuit with exponential input (cont)

- Let $R = 2\Omega$ and $L = 1\text{H}$ so that $H(w) = (jw/2 + 1)/(jw + 1)$.

- So, if the input of this circuit is $f(t) = 4e^{-3t}u(t)$, $t \in \mathbb{R}$, then the Fourier transform of the ZSR $y$ is, $\forall w \in \mathbb{R}$,

$$Y(w) = H(w)F(w) = \frac{jw/2 + 1}{jw + 1} \cdot \frac{4}{jw + 3} = \frac{2jw + 4}{(jw + 1)(jw + 3)}$$

where the last equality is a partial-fraction expansion (PFE) of $Y$.

- Thus, the ZSR $y(t) = (e^{-t} + e^{-3t})u(t)$.

- Note that $y$ is the sum of a system characteristic mode $e^{-t}$ and the forced response, $H|_{jw=-3}f(t) = (1/4)4e^{-3t} = e^{-3t}$.

- We will subsequently study in detail the PFE of rational polynomials in $s = jw$ for purposes of computing the inverse Laplace transform.
Fourier transform properties - time shifts

- The time-shift property is: \( \forall t_o, w \in \mathbb{R}, \)
  \[
  (\mathcal{F}\{\Delta_{t_o}x\})(w) = \int_{-\infty}^{\infty} (\Delta_{t_o}x)(t)e^{-jwt}dt
  = \int_{-\infty}^{\infty} x(t - t_o)e^{-jwt}dt
  = \int_{-\infty}^{\infty} x(t')e^{-jw(t+t_o)}dt' \quad (t' := t - t_o)
  = e^{-jwt_o}\int_{-\infty}^{\infty} x(t')e^{-jwt}dt'
  = e^{-jwt_o}X(w)
  \]
  
- So a time shift by \( t_o \) corresponds to multiplication in the frequency domain by a complex-exponential sinusoid with period \( 2\pi/t_o \).

Fourier transform properties - frequency shifts

- The frequency-shift (modulation) property is: \( \forall w_o, t \in \mathbb{R}, \)
  \[
  (\mathcal{F}^{-1}\Delta_{w_o}X)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\Delta_{w_o}X)(w)e^{jwt}dw
  = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w - w_o)e^{jwt}dw
  = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w')e^{jwt}e^{jwt}dw' \quad (w' := w - w_o)
  = e^{jwt_o}\frac{1}{2\pi} \int_{-\infty}^{\infty} X(w')e^{jwt}dw'
  = e^{jwt_o}X(t).
  \]
  
- A baseband signal \( x \) multiplied by a carrier signal at frequency \( w_o \) radians/s leads to a frequency shifted (modulated) signal \( e^{jwt_o}x(t) \).
Fourier transform properties - time and frequency scaling

- If $X = \mathcal{F}x$ and scalar $\alpha \neq 0$, the time/frequency scaling property is easily shown by change-of-variables:
  \[
  (\mathcal{F} S_{1/\alpha} x)(w) \ := \ (\mathcal{F}\{x(\alpha t)\})(w) \ = \ \frac{1}{|\alpha|} X\left(\frac{w}{\alpha}\right).
  \]

- For example,
  \[
  \frac{jw}{jw/6 + 2} \xrightarrow{\mathcal{F}^{-1}} D \frac{1}{6}x(6t) = D 6e^{-12t}u(t) = -72e^{-12t}u(t) + 6\delta(t)
  \]
  where $x(t) = e^{-2t}u(t)$ and $u(6t) = u(t)$; equivalently,
  \[
  \frac{jw}{jw/6 + 2} \xrightarrow{\mathcal{F}^{-1}} 36\left(S_{1/6}D x\right)(t) = 36 S_{1/6}\left(-2e^{-2t}u(t) + \delta(t)\right)
  \]
  \[
  = -72e^{-12t}u(6t) + 36\delta(6t) = -72e^{-12t}u(t) + 6\delta(t)
  \]

Fourier transform properties - composition order

- The order of composed frequency-shift and time-shift operations matters, i.e., these operations are not commutative.

- Given $X = \mathcal{F}x$ and scalars $t_o, w_o$.

- If frequency shift (modulation) before time shift:
  \[
  x(t - t_o)e^{jw_o(t-t_o)} = \Delta_{t_o}\{x(t)e^{jw_o t}\} \xrightarrow{\mathcal{F}} X(w - w_o)e^{-jw t_o}.
  \]

- On the other hand, if time shift before frequency shift:
  \[
  x(t - t_o)e^{jw_o t} = (\Delta_{t_o} x)(t)e^{jw_o t} \xrightarrow{\mathcal{F}} X(w - w_o)e^{-j(w - w_o)t_o}.
  \]

Similarly, other operation pairs (e.g., involving differentiation, time/frequency scaling) are not commutative.
Fourier transform pairs - complex-exponentials and impulses

- Note that by the time shift property, the Fourier transform of an impulse delayed by $t_0$ is a complex-exponential sinusoid in the frequency domain with period $2\pi/t_0$:
  \[ \mathcal{F}\{\delta(t)\}(f) = e^{-jft_0} \quad f \in \mathbb{R}. \]

- Similarly, an impulse located at $w_0$ in the frequency domain corresponds to a sinusoid in the time-domain with frequency $w_0$:
  \[ \mathcal{F}^{-1}\{\delta(w)\}(t) = \frac{1}{2\pi}e^{jw_0t} \quad t \in \mathbb{R}. \]

- The presence of impulses in the frequency domain indicates discrete frequency components, e.g., of periodic signals.

- Again note that a complex exponential (over a time or frequency domain) cannot be directly integrated.

Fourier transform pairs - sine and cosine

- By direct integration we can verify that
  \[
  \mathcal{F}^{-1}\{\pi \delta(w) + \pi \delta(-w)\}(t) = \cos(w_0 t) \quad t \in \mathbb{R} \\
  \mathcal{F}^{-1}\{-j\pi \delta(w) + j\pi \delta(-w)\}(t) = \sin(w_0 t) \quad t \in \mathbb{R}.
  \]

- By the Euler-De Moivre identity, the aperiodic, one-sided functions (real $\alpha > 0$)
  \[
  e^{-\alpha t} \cos(w_0 t) u(t) = \frac{1}{2}e^{(-\alpha+jw_0)t}u(t) + \frac{1}{2}e^{(-\alpha-jw_0)t}u(t), \\
  e^{-\alpha t} \sin(w_0 t) u(t) = \frac{1}{2j}e^{(-\alpha+jw_0)t}u(t) - \frac{1}{2j}e^{(-\alpha-jw_0)t}u(t)
  \]

- Recalling the FT for such one-sided exponential functions, we can show by linearity that for $\alpha > 0$, $w \in \mathbb{R}$,
  \[
  \mathcal{F}\{e^{-\alpha t} \cos(w_0 t)\}(w) = \frac{1}{2} \cdot \frac{1}{\alpha + j(w - w_0)} + \frac{1}{2} \cdot \frac{1}{\alpha + j(w + w_0)} = \frac{\alpha + jw}{(\alpha + jw)^2 + w_0^2} \\
  \mathcal{F}\{e^{-\alpha t} \sin(w_0 t)\}(w) = \frac{1}{2j} \cdot \frac{1}{\alpha + j(w - w_0)} - \frac{1}{2j} \cdot \frac{1}{\alpha + j(w + w_0)} = \frac{w_0}{(\alpha + jw)^2 + w_0^2}
  \]
Fourier transform of $f(t) = \cos(\omega_0 t)$, $t \in \mathbb{R}$

- Recall that SISO LTI systems have characteristic modes of the form $t^ke^{s_0 t}u(t)$ where $k \geq 0$ is an integer and $s_0 \in \mathbb{C}$ a characteristic value of the system.

- As we are considering only asymptotically stable systems in our discussion of Fourier transforms, we can assume $\text{Re}\{s_0\} < 0$.

- To evaluate the Fourier transform of such a characteristic mode, we need to integrate by parts: for $k > 0$ and $\text{Re}\{s_0\} < 0$:

\[
\mathcal{F}\{t^ke^{s_0 t}u(t)\}(w) = \int_0^\infty t^ke^{s_0 t}e^{-jw t}dt = \frac{1}{s_0 - jw} \int_0^\infty t^k de^{(s_0 - jw)t} \\
= \frac{1}{s_0 - jw} t^k e^{(s_0 - jw)t} \bigg|_0^\infty - \frac{1}{s_0 - jw} \int_0^\infty e^{(s_0 - jw)t}dt^k \\
= \frac{k}{jw - s_0} \int_0^\infty t^{k-1} e^{(s_0 - jw)t}dt \\
= \frac{k}{jw - s_0} \mathcal{F}\{t^{k-1}e^{s_0 t}u(t)\}(w) \quad w \in \mathbb{R}.
\]

- So by induction, we can show that for integers $k > 0$ and $\text{Re}\{s_0\} < 0$:

\[
\mathcal{F}\{t^ke^{s_0 t}u(t)\}(w) = \frac{k!}{(jw - s_0)^{k+1}} \quad \forall w \in \mathbb{R}.
\]
Fourier transform pairs - sinc and rectangle pulse

- Consider the even rectangular pulse of height 1 and width $2w_o$ in the frequency domain:
  \[ R(w) = u(w + w_o) - u(w - w_o) \]

- Its inverse Fourier transform is
  \[
  r(t) = \mathcal{F}^{-1}\{R\}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(w) e^{jwt} dw
  \]
  \[
  = \frac{1}{2\pi} \int_{-w_o}^{w_o} e^{jwt} dw
  \]
  \[
  = \frac{1}{2\pi} \cdot \frac{1}{jt} (e^{jw_o t} - e^{-jw_o t})
  \]
  \[
  = \frac{w_o}{\pi} \cdot \frac{e^{jw_o t} - e^{-jw_o t}}{2jw_o t}
  \]
  \[
  = \frac{w_o}{\pi} \cdot \text{sinc}(w_o t).
  \]

  where
  \[
  \text{sinc}(v) := \frac{\sin(v)}{v}.
  \]

- Note that by L’Hopital’s rule, \( \lim_{v \to 0} \text{sinc}(v) = 1 \).

Fourier transform pairs - sinc and rectangle pulse (cont)

- The rectangular pulse has finite support but its FT pair, the sinc, has infinite support.

- Note that as the frequency $w_o$ increases,
  - the rectangle pulse $R$ in the frequency domain is broader, but
  - the sinc function in the time domain is narrower; indeed, the width between the zero crossings of the central lobe is \( \pi/w_o - (-\pi/w_o) = 2\pi/w_o \).

- The two plotted sinc functions have $w_o \in \{1, 5\}$.

- In the limit as $w_o \to \infty$ this approaches the $\delta \leftrightarrow 1$ Fourier-transform pair.
Duality

- We have noted several "dual" FT properties and pairs, e.g., time and frequency shifts, sinusoids and impulses.

- Duality is a consequence of the similarity of the Fourier and inverse Fourier transforms.

- If \( X = \mathcal{F}x \), i.e.,

\[
X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt \quad \forall w \in \mathbb{R},
\]

then we can consider the Fourier transform (not inverse FT) of \( X \):

\[
(\mathcal{F}X)(w) = \int_{-\infty}^{\infty} X(t) e^{-jwt} dt = 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{j(-w)t} dt
\]

\[
= 2\pi (\mathcal{F}^{-1}X)(-w) = 2\pi x(-w) \quad \forall w \in \mathbb{R},
\]

noting how we have used the definition of inverse FT for the last equality and have switched the roles of frequency \( w \) and time \( t \).

