Towards an Axiomatization of Privacy and Utility

Daniel Kifer
Bing-Rong Lin

Department of Computer Science & Engineering
Penn State University
Motivation

- Database
- Census
- Search logs
- Network Traces

Preserves Privacy?

Published Data

Is it useful?

Researchers: External and Internal
<table>
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<tr>
<th>SSN</th>
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<th>Age</th>
<th>Zip Code</th>
<th>Disease</th>
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Guiding Principles?

- We know this is not enough

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Aug 6, 2006 - AOL releases data
- 20 Million Search Queries from 3 months
- 650,000 users

How is data protected: Change AOL id to a number.

What happened?
- NYT identified user # 4417749
  - People search for names of friends/relatives/self
  - People search for locations “What to do in State College”
  - Age-related searches
- Many people got fired.
Introduction

2 Axiomatizing Privacy
   - A framework
   - Privacy Axioms
   - Application to Differential Privacy

3 Axiomatizing Utility
   - Counterexample
   - Axioms and Examples
   - Insights
Art of turning sensitive data into nonsensitive data suitable for public release.

Sensitive data:
- Cannot release sensitive data directly.
- Detailed information about individuals (search logs, health records, census/tax data, etc.)
- Proprietary secrets (search logs, network traces, machine debug info)

Want to release useful but non-private information from this data.
- Typical user web search behavior
- Demographics
- Information that can be used to build models
- Information that can be used to design & evaluate algorithms

Mechanism: a (randomized) algorithm that converts sensitive into nonsensitive data.

Goal: Design a mechanism that protects privacy and provides utility.
What does privacy mean?
- Many, many privacy definitions in the literature.
- How do I compare them?
- How do I identify strengths and weaknesses?
- How do I customize them (for an application)?
- How do I design one?
- Does it really do what I want it to do?
- What statements are/aren’t privacy definitions?

What does utility mean?
- Many, many measures of utility in the literature:
  - KL-divergence.
  - Expected (Bayesian) utility.
  - Minimax estimation error.
  - Task-specific measures.
- Which one should I choose?
- Does it do what I want it to do?
- How do I design one?
- Does it make sense in statistical privacy?
A Common Approach

1. Start with a privacy mechanism.
   - Generalization (e.g. coarsen “state college” → “Pennsylvania”)
   - Suppression (remove parts of data items)
   - Add random noise

2. Create privacy definition that feels most natural with this privacy mechanism.

3. Create utility measure that feels most natural for this mechanism.
   - # of generalizations
   - # of suppressions
   - variance of noise
   - anything we can borrow from statistics
   - often can’t compare utility across mechanisms

4. (Usually) Find flaws, revise steps 2 and 3.
What if we did this in reverse? For a given application:

1. Identify properties we think a privacy definition should satisfy.
2. Identify properties we think a utility metric should satisfy.
3. Find a privacy mechanism that satisfies those properties.

Benefits of axiomatization:

- Apples to apples comparison of properties of privacy definitions.
- Small set of axioms easier to study than large set of privacy definitions.
- Abstract approaches yield general results and insights (e.g. group theory, vector spaces, etc.)
- Can study relationships between axioms.
- Easier to identify weaknesses.
- Design mechanisms by picking axioms depending on application.
- Can study consequences of omitting axioms.

Is it really necessary for privacy and utility?

- Let’s look at some illustrative results.
Outline

1. Introduction

2. Axiomatizing Privacy
   - A framework
   - Privacy Axioms
   - Application to Differential Privacy

3. Axiomatizing Utility
   - Counterexample
   - Axioms and Examples
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Hard to create a good privacy definition.

Simple things usually don’t work.

Different applications have different privacy requirements.

Instead of starting from a privacy definition:
- Identify axioms you want it to support.
- Determine the privacy definition implied by axioms.
- Let axioms be the building blocks.

It is easier to reason about axioms that about entire privacy definitions.

Efficiency: insights into 1 axiom lead to insights into many privacy definitions.

Example: how to relax differential privacy.
Some definitions

- Abstract input space $\mathcal{I}$ (all possible data).
  - Semantics (e.g. neighboring databases in differential privacy) should be given by axioms.
- Abstract output space $\mathcal{O}$.
  - Semantics (e.g. query answers, synthetic data, utility) should be given by axioms.

**Definition (Randomized Algorithm)**

A randomized algorithm $\mathcal{A}$ is a regular conditional probability distribution $P(O \mid I)$ with $O \subset \mathcal{O}$ and $I \subset \mathcal{I}$.

- Privacy definition: intentionally undefined (all parameters must be instantiated).

**Definition (Privacy Mechanism for $D$)**

A privacy mechanism $\mathcal{M}$ is a randomized algorithm that satisfies privacy definition $D$. 

Two Simple Privacy Axioms

- Intuition: postprocessing the output of a privacy mechanism should still maintain privacy.

**Axiom (Transformation Invariance)**

Given a privacy mechanism $M$ and a randomized algorithm $A$ (independent of the data and $M$), the composition $A \circ M$ is a privacy mechanism.

- Intuition: it does not matter which privacy mechanism I choose.

**Axiom (choice)**

If $M_1$ and $M_2$ are privacy mechanisms for $D$, then the process of choosing $M_1$ with probability $c$ and $M_2$ with probability $1 - c$ (with randomness independent of the data, $M_1$, and $M_2$) results in a privacy mechanism for $D$. 
Two Simple Privacy Axioms

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Given a privacy mechanism $M$ and a randomized algorithm $A$ (independent of the data and $M$), the composition $A \circ M$ is a privacy mechanism.