- Duality of Fourier transforms can be stated more compactly as:

\[
X = \mathcal{F}x \iff \mathcal{F}X = 2\pi S_{-1} x,
\]

recalling \( S_{-1} \) is the domain-inversion (reflection in y-axis) operation.

---

Duality - example

- For example, recall that

\[
r(t) = \frac{w}{\pi} \text{sinc}(wt) , \quad t \in \mathbb{R} \rightarrow \mathcal{F} \quad R(w) = u(w - w_0) - u(w + w_0) , \quad w \in \mathbb{R}.
\]

- By duality, we expect (replacing parameter \( w \) with \( t_o \) now representing a time value):

\[
u(t - t_o) - u(t + t_o) , \quad t \in \mathbb{R} \rightarrow \mathcal{F} \quad 2\pi \frac{t_o}{\pi} \text{sinc}(-wt_o) = 2t_o \text{sinc}(wt_o) , \quad w \in \mathbb{R},
\]

where we note that the sinc function is also even.

- **Exercise:** Verify this by directly integrating to compute \( \mathcal{F}R \).

- **Exercise:** Suppose the "procedure" or "function" \( \mathcal{F}(\zeta,v) \) returns \( \int_{-\infty}^{\infty} \zeta(a)e^{-jva} da \) when passed a signal \( \zeta : \mathbb{R} \rightarrow \mathbb{R} \) and real number \( v \). Show how to use this procedure to compute (i) \( (\mathcal{F}x)(w) \) and (ii) \( (\mathcal{F}^{-1}X)(t) \).
Signal energy by Parseval’s theorem

- Consider an energy signal \( x : \mathbb{R} \to \mathbb{R} \) with Fourier transform \( X = \mathcal{F}x \),

\[
E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)x(\bar{t}) dt.
\]

- Substituting \( x = \mathcal{F}^{-1}X \), using Fubini’s theorem, and then \( \mathcal{F}\delta \), we get Parseval’s theorem:

\[
E_x = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(v)e^{jvt} dv \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)e^{jwt} dw \ dt
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) \int_{-\infty}^{\infty} X(v) \left( \int_{-\infty}^{\infty} \frac{e^{jvt}}{2\pi} e^{-jwt} dt \right) dv dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) \int_{-\infty}^{\infty} X(v) \delta(w - v) dv dw
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)|^2 dw
\]

Parseval’s theorem - example

- Recall that if \( X(w) = u(w + w_o) - u(w - w_o) \), \( w \in \mathbb{R} \) (i.e., an even rectangle pulse), then its inverse Fourier transform is \( x(t) = \frac{w_o}{\pi} \text{sinc}(w_o t) \), \( t \in \mathbb{R} \).

- The energy of this signal in the time domain cannot be directly computed, but by using Parseval’s theorem,

\[
E_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)|^2 dw
\]

\[
= \frac{1}{2\pi} \int_{-w_o}^{w_o} dw
\]

\[
= \frac{w_o}{\pi}
\]

- Equating to \( \int_{-\infty}^{\infty} |x(t)|^2 dt \) and substituting \( \tau = w_o t \), we get an interesting result:

\[
\int_{-\infty}^{\infty} \text{sinc}^2(\tau) d\tau = \pi.
\]
Suppose we want to find the energy of the response \( y \) of an ideal band-pass filter to a given input signal \( f \), with filter pass-band \([w_1, w_2]\) such that \( 0 < w_1 < w_2 \) radians/s.

\( Y = \mathcal{F}y \) satisfies

\[
|Y(w)| = \begin{cases} |F(w)| & \text{if } w \in [w_1, w_2] \text{ or } w \in [-w_2, -w_1] \\ 0 & \text{else} \end{cases}
\]

So, by Parseval’s theorem,

\[
E_y = \frac{1}{2\pi} \int_{-w_1}^{w_1} |F(w)|^2dw + \frac{1}{2\pi} \int_{w_2}^{w_2} |F(w)|^2dw
\]

\[
= \frac{1}{\pi} \int_{w_1}^{w_2} |F(w)|^2dw,
\]

where the last equality is because \( f \) is \( \mathbb{R} \)-valued (and hence \( F = \mathcal{F}f \) is an even function).

Exercise: Find the transfer function \( H \) and impulse response \( h = \mathcal{F}^{-1}H \) of this band-pass filter.

---

### Ideal Amplifier

An ideal amplifier has impulse response \( h = A \Delta_\tau \delta \) for delay \( \tau > 0 \), and hence transfer function \( H(w) = Ae^{j\tau}w \), \( w \in \mathbb{R} \).

Thus, the ZSR of an ideal amplifier to any (Dirichlet) input signal \( f \) is, for some \( \tau \in \mathbb{R} \),

\[
y(t) = (f * h)(t) = Af(t - \tau), \quad t \in \mathbb{R},
\]

i.e., an amplified (by \( A \)) and undistorted (though potentially delayed by \( \tau \)) version of the input signal; equivalently,

\[
Y(w) = F(w)H(w) = AF(w)e^{-j\tau}, \quad w \in \mathbb{R}.
\]

Such a constant response \( (A = |H(w)| \forall w \in \mathbb{R}) \) across the whole frequency band is not feasible.

Typically, only an approximately distortionless amplification is possible over a limited frequency band.
Similarly, we may be interested in “filtering out” all frequencies \( w \) \((H(w) = 0)\) of an input signal \( f \) except for a certain frequency band, i.e., except for the pass-band of frequencies \( w \) where \( H(w) = 1 \) \((or\ H(w) = e^{-j\omega \tau} \text{ for delay } \tau)\).

Thus, the ZSR

\[
Y(w) = F(w)H(w) = \begin{cases} 
F(w) & \text{if } w \text{ is in the pass-band} \\
0 & \text{else} 
\end{cases}
\]

If the pass-band is of the form:

- \([0, w_0]\) for \(w_0 > 0\) (more precisely \([-w_0, w_0]\) in the Fourier domain), then the filter is a low-pass filter, and \(w_0\) (or sometimes \(2w_0\)) is called the bandwidth of the filter;
- \((-\infty, -w_0] \cup [w_0, -\infty)\) for \(w_0 > 0\), then the filter is a high-pass filter;
- \([-w_1, -w_0] \cup [w_0, w_1]\) for \(w_1 > w_0 > 0\), then the filter is a band-pass filter.

Feasibility also requires causality, equivalently the Paley-Wiener criterion in the frequency domain which precludes \(H(w) = 0\) on whole intervals of the real line.
Feasible low-pass filter (LPF) or amplifier

- We therefore see that an ideal low-pass filter (without delay) will have transfer function in the form of an even rectangle pulse,
  \[ H(w) = A(u(w + w_0) - u(w - w_0)) \quad \forall w \in \mathbb{R}, \]
  where \( A > 1 \) if it’s an ideal amplifier over the “low” frequency band \([-w_0, w_0]\).

- Thus, the impulse response is
  \[ h(t) = \frac{A w_0}{\pi} \text{sinc}(w_0 t), \quad t \in \mathbb{R}. \]

- Note that a non-causal impulse response \( h \) that is not anti-causal can be delayed to a causal impulse response \( \Delta \tau h \), where \( h(t) = 0 \ \forall t < \tau \);

- thus, the ZSR to \( f \) of the causal LTI system is \( f \ast \Delta \tau h = \Delta \tau (f \ast h) \).

- But the impulse response of an ideal LPF is anti-causal (meaning that the support of \( h \) extends to \(-\infty\)).

- Hence this ideal filter is anticausal and infeasible.

Approximating an ideal LPF \( H \) with a causal LPF \( \tilde{H} \)

- Consider again the ideal LPF \( H(w) = u(w + w_0) - u(w - w_0), w \in \mathbb{R} \).

- Suppose the anticausal \( \text{sinc} \) impulse response \( h = \mathcal{F}^{-1} H \) is delayed by \( \tau > \pi/w_0 > 0 \) then truncated to yield a causal impulse response
  \[ \tilde{h} = u \Delta \tau h \quad \Rightarrow \quad \tilde{H} = (2\pi)^{-1} U \ast \{H(w)e^{-j\omega t}\} \quad \text{where} \ U = \mathcal{F} u. \]

- The idea is that \( \tau > 0 \) is sufficiently large so that the square error
  \[ \int_{-\infty}^{\infty} |h(t - \tau) - \tilde{h}(t)|^2 dt = \int_{-\infty}^{-\tau} |h(t)|^2 dt \]
  is small.

- If this is the case, the transfer function of the causal LPF \( \tilde{H} = \mathcal{F} \tilde{h} \) will be a slightly distorted version of \( H \) (both in the pass and non-pass bands) with a roughly linear phase of negative slope due to the time delay.

- Note that just the delayed version of the impulse response \( \Delta \tau h \) still corresponds to an ideal, anticausal LPF with transfer function \( H(w)e^{-j\omega \tau}, w \in \mathbb{R} \).

- In a subsequent course on signal processing, you will see how to feasibly trade-off lower distortion for complexity \( \text{e.g., degree } n \) of the causal filter’s characteristic polynomial) through the use of special polynomials, \( \text{e.g., from the Butterworth or Chebyshev families.} \)
Modulation of a baseband signal

- A signal \( f = \mathcal{F}^{-1}F \) is said to be a baseband signal if it is band limited, i.e., if there is frequency \( w_0 > 0 \) such that \( F(w) = 0 \ \forall |w| > w_0 \).

- For example, \( w_0 \approx 20kHz = 20000 \cdot 2\pi \) radians/s for an audio signal discernible to the human ear.

- Suppose the audio signal is to be (wireless) transmitted over the air in a frequency band assigned to a radio station,

\[
[-w_c - w_0, -w_c + w_0] \cup [w_c - w_0, w_c + w_0]
\]

where typically \( w_c \gg w_0 \).

- So, the audio signal needs to be frequency shifted to the carrier frequency \( w_c \) prior to transmission over the air.

- Again, frequency shifting is accomplished by multiplication by a sinusoid in the time domain,

\[
f(t) \cos(w_c t), \quad t \in \mathbb{R}.
\]

which is the modulated signal traveling over the air.
Demodulation of a baseband signal

- The radio receiver needs to demodulate the signal $f(t) \cos(w_c t)$ after it has been received over the air.

- Demodulation is accomplished by first applying the same frequency shift as modulation,

\[
\begin{align*}
    f(t) \cos(w_c t) \cdot \cos(w_c t) &= f(t) \cos^2(w_c t) \\
    &= f(t) \left( \frac{1}{2} + \frac{1}{2} \cos(2w_c t) \right), \quad t \in \mathbb{R}.
\end{align*}
\]

and then applying a low-pass filter to eliminate the high-frequency components (at $\pm 2w_c$) to recover the baseband signal,

\[
\frac{1}{2} f.
\]

- In later courses on communications, you will learn of different modulation techniques, including those that involve first sampling and quantizing the continuous-time baseband signal.
Sampling - preliminaries

- Sampling a continuous-time, band-limited signal $f$ is a first step to processing it digitally (i.e., applying a digital filter to its sample sequence).

- Recall the ideal sampling property of the unit impulse: if $f$ is continuous at $\tau$ then
  \[
  \int_{-\infty}^{\infty} \delta(t - \tau) f(t) \, dt = f(\tau),
  \]
  i.e., the sample at time $\tau$, $f(\tau)$, is extracted from the signal $f$ by multiplying the signal with an impulse at $\tau$ and integrating.

- Suppose the continuous-time signal $f$ is being sampled every $T_s = 2\pi/w_s$ seconds to obtain the sequence $\{f(kT_s)\}_{k\in\mathbb{Z}}$.

Sampling - picket-fence function

- To capture such periodic sampling at frequency $w_s$, we multiply $f$ by the “picket-fence” function,
  \[
  p_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \quad \forall t \in \mathbb{R},
  \]
  to obtain
  \[
  p_{T_s}(t)f(t) = \sum_{k=-\infty}^{\infty} f(t)\delta(t - kT_s) = \sum_{k=-\infty}^{\infty} f(kT_s)\delta(t - kT_s) \quad \forall t \in \mathbb{R},
  \]
  i.e., the impulse $\delta(t - kT_s)$ is weighted by the $f$-sample $f(kT_s)$. 
Impact of sampling in the frequency domain - Poisson’s identity

- To visualize the impact of sampling on the signal, we now derive $\mathcal{F}\{p_T f\}$.
- Obviously, $p_T$ is $T_s$-periodic.
- The complex-exponential Fourier series of $p_T$ has coefficients $D_k = T_s^{-1} \forall k \in \mathbb{Z}$; this leads to Poisson’s identity:
  \[
  p_T(t) := \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jkw_s t}, \quad \forall t \in \mathbb{R}
  \]
- Thus, if $F = \mathcal{F} f$ then by the linearity and frequency-shift properties,
  \[
  \mathcal{F}\{p_T f\}(w) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} \mathcal{F}\{f(t)e^{jkw_s t}\}(w) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(w - kw_s), \quad \forall w \in \mathbb{R}.
  \]
- So, sampling has the effect of superposing copies of $F$ frequency-shifted by all integer multiples of the sampling frequency, $w_s = 2\pi/T_s$. 

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Nyquist sampling rate for band-limited signals

- Suppose the signal $f$ is band-limited, i.e., there is a finite $w_B > 0$ such that $F(w) = 0$ for $|w| > w_B$.
- The frequency-shifted copies of $F$ that constitute $F_{PT_f}$ will not overlap if
  \[ w_S - w_B > w_B \]
  \[ \Rightarrow w_S > 2w_B. \]
- If this is the case, note that we can recover the original signal $f$ from $p_{T_s} f$ by applying a low-pass filter (with filter bandwidth between $w_B$ and $w_c = w_B$).
- Thus, if $w_S > 2w_B$, then the complete "information" in $f$ is preserved in its sample-sequence $p_{T_s} f$, equivalently (in terms of information) the samples \( \{f(kT_s)\}_{k=-\infty}^{\infty} \).
- $2w_B$ is called the Nyquist sampling frequency of $f$.
- Sampling at less than the Nyquist sampling frequency may prevent recovery of $f$ from $p_{T_s} f$, e.g., aliasing.

Sampling band-limited signals - example and exercise

- Consider the signal $x(t) = A \text{sinc}(w_o t)$ with
- \[ X(w) := (F x)(w) = (\pi A/w_o)(u(w + w_o) - u(w - w_o)) \] (check $F^{-1} X = x$).
- So, the Nyquist sampling frequency of $x$ is $2w_o$ radians/s, i.e., the Nyquist sampling period is $2\pi/(2w_o) = \pi/w_o$ seconds.
- Consider again the ideal sampling (picket fence) function $p(t) = \sum_{k=\infty}^{\infty} \delta(t - kT)$ for sampling period $T > 0$.
- Recall that \( (F x p)(w) = \sum_{r=\infty}^{\infty} X(w - k2\pi/T)/T. \)
- Plot $F x p$ and show that, in particular, if $T = \pi/w_o$ (the Nyquist sampling period) then \( (F x p)(w) = (\pi A/w_o)/(\pi/w_o) = A \) (a constant).
- So by inverse Fourier transform, $x(t)p(t) = A \delta(t)$.
- Does this make sense in light of $x(t)p(t) = \sum_{k=-\infty}^{\infty} x(kT) \delta(t - kT)$?
Nyquist sampling \( (w_s > 2w_b) \) and aliasing

Unilateral Laplace transform for transient response - Outline

- Laplace transform definition and region of convergence.
- Basic LT pairs and properties.
- Inverse LT of rational polynomials by partial fraction expansion.
- Total response of SISO LTIC systems \( Q(D)y = P(D)f \).
- The steady-state eigenresponse revisited.
- Complex-frequency(s)-domain equivalent circuits.
- System composition and canonical realizations.
Unilateral Laplace transform

- We define the unilateral Laplace transform (LT) of time-domain signal $x : [0, \infty) \to \mathbb{C}$ as
  \[ X(s) := \int_{0-}^{\infty} x(t) e^{-st} \, dt \]
  and:
  - the LT is defined for complex frequencies $s \in \mathbb{C}$ for which this integral converges,
  - $0-$ is just before the chosen origin of time, $0$. 

- The lower limit of integration is taken as “$0-$” so that:
  - the LT of the impulse is well defined, i.e., we’re not “splitting hairs”,
    \[ 1 = (\mathcal{L}\delta)(s), \quad \forall s \in \mathbb{C}, \quad \text{and} \]
  - the LT can accommodate “initial conditions.”

- Input functions of the form $f = gu$ ( $u$ the unit step), will have zero initial conditions by definition, i.e.,
  \[ (D^k f)(0-) = 0, \quad \text{for all integers } k \geq 0. \]

Region of convergence for Laplace transform

- Suppose that the signal $x : [0, \infty) \to \mathbb{C}$ is bounded by an exponential:
  \[ \exists A > 0, \alpha \in \mathbb{R} \quad \text{such that} \quad \forall t \geq 0-, \quad |x(t)| \leq Ae^{\alpha t}, \]
  \[ \text{i.e.,} \quad \forall t \geq 0-, \quad -Ae^{\alpha t} \leq |x(t)| \leq Ae^{\alpha t}. \]

- If $x$ is bounded by an exponential $Ae^{\alpha t}$, $t \geq 0-$, then the region of convergence (RoC) of $\mathcal{L}x$ contains a portion of $\mathbb{C}$ that is an open right-half plane.

- To see why, recall absolute integrability $\Rightarrow$ integrability, and:
  \[ \int_{0-}^{\infty} |x(t)| e^{-st} \, dt = \int_{0-}^{\infty} |x(t)| e^{-Re{s}t} \, dt \leq \int_{0-}^{\infty} Ae^{(\alpha-\text{Re}\{s\})t} \, dt \quad \text{by exp. bdd.} \]
  \[ = \frac{Ae^{(\alpha-\text{Re}\{s\})t}}{\alpha - \text{Re}\{s\}} |_{0-}^{\infty} = \frac{A}{\text{Re}\{s\} - \alpha} < \infty. \]

- Thus, if $\alpha - \text{Re}\{s\} < 0$ then $X(s)$ exists and $|X(s)| < \infty$ by the triangle inequality.

- Note that $\alpha - \text{Re}\{s\} < 0 \Leftrightarrow \text{Re}\{s\} > \alpha \in \mathbb{R}$, i.e., $s$ is an element of an open RHP $\subset \mathbb{C}$. 

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Laplace transforms of exponential functions of time and their RoCs

- Consider the exponential signal \( x(t) = e^{s_o t}u(t), \ t \in \mathbb{R} \), with a particular \( s_o \in \mathbb{C} \) (which is bounded by \( 1e^{\text{Re}(s_o) t} \), \( \text{i.e.,} \ \alpha = \text{Re}(s_o) \)).

- The Laplace transform of the above exponential signal is

\[
X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{0^-}^{\infty} e^{-(s-s_o)t}dt = \left. \frac{1}{s-s_o}e^{-(s-s_o)t} \right|_{0^-}^{\infty} = \frac{1}{s-s_o}, \ \text{for} \ \text{Re}\{s\} > \text{Re}\{s_o\}.
\]

- Note: \( x(t) \) has no \( \delta(t) \) term, so it doesn’t matter whether we integrate from \( 0^- \) or \( 0 \).

Laplace transforms pairs - \((\mathcal{L}u)(s) = 1/s\) when \( \text{Re}\{s\} > 0 \).

- The Laplace transform of the unit step is just a special case of that of an exponential function.

\[
(\mathcal{L}u)(s) = \frac{1}{s} \ \text{when} \ \text{Re}\{s\} > 0.
\]

- If \( x(t) = 1 \) for \( t \geq 0^- \) then \( x \neq u \) since \( u(0^-) = 0 \) but

\[
(\mathcal{L}x)(s) = \frac{1}{s} \ \text{when} \ \text{Re}\{s\} > 0,
\]

\( \text{i.e.,} \ \mathcal{L}x = \mathcal{L}u. \)
Ambiguity of inverse Laplace transform, $L^{-1}$

- For Dirichlet signals $x$ that do not have an impulse $\delta$ component at the origin $0$,
  \[ X = Lx = Xu \]

- For example, if $x(t) = e^{s_0 t}$ for $t \geq 0^-$, then $(Lx)(s) = (Lxu)(s) = 1/(s - s_0)$ for $\text{Re}(s) > \text{Re}(s_0)$, but note that
  \[ x(0^-) = 1 \quad \text{while} \quad (xu)(0^-) = 0, \]
  i.e., $x$ and $xu$ differ in their initial values.

- So, there is some ambiguity when inverting $L^{-1}X$ to find the corresponding time-domain signal.

- This ambiguity is important as, generally, we want
  - zero initial conditions when inverting $s$-domain terms $X$ of the ZSR (i.e., $L^{-1}X = xu$), and
  - possibly non-zero initial conditions when inverting $s$-domain terms of the ZIR (i.e., $L^{-1}X = x$ without the step $u$ factor so that $(D^k x)(0^-) \neq 0$ is possible).

- As a default, we will use the form $L^{-1}X = xu$ in the following.

---

Laplace transform properties - linearity

- Just as the Fourier transform, the Laplace transform is linear.

- With the Laplace transform, we must be sure to stipulate the RoC for the linear combination as the intersection (smallest open RHP) of those of the component signals.

- For example, if $x(t) = (\alpha_0 e^{s_0 t} + \alpha_1 e^{s_1 t})u(t)$ for $\alpha_0, \alpha_1, s_0, s_1 \in \mathbb{C}$ and $t \geq 0^-$, then $X = Lx$ is
  \[
  X(s) = \alpha_0 \cdot \frac{1}{s - s_0} + \alpha_1 \cdot \frac{1}{s - s_1}, \quad \text{when } \text{Re}(s) > \max\{\text{Re}(s_0), \text{Re}(s_1)\}. 
  \]
• Again, as the Fourier transform, the Laplace transform has a time-shift property.

• If we do not delay the negative-time support of a signal \( x \) into positive time, or there is no negative-time support of \( x \) (\( x \) is causal), then we can simply write

\[
(L \Delta_{\tau} x)(s) = e^{-\tau s} (Lx)(s) \quad \text{if } \tau > 0,
\]

where the RoC is unchanged by time shifting.

• For example, if \( x(t) = e^{s t} u(t) \) for \( t \geq 0 \) and fixed \( \tau > 0 \) then

\[
(L \Delta_{\tau} x)(s) = \int_{0-}^{\infty} x(t-\tau) e^{-st} dt = \int_{0-}^{\infty} e^{s(t-\tau)} u(t-\tau) e^{-st} dt \\
= \int_{-\tau}^{\infty} e^{s t'} u(t') e^{-s(t'+\tau)} dt' \quad (t' = t - \tau) \\
= e^{-\tau s} \int_{0-}^{\infty} e^{-(s-s_0)t'} dt' \quad (u \text{ and lower limit of integration}) \\
= \frac{e^{-\tau s}}{s-s_0}, \quad \text{when } \Re\{s\} > \Re\{s_0\}.
\]

Laplace transform properties - time-shifting (cont)

• However, if we delay the negative-time support of a signal to positive time, then we must subtract it out in order to relate to (unilateral) \( L x = X \).

• That is, if \( \tau > 0 \) then

\[
(L \Delta_{\tau} x)(s) = (L \{x(t-\tau); t \geq 0-\})(s) = \int_{0-}^{\infty} x(t-\tau) e^{-st} dt \\
= e^{-\tau s} \int_{-\tau}^{\infty} x(t') e^{-st'} dt' \quad \text{(no unit-step } u \text{ factor)} \\
= e^{-\tau s} \int_{0-}^{\infty} x(t') e^{-st'} dt' + e^{-\tau s} \int_{-\tau}^{0-} x(t') e^{-st'} dt' \\
= e^{-\tau s} X(s) + e^{-\tau s} \int_{-\tau}^{0-} x(t') e^{-st'} dt'.
\]

• E.g., if \( x(t) = e^{s t} \) (no unit-step \( u \) factor) for \( t \in \mathbb{R} \) and \( \tau > 0 \), then

\[
(L \Delta_{\tau} x)(s) = \frac{e^{-\tau s}}{s-s_0} - \frac{e^{-\tau s} - e^{-\tau s_0}}{s-s_0} = \frac{e^{-\tau s}}{s-s_0}, \quad \text{when } \Re\{s\} > \Re\{s_0\}.
\]

• Similarly, if we advance \( \tau < 0 \) a signal \( x \)'s positive time support into negative time, we need to add it back to relate the result to \( X = L x \).
Laplace transform pairs - sinusoids

- In the special case of an exponential function that is a sinusoid, \( x(t) = e^{itw_o}u(t) \) for \( w_o \in \mathbb{R} \),
  \[
  X(s) = \frac{1}{s - jw_o}, \text{ when } \text{Re}\{s\} > 0.
  \]

- By Euler-De Moivre identity and linearity, we get that
  \[
  \mathcal{L}\{\cos(w_o t)u(t)\}(s) = \frac{s}{s^2 + w_o^2}, \text{ when } \text{Re}\{s\} > 0
  \]
  \[
  \mathcal{L}\{\sin(w_o t)u(t)\}(s) = \frac{w_o}{s^2 + w_o^2}, \text{ when } \text{Re}\{s\} > 0
  \]

Laplace transform properties - frequency shift

- As the for the Fourier transform, multiplication by an exponential function in time results in a frequency shift.
- That is, if \( X = \mathcal{L}x \) and \( s_o \in \mathbb{C} \), then
  \[
  \mathcal{L}\{e^{s_o}x(t)\}(s) = X(s - s_o),
  \]
  for all \( s \in \mathbb{C} \) such that \( s - s_o \) is in the RoC of \( X \).
- For example, for all \( \alpha, w_o \in \mathbb{R} \),
  \[
  \mathcal{L}\{e^{\alpha t}\cos(w_o t)u(t)\}(s) = \frac{s - \alpha}{(s - \alpha)^2 + w_o^2}, \text{ when } \text{Re}\{s\} > \alpha
  \]
  \[
  \mathcal{L}\{e^{\alpha t}\sin(w_o t)u(t)\}(s) = \frac{w_o}{(s - \alpha)^2 + w_o^2}, \text{ when } \text{Re}\{s\} > \alpha
  \]
LT pairs - \( \mathcal{L}\{t^k e^{s_0 t} u(t)\}(s) = k!/(s-s_0)^{k+1} \) when \( \text{Re}\{s\} > \text{Re}\{s_0\} \)

- Again recall that SISO LTI systems have characteristic modes of the form \( t^k e^{s_0 t} u(t) \) where \( k \geq 0 \) is an integer and \( s_0 \in \mathbb{C} \) a characteristic value of the system.

- As for the Fourier transform, to evaluate the Laplace transform of such a characteristic mode, we need to integrate by parts: for \( k > 0 \) and \( \text{Re}\{s\} > \text{Re}\{s_0\} \):

\[
\mathcal{L}\{t^k e^{s_0 t} u(t)\}(s) = \int_{0-}^{\infty} t^k e^{s_0 t} e^{-st} dt = \frac{1}{s_0 - s} \int_{0-}^{\infty} t^k e^{(s_0-s)t} dt
\]

\[
= \frac{1}{s_0 - s} s^k e^{s_0 t} \bigg|_{0-}^{\infty} - \frac{1}{s_0 - s} \int_{0-}^{\infty} e^{(s_0-s)t} dt^k
\]

\[
= \frac{k}{s - s_0} \int_{0-}^{\infty} t^{k-1} e^{(s_0-s)t} dt
\]

\[
= \frac{k}{s - s_0} \mathcal{L}\{t^{k-1} e^{s_0 t} u(t)\}(s)
\]

- So by induction, we can show that for integers \( k > 0 \) and \( \text{Re}\{s\} > \text{Re}\{s_0\} \):

\[
\mathcal{L}\{t^k e^{s_0 t} u(t)\}(s) = \frac{k!}{(s-s_0)^{k+1}}.
\]

---

**Laplace transform properties - differentiation**

- For all differentiable (Dirichlet) signals \( x \), with \( X := \mathcal{L}\{x\} \), the Laplace transform of a time derivative is (recall \( D := \frac{d}{dt} \)),

\[
(\mathcal{L} D x)(s) = sX(s) - x(0-).
\]

- This property is proved by simple integration by parts: \( \forall s \) in the RoC of \( X = \mathcal{L}x \),

\[
(\mathcal{L} D x)(s) = \int_{0-}^{\infty} (Dx)(t)e^{-st} dt
\]

\[
= \int_{0-}^{\infty} e^{-st} dx(t)
\]

\[
= x(t)e^{-st} \bigg|_{0-}^{\infty} - \int_{0-}^{\infty} x(t) e^{-st} dt
\]

\[
= -x(0-) + s \int_{0-}^{\infty} x(t) e^{-st} dt \quad (*)
\]

\[
= -x(0-) + sX(s),
\]

where \( (*) \) because \( \lim_{t \to \infty} x(t)e^{-st} = 0 \) is a necessary condition for the convergence of \( X \) (i.e., for \( s \) to be in the RoC of \( x \)).
Laplace transform properties - differentiation (cont)

• For two derivatives:

\[ (\mathcal{L}D^2 x)(s) = (\mathcal{L}D(Dx))(s) = s(\mathcal{L}Dx)(s) - (Dx)(0-) = s(sX(s) - x(0-)) - (Dx)(0-) = s^2X(s) - sx(0-) - (Dx)(0-). \]

• So, the derivative property can be inductively generalized to: \( \forall s \in \text{RoC of } X = \mathcal{L}x \) and integers \( k \geq 1 \),

\[ (\mathcal{L}D^k x)(s) = s^kX(s) - \sum_{l=0}^{k-1} s^l(D^{k-1-l}x)(0-) \]

Laplace transform properties - Initial Value Theorem

• Suppose \( Y = \mathcal{L}y \) where \( y \) is Dirichlet and continuous at the time 0.

• The initial value theorem states that

\[ y(0) = \lim_{s \to \infty} sY(s). \]

• To see why:

\[ sY(s) = \int_{0-}^{\infty} y(t)se^{-st}dt \quad \text{and} \quad \lim_{s \to \infty} se^{-st} = 0. \]

• So, the IVT follows from the ideal sampling property of the impulse.

• If \( Y = \tilde{P}/Q \) is a rational polynomial with the degree of \( \tilde{P} < -1 + \text{degree of } Q \),

– then \( sY(s) \) is still a strictly proper rational polynomial with degree of \( s\tilde{P}(s) < \text{degree of } Q(s) \),

– so \( y(0) = 0 \) by L’Hôpital’s rule on \( sY(s) \) and the IVT.
Laplace transform properties - Initial Value Theorem - example

- For example, if \( y(t) = 2e^{-2t} - 2e^{-3t} \) then \( y(0) = 0 \) and
  \[
  Y(s) = \frac{2}{s+2} - \frac{2}{s+3} = \frac{2}{(s+2)(s+3)},
  \]
i.e., \( 0 = \) the degree of \( P < -1 + \) degree of \( Q = -1 + 2 \) so that
  \[
  \lim_{s \to \infty} sY(s) = 0 = y(0).
  \]

- For another example, if \( y(t) = (3 + e^{-2t})u(t) \) then \( y(0) = 4 \) and
  \[
  Y(s) = \frac{4s + 6}{s^2 + 2s};
  \]
  thus
  \[
  \lim_{s \to \infty} sY(s) = \lim_{s \to \infty} \frac{4s + 6}{s + 2} = 4 = y(0).
  \]

Laplace transform properties - Final Value Theorem

- The final value theorem states that if \( \lim_{t \to \infty} y(t) \) exists, then
  \[
  \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s).
  \]
- To see why, recall that
  \[
  sY(s) - y(0-) = (\mathcal{L} \cdot y)(s) = \int_0^\infty (D_y(t)e^{-st}dt = \int_0^\infty e^{-st}dy(t)
  \]
  \[
  \Rightarrow \lim_{s \to 0} sY(s) - y(0-) = \int_0^\infty dy(t) = y(\infty) - y(0-)
  \]
- For example, if \( y(t) = (3 + e^{-2t})u(t) \) then \( y(\infty) = 3 \) and
  \[
  \lim_{s \to 0} sY(s) = \lim_{s \to 0} s \left( \frac{3}{s} + \frac{1}{s + 2} \right) = \lim_{s \to 0} 3 + \frac{s}{s + 2} = 3 = y(\infty).
  \]
Laplace transforms applied to SISO systems given by ODE

- We can use the Laplace transform to analyze the total transient response of the SISO, LTIC system with input and output signals, respectively, \( f : [0, \infty) \to \mathbb{R}, y : [0-\infty) \to \mathbb{R} \):
  \[
P(D)f = Q(D)y.
  \]

- We assume Dirichlet input functions of the form \( f = gu \), \( u \) the unit step, with zero initial conditions:
  \[
  (D^k f)(0-) = 0, \text{ for all integers } k \geq 0;
  \]
  so, by the derivative property, \( (\mathcal{L}D^k f)(s) = s^k(\mathcal{L}f)(s) \).

- Recall \( Q(D) = \sum_{k=0}^{n} a_k D^k \) and \( P(D) = \sum_{k=0}^{m} b_m D^m \), where \( \forall k, \ a_k, b_k \in \mathbb{R} \) and \( a_n = 1 \).

- Again, let \( Y = \mathcal{L}y \) and \( F = \mathcal{L}f \).

Laplace transforms applied to SISO systems given by ODE (cont)

- By the derivative property for the Laplace transform of the ODE,
  \[
P(s)F(s) = Q(s)Y(s) - \sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} s^i(D^{k-1-i}y)(0-).
  \]
  for \( s \) in the intersection of the \( RoC \) of \( f \) and the system characteristic modes.

- Thus, the total response,
  \[
  Y(s) = \frac{P(s)}{Q(s)} F(s) + \frac{\sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} s^i(D^{k-1-i}y)(0-)}{Q(s)}
  \]
Thus, the total response

\[ Y(s) = Y_{ZS}(s) + Y_{ZI}(s) \]

where

\[ Y_{ZS}(s) = \frac{P(s)}{Q(s)} F(s) = H(s) F(s), \]

\[ H(s) = \frac{P(s)}{Q(s)} \]

is the transfer function of the system, and

\[ Y_{ZI}(s) = \sum_{k=1}^{n} a_k \sum_{l=0}^{k-1} s^l (D^{k-1-l} y)(0-) \]

\[ \frac{Q(s)}{s^2} \]

Laplace analysis of RL circuit with exponential input

The nodal equation at \( y \) for this circuit is, by KCL,

\[ \frac{f(t) - y(t)}{R} + i(0-) + \int_{0-}^{t} \frac{f(\tau) - y(\tau)}{L} d\tau + \frac{0 - y(t)}{R} = 0, \quad t \geq 0. \]

Differentiating this equation, multiplying by \( R/2 \), and rearranging gives the standard form:

\[ D_y + \frac{R}{2L} y = \frac{1}{2} D f + \frac{R}{2L} f. \]
Laplace analysis of RL circuit with exponential input (cont)

• Taking Laplace transform gives (with \( f(0^-) = 0 \)):

\[
\begin{align*}
\mathcal{L}\{y\} - y(0^-) + \frac{R}{2L} \mathcal{L}\{y\} &= \frac{1}{2} s \mathcal{L}\{F\} + \frac{R}{2L} \mathcal{L}\{F\} \\

\Rightarrow \quad Y(s) &= H(s) F(s) + \frac{y(0^-)}{Q(s)}, \text{ where}
\end{align*}
\]

\[
H(s) = \frac{Y_{ZS}(s)}{F(s)} = \frac{P(s)}{Q(s)} = \frac{s/2 + R/(2L)}{s + R/(2L)} = \frac{1}{2} + \frac{R/(4L)}{s + R/(2L)}.
\]

• Note that by linearity, the impulse response of this system is

\[
h(t) = (\mathcal{L}^{-1} H)(t) = \frac{1}{2} \delta(t) + \frac{R}{4L} e^{-Rt/(2L)} u(t) \; \forall t \geq 0^-.
\]

Laplace analysis of RL circuit with exponential input (cont)

• Suppose \( R = 4\Omega, L = 1H \), the initial condition \( y(0^-) = 2V \), and the input signal \( f(t) = 2e^{-3t} u(t) \) for \( t \geq 0^- \), i.e., \( F(s) = 2/(s + 3) \) for \( \text{Re}\{s\} > -3 \).

• Thus, the total response

\[
Y(s) = H(s) F(s) + \frac{y(0^-)}{Q(s)}
\]

\[
= \frac{s/2 + 2}{s + 3} \cdot \frac{2}{s} + \frac{2}{s + 2} = \frac{s + 4}{(s + 2)(s + 3)} + \frac{2}{s + 2}
\]

\[
= \left(-\frac{1}{s + 3} + \frac{2}{s + 2}\right) + \frac{2}{s + 2}
\]

where we have “expanded the fraction” for the last equality, and the RoC for \( Y \) is \( \text{Re}\{s\} > \max\{-3, -2\} = -2 \).

• Thus, by linearity, the total response

\[
\begin{align*}
y(t) &= (-e^{-3t} + 2e^{-2t}) u(t) + 2e^{-2t} \; \forall t \geq 0^- \\
y_{ZS}(t) &= (-e^{-3t} + 2e^{-2t}) u(t) \text{ and } y_{ZI}(t) = 2e^{-2t} \; \forall t \geq 0^-.
\end{align*}
\]

• Note that \( y_{ZS}(0^-) = 0 \) while \( y_{ZI}(0^-) = 2 = y(0^-) \), and the forced/eigen-response is \( H(-3) f(t) = -e^{-3t} u(t) \).
Laplace analysis of a circuit - approach

• In the following, we will solve transient total response of circuits by
  – first finding a complex-frequency($s$)-domain equivalent circuit that includes its initial conditions,
  – then solving the circuit for the total transient response $Y(s)$,
  – then performing partial fraction expansion separately on the ZIR and ZSR,
  – finally inverting the Laplace transform separately for the ZSR and ZIR using known Laplace-transform pairs.

• Regarding Laplace-transform pairs, we will
  – use the unit-step $u$ factor in the time-domain for ZSR terms (so that the ZSR has zero initial conditions at $0^-$),
  – otherwise not for ZIR terms (so that the ZIR is continuous at time $0^-$ and meets the given initial conditions at $0^-$).

Partial Fraction Expansion (PFE)

• In the previous example we inverted the Laplace transform of a strictly proper rational polynomial of the form $M(s)/N(s)$ by using a PFE.

• A rational polynomial $M(s)/N(s)$ is strictly proper if the degree of $M < \text{ that of } N$.

• Use long division to obtain a strictly proper rational polynomial from one that is not.

• For example,
  \[
  \frac{3s + 2}{s + 7} = 3 - \frac{19}{s + 7}
  \]

  \[
  \Rightarrow \mathcal{L}^{-1} \left( \frac{3s + 2}{s + 7} \right) (t) = 3\delta(t) - 19e^{-7t}u(t), \ t \geq 0.
  \]

• Here, $(3s + 2)/(s + 7)$ is not a strictly proper rational polynomial while $19/(s + 7)$ is.

• If a transfer function $H$ is proper but not strictly so, then: $h = \mathcal{L}^{-1}H$ will have an impulse component $(b_0\delta)$, the ZSR may not be continuous at the origin, and there is “direct coupling” between input and output.
In the following, we will consider the strictly proper rational polynomial \( M(s)/N(s) \) with all coefficients \( \in \mathbb{R} \) and \( K \) poles (roots of the polynomial \( N \)).

That is, \( K := \text{degree of } N > \text{degree of } M \), and so we can write

\[
N(s) = \prod_{k=1}^{K} (s - p_k),
\]

i.e., we can also assume \( N \)'s coefficient of \( s^K \) is one without loss of generality.

We also assume \( M \) and \( N \) have no common roots, i.e., no "pole-zero cancellation" issue to consider.

Note that the RoC for \( M(s)/N(s) \) is the open RHP \( \{ s \in \mathbb{C} \mid \text{Re}\{s\} > \max_k \text{Re}\{p_k\} \} \).

Suppose there are no repeated "poles" for \( M/N \) (roots of \( N \)), i.e.,

\[
\forall k \neq \ell, \quad p_k \neq p_\ell.
\]

In this case, we can write the PFE of \( M/N \) as

\[
\frac{M(s)}{N(s)} = \sum_{\ell=1}^{K} \frac{c_\ell}{s - p_\ell}
\]

where the scalars (Heaviside coefficients) \( c_\ell \in \mathbb{C} \) are

\[
c_\ell = \frac{M(s)}{\prod_{k \neq \ell} (s - p_k)} \bigg|_{s = p_\ell} = \lim_{s \to p_\ell} \frac{M(s)}{N(s)}(s - p_\ell) = \frac{M(s)}{N(s)}(s - p_\ell) \bigg|_{s = p_\ell}.
\]

That is, to find the Heaviside coefficient \( c_\ell \) over the term \( s - p_\ell \) in the PFE, we have removed (covered up) the term \( s - p_\ell \) from the denominator \( N(s) \) and evaluated the remaining rational polynomial at \( s = p_\ell \).

This approach, called the Heaviside cover-up method, works even when \( p \) is \( \mathbb{C} \)-valued.

Given the PFE of \( M/N \), \( (\mathcal{L}^{-1} M/N)(t) = \sum_{\ell=1}^{K} c_\ell e^{p_\ell t} u(t) \).

\[
(\mathcal{L}^{-1} M/N)(t) = \sum_{\ell=1}^{K} c_\ell e^{p_\ell t} u(t).
\]
To prove that the above formula for the Heaviside coefficient $c_t$ is correct, note that the claimed PFE of $M/N$ is
\[
\sum_{t=1}^{K} c_t \frac{c_t \prod_{k \neq t}(s - p_k)}{N(s)}
\]
Thus, the PFE equals $M/N$ if and only if the numerator polynomials are equal, i.e., iff
\[
M(s) = \sum_{t=1}^{K} c_t \prod_{k \neq t}(s - p_k) =: \tilde{M}(s).
\]
Again, two polynomials are equal if their degrees, $L$, are equal and either:
- their coefficients are the same, or
- they agree at $L + 1$ (or more) different points, e.g., two lines ($L = 1$) are equal if they agree at 2 ($= L + 1$) points.

Since $M$ is a polynomial of degree $< K$, it suffices to check that whether $M = \tilde{M}$ for all $s = p_k, k \in \{1, 2, ..., K\}$, i.e., this would create $K$ equations in $< K$ unknowns (the coefficients of $M$).

\[\text{Q.E.D.}\]
PFE - the case of no repeated poles - example

- To find the inverse Laplace transform of a strictly proper rational polynomial \( X = M/N \), first factor its denominator \( N \), e.g.,

\[
X(s) = \frac{s^2 + 5s}{s^3 + 9s^2 + 26s + 24} = \frac{s^2 + 5s}{(s + 4)(s + 3)(s + 2)}, \quad \text{for} \ Re\{s\} > -2.
\]

- So, by PFE

\[
X(s) = \frac{c_4}{s + 4} + \frac{c_3}{s + 3} + \frac{c_2}{s + 2} = \frac{c_4(s + 3)(s + 2) + c_3(s + 4)(s + 2) + c_2(s + 4)(s + 3)}{(s + 4)(s + 3)(s + 2)}
\]

\[
\Rightarrow M(s) = 1s^2 + 5s + 0 = c_4(s + 3)(s + 2) + c_3(s + 4)(s + 2) + c_2(s + 4)(s + 3) = \tilde{M}(s).
\]

- We can solve for the 3 constants \( c_k \) by comparing the 3 coefficients of quadratic \( M \) and \( \tilde{M} \).

- The Heaviside cover-up method suggests we try \( s = -2, -3, -4 \) to solve for \( c_2, c_3, c_4 \):

\[
c_4 = \frac{s^2 + 5s}{(s + 3)(s + 2)} \bigg|_{s = -4} = -2, \quad c_3 = \frac{s^2 + 5s}{(s + 4)(s + 2)} \bigg|_{s = -3} = 6, \quad c_2 = \frac{s^2 + 5s}{(s + 4)(s + 3)} \bigg|_{s = -2} = -3
\]

- Thus, \( x(t) = (L^{-1}X)(t) = (-2e^{-4t} + 6e^{-3t} - 3e^{-2t})u(t) \).

\[
\]

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PFE - the case of a non-repeated, complex-conjugate pair of poles

- Again, recall that for polynomials with all coefficients \( \in \mathbb{R} \), all complex poles will come in complex-conjugate pairs, \( p_1 = \overline{p_2} \).

- The case of non-repeated poles \( p_1, p_2 = \alpha \pm j\beta \) \( (\alpha, \beta \in \mathbb{R}, j := \sqrt{-1}) \) that are complex-conjugate pairs can be handled as above, leading to corresponding complex-conjugate Heaviside coefficients \( c_1, c_2 \), i.e., \( c_1 = \overline{c_2} \).

- In the PFE, we can alternatively combine the terms

\[
\frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} = \frac{r_1s + r_0}{(s - \alpha)^2 + \beta^2}
\]

where by equating the two numerator polynomials’ coefficients,

\[
r_0 = -c_1p_2 - c_2p_1 = -2\Re\{c_1p_2\} \in \mathbb{R} \quad \text{and} \quad r_1 = c_1 + c_2 = 2\Re\{c_1\} \in \mathbb{R}.
\]

- Note that

\[
L^{-1} \left\{ \frac{r_1s + r_0}{(s - \alpha)^2 + \beta^2} \right\}(t) = L^{-1} \left\{ \frac{r_1}{(s - \alpha)^2 + \beta^2} + \frac{r_0 + r_1\alpha}{\beta} \cdot \frac{\beta}{(s - \alpha)^2 + \beta^2} \right\}(t)
\]

\[
= r_1e^{-\alpha t}\cos(\beta t)u(t) + \frac{r_0 + r_1\alpha}{\beta}e^{-\alpha t}\sin(\beta t)u(t), \quad t \geq 0.
\]

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To find the inverse Laplace transform of

\[ X(s) = \frac{3s + 2}{s^3 + 5s^2 + 10s + 12}, \]

first factor the denominator to get

\[ X(s) = \frac{3s + 2}{(s^2 + 2s + 4)(s + 3)}. \]

Note that the poles of \( X \) are \(-3\) and \(-1 \pm j\sqrt{3}\) (so \( X \)'s RoC is \( \text{Re}\{s\} > -1 \)).

So, we can expand \( X \) to

\[ X(s) = \frac{r_1 s + r_0}{s^2 + 2s + 4} + \frac{c_3}{s + 3}, \]

where by the Heaviside cover-up method,

\[ c_3 = \frac{\left. \frac{3s + 2}{s^2 + 2s + 4} \right|_{s = -3}}{s + 3} = -1. \]

To find \( r_1, r_0 \), we will compare coefficients of the numerator polynomials of \( X \) and its PFE, \( i.e., \)

\[ 0s^2 + 3s + 2 = (r_1 s + r_0)(s + 3) + c_3(s^2 + 2s + 4) \quad (*) \]

\[ = (r_1 - 1)s^2 + (3r_1 + r_0 - 2)s + 3r_0 - 4. \]

Thus, by comparing coefficients

\[ 0 = r_1 - 1, \quad 3 = 3r_1 + r_0 - 2, \quad 2 = 3r_0 - 4 \]

we get

\[ r_0 = 2 \quad \text{and} \quad r_1 = 1. \]

Note how \( s = -3 \) in (*) gives \( c_3 = -1 \) as Heaviside cover-up did.

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Thus, by substituting, then completing the square in the denominator and rearranging the numerator,

\[ X(s) = \frac{s + 2}{s^2 + 2s + 4} + \frac{-1}{s + 3} = \frac{s + 1}{(s + 1)^2 + (\sqrt{3})^2} + \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{(s + 1)^2 + (\sqrt{3})^2} - \frac{1}{s + 3}. \]

So,

\[ x(t) = (L^{-1}X)(t) = \left( e^{-t} \cos(\sqrt{3}t) + \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t) - e^{-3t} \right) u(t). \]

PFE - the general case of repeated poles

If a particular pole \( p \) of \( M(s)/N(s) \) is of order \( r \geq 1 \), i.e., \( N(s) \) has a factor \( (s - p)^r \), then the PFE of \( M(s)/N(s) \) has the terms

\[ \frac{c_1}{s - p} + \frac{c_2}{(s - p)^2} + \ldots + \frac{c_r}{(s - p)^r} = \sum_{k=1}^{r} \frac{c_k}{(s - p)^k} = \frac{M(s)}{N(s)} - \Phi(s) \]

with \( c_k \in \mathbb{C} \forall k \in \{1, 2, \ldots, r\} \), where \( \Phi(s) \) represents the other PFE terms of \( M(s)/N(s) \) (i.e., corresponding to poles \( \neq p \)).

Note that equating \( M/N \) to its PFE and multiplying by \( (s - p)^r \) gives

\[ \frac{M(s)}{N(s)}(s - p)^r = c_r + \sum_{k=1}^{r-1} c_k (s - p)^{r-k} + \Phi(s)(s - p)^r \]

\[ \Rightarrow \left. \frac{M(s)}{N(s)}(s - p)^r \right|_{s = p} = c_r, \]

i.e., Heaviside cover-up (of the entire term \( (s - p)^r \)) works for \( c_r \).
• To find \( c_{r-1} \), we differentiate the previous display to get

\[
\frac{d}{ds} M(s)(s-p)^r = \sum_{k=1}^{r-1} c_k (r-k)(s-p)^{r-1-k} + \frac{d}{ds} \Phi(s)(s-p)^r
\]

\[
= c_{r-1} + \sum_{k=1}^{r-2} c_k (r-k)(s-p)^{r-1-k} + \frac{d}{ds} \Phi(s)(s-p)^r
\]

\[
\Rightarrow c_{r-1} = \left( \frac{d}{ds} M(s)(s-p)^r \right) \bigg|_{s=p}
\]

• If we differentiate the original display \( k \in \{0, 1, 2, ..., r-1\} \) times and then substitute \( s = p \), we get (with \( 0! := 1 \))

\[
\left( \frac{d^k}{ds^k} M(s)(s-p)^r \right) \bigg|_{s=p} = k! c_{r-k}
\]

\[
\Rightarrow c_{r-k} = \frac{1}{k!} \left( \frac{d^k}{ds^k} M(s)(s-p)^r \right) \bigg|_{s=p}.
\]

PFE - the general case of repeated poles - example

• To find the inverse Laplace transform of

\[
X(s) = \frac{3s + 2}{(s+1)(s+2)^3},
\]

write the PFE of \( X \) as

\[
X(s) = \frac{c_1}{s+1} + \frac{c_{2,1}}{s+2} + \frac{c_{2,2}}{(s+2)^2} + \frac{c_{2,3}}{(s+2)^3},
\]

so clearly the RoC of \( X \) is \( \text{Re}\{s\} > -1 \).

• By Heaviside cover-up

\[
c_1 = \left. \frac{3s + 2}{(s+2)^3} \right|_{s=-1} = -1 \quad \text{and} \quad c_{2,3} = \left. \frac{3s + 2}{s+1} \right|_{s=-2} = 4.
\]
• Also,
\[
\begin{align*}
\text{c}_{2,2} &= \left. \frac{1}{1!} \left( \frac{d}{ds} \frac{3s + 2}{s + 1} \right) \right|_{s = -2} = \frac{1}{1!(s + 1)^2} \bigg|_{s = -2} = 1 \\
\text{c}_{2,1} &= \left. \frac{1}{2!} \left( \frac{d^2}{ds^2} \frac{3s + 2}{s + 1} \right) \right|_{s = -2} = \frac{1}{2!(s + 1)^3} \bigg|_{s = -2} = 1
\end{align*}
\]

• Thus,
\[
X(s) = \frac{-1}{s + 1} + \frac{1}{s + 2} + \frac{1}{(s + 2)^2} + \frac{4}{(s + 2)^3} \quad \forall \text{Re}\{s\} > -1
\]
\[
\Rightarrow x(t) = (L^{-1}X)(t) = (-e^{-t} + e^{-2t} + te^{-2t} + 2t^2e^{-2t})u(t).
\]

• Exercise: Find the ZSR \( y \) to input \( f(t) = 2e^{j3t}u(t) \) of the marginally stable system \( H(s) = \frac{4}{(s^2 + 9)} \).

PFE and eigenresponse for asymptotically stable systems

• The total response of a SISO LTI system to input \( f \) is of the form
\[
Y(s) = H(s)F(s) + \frac{P_1(s)}{Q(s)} = \frac{P(s)}{Q(s)}F(s) + \frac{P_1(s)}{Q(s)} = Y_{\text{ZS}}(s) + Y_{\text{ZI}}(s).
\]
where \( P_1 \) depends on the initial conditions and the RoC is the intersection of that of input \( F = Lf \) and the system characteristic modes.

• Suppose the system is asymptotically stable (equivalently BIBO stable) and the input is a sinusoid at frequency \( w_0 \), \( f(t) = Ae^{j(w_0 t + \phi)}u(t) \) with \( A > 0 \) \( \Rightarrow F(s) = Ae^{j\phi}/(s - jw_0) \).

• Since \( jw_0 \) cannot be a system pole (owing to asymptotic stability all poles have strictly negative real part), we can use Heaviside cover-up on
\[
Y_{\text{ZS}}(s) = H(s)F(s) = \frac{P(s)}{Q(s)(s - jw_0)}Ae^{j\phi}
\]
to get that the total response is of the form
\[
Y_{\text{ZS}}(s) = \frac{H(jw_0)}{s - jw_0}Ae^{j\phi} + \text{modes} = H(jw_0)F(s) + \text{modes},
\]
where \( Ae^{j\phi} \) is the phasor of \( f \).
• Thus, the total response of an asymptotically stable system to a sinusoidal input $f$ at frequency $w_0 \geq 0$ is

$$y(t) = H(jw_0)f(t) + \text{linear combination of characteristic modes.}$$

• So by asymptotic stability, the steady-state response is the eigenresponse, i.e., as $t \to \infty$,

$$y(t) \sim H(jw_0)f(t) = H(jw_0)Ae^{j(w_0t+\phi)} = |H(jw_0)|Ae^{j(w_0t+\phi+\angle H(jw_0))}.$$
The current-voltage characteristic of an inductor is \( v = L \frac{di}{dt} \), so in the s-domain,
\[
V(s) = sLI(s) - Li(0^-) \quad \text{or} \quad I(s) = \frac{1}{sL} V(s) + \frac{1}{s} i(0^-).
\]

The current-voltage characteristic of a capacitor is \( i = C \frac{dv}{dt} \), so in the s-domain,
\[
I(s) = sCV(s) - Cv(0^-) \quad \text{or} \quad V(s) = \frac{1}{sC} I(s) + \frac{1}{s} v(0^-).
\]

Recall
\[
L^{-1} \left\{ \frac{1}{s} \right\} = u, \quad \text{the unit step},
\]
but when considering initial conditions (ZIR terms), we take
\[
L^{-1} \left\{ \frac{1}{s} \right\} (t) = 1 \ \forall t \geq 0 -.
\]

The Thevinin or Norton equivalent may be more convenient depending on the circuit.
- The $s$-domain equivalent of a first-order op-amp circuit is depicted above for $v_C(0^-) = 2V$ and the Norton equivalent $s$-domain capacitor - so that $-Cv_C(0^-) = -0.5 \cdot 2 = -1$.

- By KCL at the negative terminal of the op-amp and at $v_C$:

$$0 + \frac{Y-F}{1} + \frac{V_c-F}{2} = 0$$

$$\frac{F-V_c}{2} + 0.5s(0-V_c) - (-1) = 0 \text{ (impedance of cap. is } 1/(0.5s))$$

- Eliminating $V_c = (F+2)/(s+1)$ we get

$$Y(s) = \frac{3s+2}{2(s+1)}F(s) - \frac{1}{s+1} = H(s)F(s) + Y_ZI(s).$$

$s$-domain equivalent circuits - example (cont)

- So, if the input $f(t) = 2e^{-3t}u(t)$, $t \geq 0^-$, then $F(s) = 2/(s+3)$ and the total response,

$$Y(s) = \frac{3s+2}{(s+1)(s+3)} - \frac{1}{s+1} = Y_ZS(s) + Y_ZI(s)$$

$$= \frac{c_1}{s+1} + \frac{c_3}{s+3} - \frac{1}{s+1}$$

where by Heaviside cover-up for the PFE,

$$c_1 = [(3s+2)/(s+3)]_{s=-1} = -0.5 \quad \text{and} \quad c_3 = [(3s+2)/(s+1)]_{s=-3} = 3.5.$$  

- Thus,

$$y(t) = (-0.5e^{-t} + 3.5e^{-3t})u(t) - e^{-t}$$

$$= (-0.5e^{-t}u(t) + H(-3)f(t)) - e^{-t}$$

$$= y_ZS(t) + y_ZI(t), \quad t \geq 0^-$$

- Regarding initial conditions, note that

- by KCL at the negative terminal in the time-domain,

$$0 = y(0^-) - f(0^-) + (v_C(0^-) - f(0^-))/2 = y(0^-) + v_C(0^-)/2 = y(0^-) + 1,$$

so that, consistent with the above solution,

$$y(0^-) = 1 = y_ZI(0^-).$$
• The switch is thrown at time $t = 0$ after a long time (i.e., when $t < 0$).

• The input is DC so that at time $0^-$, the circuit (consisting of just the battery, the 2H inductor and 1Ω resistor) is in DC steady-state; check that for $t < 0$, this first-order circuit is asymptotically stable with transfer function $1/(2s + 1)$, i.e., with system pole (characteristic value) $-1/2$.

• An inductor has zero DC impedance, so the output $y(0^-) = 10V/1Ω = 10A$ by Ohm's law.

• For time $t \geq 0^-$, the input $f(t) = 10u(t)$ so that $f(0^-) = 0$; the negative-time evolution of the circuit is accounted for by the initial system states, e.g., $y(0^-) = 10$.

• Because the circuit is closed for time $t < 0$, the initial (time $0^-$) current through the 1H inductor is zero and the initial voltage of the 1F capacity is zero.

Circuit example with switch thrown at time $t = 0$ (cont)

• Writing KVL in the time domain for time $t \geq 0$, we get

$$f = 10u = 2Dy + v_c + D y + y,$$

where $y = D v_c$ and $v_c$ is the capacitor voltage.

• Instead of simply differentiating KVL to eliminate $v_c$ (and using $D f = 10\delta$), we will take Laplace transform of these two equations taking care to including the initial values of the 2H inductor but to use zero initial values for the 1H inductor and 1F capacitor, i.e.,

$$F(s) = \frac{10}{s} = 2(sY(s) - y(0^-)) + V_c(s) + sY(s) + Y(s) \quad \text{and} \quad Y(s) = sV_c(s)$$

$\Rightarrow \quad Y(s) = \frac{10}{3s^2 + s + 1} + \frac{20s}{3s^2 + s + 1} = Y_{2S}(s) + Y_{2I}(s)$ where

$$Y_{2S}(s) = \frac{10}{3s^2 + s + 1} = \frac{s}{3s^2 + s + 1} \cdot \frac{10}{s} = H(s)F(s)$$

and

$$Y_{2I}(s) = \frac{20s}{3s^2 + s + 1} = \frac{2y(0^-)}{3s^2 + s + 1}.$$
Circuit example involving a switch thrown at time $t = 0$ - ZSR

- Note that the poles of $Y_{ZS}$ (roots of $3s^2 + s + 1$) are complex:
  \[ p = \frac{-1 + j\sqrt{11}}{6} \quad \text{and} \quad \bar{p} = \frac{-1 - j\sqrt{11}}{6}. \]

- To obtain the total response in the time domain, we can either complete the square $(3s^2 + s + 1 = 3((s + 1/6)^2 + 35/108))$ and then use the known Laplace transforms of sine and cosine, or just use Heaviside cover-up on the complex-conjugate poles:
  \[ Y_{ZS}(s) = \frac{10}{3s^2 + s + 1} = \frac{10}{(s - p)(s - \bar{p})} = \frac{c}{s - p} + \frac{\bar{c}}{s - \bar{p}}, \]
  where $c = \left. \frac{10/3}{s - \bar{p}} \right|_{s = p} = \frac{10/3}{p - \bar{p}} = -\frac{j10}{3\sqrt{11}}$.

  Thus, $\bar{c} = j\frac{10}{3\sqrt{11}}$ and
  \[ y_{ZS}(t) = ce^{pt}u(t) + \bar{c}e^{pt}u(t), \quad \forall t \geq 0. \]

- Note that there is no forced response (eigenresponse) component $H(0)f(t)$ for the DC input $f$ since the transfer function $H$ has a DC zero, i.e., $H(0) = 0$.

---

Circuit example with switch thrown at time $t = 0$ - ZIR and total resp.

- Similarly, we can perform PFE on $Y_{ZI}$ to get
  \[ Y_{ZI}(s) = \frac{20s}{3s^2 + s + 1} = \frac{(20/3)s}{(s - p)(s - \bar{p})} = \frac{a}{s - p} + \frac{\bar{a}}{s - \bar{p}}, \]
  where $a = \left. \frac{(20/3)s}{s - \bar{p}} \right|_{s = p} = \frac{20p/3}{p - \bar{p}}$.

  So, again noting no step-function $u$ factors,
  \[ y_{ZI}(t) = ae^{pt} + \bar{a}e^{pt}, \quad \forall t \geq 0. \]

- The total response is clearly
  \[ y(t) = y_{ZS}(t) + y_{ZI}(t) = ce^{pt}u(t) + \bar{c}e^{pt}u(t) + ae^{pt} + \bar{a}e^{pt}, \quad \forall t \geq 0. \]

  Exercise: check the computed $y(0^-) = y_{ZI}(0^-) = 2\text{Re}\{a\} = 10$.

  This example is Lathi Problem 6.4-5, p. 465.
Circuit example with switch thrown at time \( t = 0 \) - s-domain equiv.

- The \( s \)-domain equivalent circuit would have input voltage \( F(s) = 10/s \), output current \( Y(s) \), inductive impedances \( 2s \) and \( s \), capacitive impedance \( 1/s \), and resistance 1.
- To preserve the single loop, we use Thevinin equivalent \( s \)-domain (reactive) devices for time \( t \geq 0 \).
- Since the initial conditions are zero for the 1H inductor and 1F capacitor, their \( s \)-domain representations (of \( s \) and \( 1/s \) impedances, respectively) will not have (i.e., have zero) associated independent voltage sources.
- The 2H inductor will have a \(-LY(0-) = -20\) battery referenced positive counterclockwise.
- Thus, by KVL in the \( s \)-domain

\[
F(s) = \frac{10}{s} = 2sY(s) - 20 + s^{-1}Y(s) + sY(s) + Y(s)
\]

\[
\Rightarrow Y(s) = \frac{10}{3s^2 + s + 1} + \frac{20s}{3s^2 + s + 1} = \frac{Y_{ZS}(s) + Y_{ZI}(s)}{Z_{S}}
\]

as computed above by applying Laplace transform to time-domain KVL.

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\textit{s-domain equivalent circuits - RLC circuit example (w/o switch)}

- For the series RLC circuit, we’ve used Thevinin equivalent devices for the \( s \)-domain equivalent circuit so that there remains a single loop.
- By KVL (voltage increases counterclockwise),

\[
0 = \frac{v_c(0-)}{s} + \frac{1}{sC}I(s) - LI(0-) + sLI(s) + Y(s) - F(s)
\]

\[
I(s) = \frac{Y(s)}{R}
\]

\[
\Rightarrow 0 = \frac{v_c(0-)}{s} + \frac{1}{sRC}Y(s) - LI(0-) + \frac{sL}{R}Y(s) + Y(s) - F(s) \quad \text{(eliminating} \ I)\]

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Thus, the total response is

\[ Y(s) = \frac{F(s) + Li(0-) - v_c(0-)s^{-1}}{(sRC)^{-1} + sLR^{-1} + 1} = \frac{sRL^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}} \]

\[ = F(s) + \frac{sRi(0-) - v_c(0-)RL^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}} \]

\[ = Y_{ZS}(s) + Y_{ZI}(s), \text{ where} \]

\[ Y_{ZS}(s) = \frac{sRL^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}} \quad \text{and} \quad \]

\[ Y_{ZI}(s) = \frac{sRi(0-) - v_c(0-)RL^{-1}}{s^2 + sRL^{-1} + (LC)^{-1}} \]

\[ = \frac{P(s)}{Q(s)} F(s) = H(s)F(s), \text{ and} \]

\[ P(s) = \frac{P_1(s)}{Q(s)}. \]

In the following, we will take \( L = 1H, C = 0.25F, v_c(0-) = 2V, i(0-) = 1A. \)

For an overdamped case, take \( R = 5\Omega, \) so that \( Q(s) = s^2 + 5s + 4 = (s + 1)(s + 4). \)

If the input \( f(t) = 3e^{2t}u(t) \Rightarrow F(s) = 3/(s - 2), \) then the total response is

\[ Y(s) = 15s \quad \text{by PFE} \]

\[ = \frac{5s - 10}{(s + 1)(s + 4)(s - 2)} + \frac{5}{s + 1} \]

\[ \Rightarrow y(t) = \left( \frac{5}{3}e^{2t} + \frac{5}{3}e^{-t} - \frac{10}{3}e^{-4t} \right) u(t) - 5e^{-t} + 10e^{-4t} \]

\[ = y_{ZS}(t) + y_{ZI}(t), \forall t \geq 0 - . \]

To check this solution is satisfies the initial conditions:

- By Ohm’s law, \( y(0-) = Ri(0-) = 5 \cdot 1 = 5 = y_{ZI}(0-). \)
- By KVL in the time-domain,

\[ 0 = v_c(0-) + LR^{-1}(Dy)(0-) + y(0-) - f(0-) = 2 + 0.2(Dy)(0-) + 5 - 0 \]

\[ \Rightarrow (Dy)(0-) = -35 = (Dy_{ZI})(0-). \]
Now consider the marginally stable case with $R = 0$ and the output redefined to be $Y := V_c$ so that KVL is

$$0 = Y + sLI - Li(0-) - F$$

where $I = sCY - Cy(0-)$. Then:

$$\Rightarrow Y(s) = \frac{(LC)^{-1}}{s^2 + (LC)^{-1}}F(s) + \frac{sY(0-) + i(0-) / C}{s^2 + (LC)^{-1}}$$

$$= \frac{4}{s^2 + 4} F(s) + \frac{2s + 4}{s^2 + 4}$$

So, the total response to the sinusoidal input $f(t) = 3e^{2jt}u(t)$ is

$$Y(s) = \frac{12}{(s + 2j)(s - 2j)^2} + \frac{2s + 4}{s^2 + 4}$$

$$= -\frac{3}{4s + 2j} + \frac{3}{s - 2j} + \frac{-3j}{(s - 2j)^2} + \left(2 \frac{s}{s^2 + 4} + 2 \frac{2}{s^2 + 4}\right)$$

$$\Rightarrow y(t) = \left(-\frac{3}{4}e^{-2jt} + \frac{3}{4}e^{2jt} - 3jte^{2jt}\right) u(t) + 2 \cos(2t) + 2 \sin(2t)$$

$$y(t) = y_{ZS}(t) + y_{ZI}(t), \quad t \geq 0 - .$$

Note that resonance phenomenon for the input $f$ is a sinusoid at the system resonant frequency of $2 = (LC)^{-0.5}$ radians/s, leading to an unbounded response ($te^{2jt}$) to a bounded input $f$.

Also note that $y_{ZI}(0-) = 2 = y(0-) = 1 = C(Dy)(0-) = 0.25(Dy)(0-) \Rightarrow (Dy)(0-) = 4 = y_{ZI}(0-)$. 

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System composition - for the ZSR of a LTIC system

- SISO system with input $f$ and ZSR $y$, where $F = \mathcal{L}f$, $Y = \mathcal{L}y$:

$$f \rightarrow h \rightarrow y = h^*f \rightarrow F \rightarrow H \rightarrow Y = HF$$

- An ideal amplifier for scalar $a$ (recall inverting and noninverting op-amp circuit configurations):

$$f \rightarrow a \rightarrow y = af \rightarrow F \rightarrow a \rightarrow Y = af^*F$$

- An integrator system - recall $(\mathcal{L}u)(s) = s^{-1}$ and $(f * u)(t) = \int_0^t f(\tau) d\tau, \ t \geq 0$:

$$f \rightarrow \int \rightarrow y = f^*u \rightarrow F \rightarrow \frac{1}{s} \rightarrow Y = F/s$$

- Exercise: show how an integrator can be implemented using a simple op-amp circuit with an output-feedback capacitor and an input-side resistor.

System composition - summer

- One can realize a summer with inverting and/or non-inverting op-amp circuit configurations.

$$\begin{align*}
F & \rightarrow H_1 \rightarrow Y_1 = H_1F \\
& \rightarrow H_2 \rightarrow Y_2 = H_2F \\
\rightarrow + \rightarrow Y_1 + Y_2 = (H_1 + H_2)F \\
F & \rightarrow H_1 + H_2 \rightarrow (H_1 + H_2)F
\end{align*}$$
System composition - non-inverting summer implementation

- The voltage at the positive input terminal is \( v_+ = v_- = \frac{R_b}{R_b + R_f} y \).

- With system inputs \( v_1, v_2 \), KCL at the positive input terminal is

\[
0 = \frac{0 - v_+}{R_a} + \sum_{k=1}^{2} \frac{v_k - v_+}{R_k}
\]

\[
\Rightarrow y = \sum_{k=1}^{2} \frac{\tilde{R}}{R_k} v_k,
\text{ where } \tilde{R} = \left( \frac{R_b}{R_b + R_f} \left( \frac{1}{R_a} + \sum_{k=1}^{2} \frac{1}{R_k} \right) \right)^{-1}.
\]

Canonical system realizations - single pole system

- Note that for the system with transfer function

\[
H(s) = \frac{a}{s + b} = \frac{Y(s)}{F(s)},
\]

we can write

\[
sY(s) = aF(s) - bY(s).
\]
Canonical system realizations - by PFE using single-pole systems

- Given \( n \) distinct real poles, we have by PFE of strictly proper \( H = P/Q \),

\[
H(s) = \sum_{k=1}^{n} \frac{c_k}{s - p_k}.
\]

- So, one can realize this system by summing the outputs of \( n \) single-pole systems.

System composition - tandem systems

- Recall how an op-amp has a separate power supply, decoupling input from output so that the transfer function \( H_1 \) does not depend on its load (system \( H_2 \)).
So, one can realize part of a transfer function $H$ with, say, twice-repeated (triple) pole $p$ leading to PFE terms

$$\frac{c_1}{s-p} + \frac{c_2}{(s-p)^2} + \frac{c_3}{(s-p)^3},$$

by using three tandem systems:

Letting the input of this subsystem be $F$ and the output be $Y$, note that

$$Y(s) = c_1 \frac{F(s)}{s-p} + c_2 \frac{F(s)}{(s-p)^2} + c_3 \frac{F(s)}{(s-p)^3}.$$ 

Canonical system-realizations - direct form

Consider the proper transfer function $H = P/Q$, i.e., $m \leq n$.

The direct-form realization employs the interior system state $X := F/Q$, i.e., $F = QX$ and $Y = PX$ where the former implies

$$F(s) = \sum_{k=0}^{n} a_k s^k X(s) \quad \text{with} \quad a_n = 1$$

$$\Rightarrow s^n X(s) = F(s) - \sum_{k=0}^{n-1} a_k s^k X(s).$$

For $n = 2$, there are two “system states” (outputs of integrators), $X$ and $sX$ ($x$ and $Dx$):
Now adding $Y = PX$, we finally get the direct-form canonical system-realization of $H$:

If $b_n = b_2 \neq 0$, there is direct coupling of input and output. $H$ is proper but not strictly so, $h = \mathcal{L}^{-1}H$ has an impulse component $b_2 \delta$, and the ZSR may be discontinuous at time 0.

Note that this $n = 2$ example above can be used to implement a pair of complex-conjugate poles as part of a larger PFE-based implementation (with otherwise different states); e.g., for $n = 2$, $H(s) = P(s)/Q(s)$ where

$$Q(s) = s^2 + a_1 s + a_0 = (s - \alpha)^2 + \beta^2$$

for $\alpha, \beta \in \mathbb{R}$, so the poles are $\alpha \pm j\beta$. 

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Canonical system realizations - by PFE

- In the general case of a proper transfer function, we can use partial-fraction expansion (after long division to get a strictly proper rational polynomial)
  - grouping the terms corresponding to a complex-conjugate pair of poles, i.e., a second-order denominator, and
  - using a direct-form realization for these terms.

- Besides the PFE-based and direct-form realizations, there are other (zero-state) system realizations, e.g., "observer" canonical.

- For proper rational-polynomial transfer functions $H = P/Q$, all of these realizations involve $n$ (degree of $Q$) integrators, the output of each being a different interior state variable of the system.

---

Signal tracking by a Proportional-Integral (PI) system

- "Open-loop" system is PI: $G(s) = K_p + K_i/s$.

- The goal is that the error signal $e = f - y \to 0$, i.e., the "closed-loop" output $y$ tracks the input $f$.

- $Y = GE = G(Y - F) \Rightarrow Y = FG/(1 + G)$.

- Thus, the closed-loop transfer function ($F$ to $Y$, with negative feedback) is

$$H(s) = \frac{G(s)}{1 + G(s)} = \frac{1}{1 + K_p} \left( K_p + \frac{\alpha}{s + \alpha} \right), \text{ where } \alpha := \frac{K_i}{K_i + K_p}.$$
Signal tracking by a PI system (cont)

- Now assume that $\alpha > 0$ ($H$ is stable).

- Suppose we want to see how the PI parameters $K_p, K_i$ affect tracking of a step $f = Au$, i.e., we want the (ZS) step response:

$$Y(s) = H(s)F(s) = H(s)\frac{A}{s} = \frac{A}{1 + K_p} \left( \frac{K_p}{s} + \frac{\alpha}{s(s + \alpha)} \right)$$

$$= \frac{A}{1 + K_p} \left( \frac{K_p + 1}{s} - \frac{1}{s + \alpha} \right) \text{ by PFE}$$

$$\Rightarrow y(t) = Au(t) - \frac{A}{1 + K_p} e^{-\alpha t} u(t)$$

**Exercise:** Find an expression for the time $t > 0$ at which $y(t) = 0.9A$.

**Exercise:** Derive how this closed-loop PI system would track a ramp, $f(t) = Atu(t)$.

Feedback control for stabilization by pole placement

- Consider this simple feedback control system where the fixed open-loop system (plant) $G = P/Q_o$ and constant-gain (proportional) $K$ output-feedback combine to give the closed-loop system $H$:

$$X = F - KY \quad \text{and} \quad Y = GX \quad \Rightarrow \quad Y = GF - KGY = \frac{G}{1 + KG} F = HF.$$ 

- Thus, the closed-loop characteristic polynomial $Q = Q_o + KP$ because

$$H = \frac{G}{1 + KG} = \frac{P}{Q_o + KP} = \frac{P}{Q}.$$ 

- E.g., if $P(s) = 1$, $Q_o(s) = s - 5$ and $K > 5$, then $G$ is unstable, but $Q(s) = s - 5 + K$ has a root at $5 - K < 0$ and so the closed-loop system is stable.

- So, proportional feedback control may be used to stabilize a system by pole placement.
Minimum-phase systems

- A LTI system $H, h$ is said to be minimum phase if both it and its inverse $1/H, L^{-1}\{1/H\}$ are stable and causal.
- So, a minimum-phase $H$ has all poles and zeros with negative real part.
- E.g., for real $p, z > 0$, consider the following two transfer functions
  \[ H_{nmp}(s) = \frac{1 + s/(-z)}{1 + s/p} \quad \text{and} \quad H_{mp}(s) = \frac{1 + s/z}{1 + s/p} \]
- Note that the magnitude responses for these two transfer functions is the same,
  \[ |H_{nmp}(jw)| = |H_{mp}(jw)| = \sqrt{\frac{1 + (w/z)^2}{1 + (w/p)^2}}. \]
- But for all (real) frequencies $w \geq 0$, the phase response of $H_{mp}$ is smaller:
  \[ \angle H_{mp}(jw) = \tan^{-1} \frac{w}{z} - \tan^{-1} \frac{w}{p} \leq \tan^{-1} \frac{w}{-z} - \tan^{-1} \frac{w}{p} = \pi - \tan^{-1} \frac{w}{z} - \tan^{-1} \frac{w}{p} = \angle H_{nmp}(jw) \]

Undershoot of non-minimum-phase systems

- Consider the unit-step response of the non-minimum-phase system,
  \[ Y(s) := L\{h_{nmp} * u\}(s) = H_{nmp}(s) = \frac{1/p - 1/z + 1/s}{1 + s/p} \]
  \[ \Rightarrow y(t) = (h_{nmp} * u)(t) = -(1 + p/z)e^{-tp}u(t) + u(t) \]
- Note that if $r > p$ then $y(0+) < 0$, though $\lim_{t \to \infty} y(t) = 1$;
- i.e., the non-minimum-phase system always first moves away from 1 = $H(0)$ (here becomes negative) before it converges to 1 = $H(0)$ (follows the steady-state unit-step response), i.e., it always undershoots.
- Exercise: Show that the minimum-phase system $H_{mp}$ never first undershoots.