**Axiom (choice)**

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- Consistency conditions for privacy definitions
- Thus privacy definitions should discuss how they are affected by postprocessing.
- Privacy definitions cannot focus only on deterministic mechanisms.
- Many privacy definitions do not satisfy these axioms!
Definition (Differential Privacy [Dwo06, DMNS06])

\( M \) satisfies \( \epsilon \)-differential privacy if 
\[ P(M(i_1) \in S) \leq e^\epsilon P(M(i_2) \in S) \]
for all measurable \( S \subset \mathcal{O} \) and all neighboring input databases \( i_1, i_2 \in \mathcal{I} \).

There has been interest in relaxing differential privacy. For example:

For example:

\[ P(M(i_1) \in S) \leq e^\epsilon P(M(i_2) \in S) + \delta \]
Example

\[ a = P(M(i_1) \in S) \quad b = P(M(i_2) \in S) \quad a \leq 2b \]
Example

\[ a = P(M(i_1) \in S) \quad b = P(M(i_2) \in S) \quad a \leq 2b + .1 \]
Definition (Differential Privacy [Dwo06, DMNS06])

\( \mathcal{M} \) satisfies \( \epsilon \)-differential privacy if
\[
P(\mathcal{M}(i_1) \in S) \leq e^\epsilon P(\mathcal{M}(i_2) \in S)
\]
for all measurable \( S \subset \mathcal{O} \) and all neighboring input databases \( i_1, i_2 \in \mathcal{I} \).

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For example:

\[
P(\mathcal{M}(i_1) \in S) \leq e^\epsilon P(\mathcal{M}(i_2) \in S) + \delta
\]

Definition (A Generic Version)

\( \mathcal{M} \) is a privacy mechanism if
\[
G [P(\mathcal{M}(i_1) \in S), P(\mathcal{M}(i_2) \in S)] = T
\]
for all measurable \( S \subset \mathcal{O} \) and all neighboring input databases \( i_1, i_2 \in \mathcal{I} \).

What other predicates can be used?
Definition (A Generic Version)

\( \mathcal{M} \) is a privacy mechanism if
\[
G \left[ P(\mathcal{M}(i_1) \in S), P(\mathcal{M}(i_2) \in S) \right] = T
\]
for all measurable \( S \subset \mathcal{O} \) and all neighboring input databases \( i_1, i_2 \in \mathcal{I} \).

- In principle, \( G \) could be any predicate:
  - \( G(a, b) = T \) if \( a - b \) is rational.
  - \( G(a, b) = T \) if \( a < b^2 \).
  - \( G(a, b) = T \) if \( b = \frac{(1 + \cos(2\pi a))}{2} \)

- Choice and Transformation Invariance Axioms limit the possibilities.
Example

\[ a = P(M(i_1) \in S) \quad b = P(M(i_2) \in S) \quad b = \frac{1 + \cos(2\pi a)}{2} \]
Definition (A Generic Version)

\( M \) is a privacy mechanism if \( G[P(M(i_1) \in S), P(M(i_2) \in S)] = T \) for all measurable \( S \subset \mathcal{D} \) and all neighboring input databases \( i_1, i_2 \in \mathcal{I} \).

- Replacing \( G[a, b] \) with \( G^*[a, b] \equiv G[a, b] \land G[1 - a, 1 - b] \) does not change privacy definition.

Theorem

Axioms of Transformation Invariance and Choice provide necessary and sufficient conditions on \( G^*[a, b] \). There exists a well-behaved upper envelope \( M(a) \) and lower envelope \( m(a) \) that determine \( G^* \).
\[ a = P(\mathcal{M}(i_1) \in S) \quad b = P(\mathcal{M}(i_2) \in S) \]

\[ M(a) \text{ is continuous* concave strictly increasing*} \]

\[ m(a) \text{ is determined by } M(a) \]
Definition (A Generic Version)

\( \mathcal{M} \) is a privacy mechanism if

\[ G \left[ P(\mathcal{M}(i_1) \in S), P(\mathcal{M}(i_2) \in S) \right] = T \]

for all measurable \( S \subset \mathcal{O} \) and all neighboring input databases \( i_1, i_2 \in \mathcal{I} \).

- Axioms imply a nice intuitive form for predicate \( G \).
- For every \( a \), there is interval of allowable \( b \) values
- Interval endpoints vary nicely with \( a \).
- Makes sense intuitively
  - But no need for intuition after axioms are selected
  - Avoids faulty/incomplete intuition
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Privacy axioms limit the privacy mechanisms we can consider.

How to choose among allowable mechanisms?
- $M$ as a column stochastic matrix:
  - Column $i$ of $M$ is $P_M(\cdot | i)$.
- $\mu(M)$ — how good is a privacy mechanism $M$?
  - How much information does it contain?
  - How useful are the outputs?

Do we understand utility well enough?
Example: Expected Utility

- Conducting a survey: Is this your favorite conference venue?
- Sensitive question, people may not respond truthfully.
- Idea: allow respondent to lie with certain probability (randomized response [War65]).
- Utility: expected loss (?)
  - I get a loss of 1 every time they lie (0 loss for truth)
  - I believe 75% of population could not imagine a better conference venue
  - Expected loss what do I believe my average (expected) loss is?
Example: Expected Utility

- Is this your favorite conference venue?
- Subjective prior belief: 75% yes

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<th>True Answer</th>
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<td>1/3</td>
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$$E[\text{Loss}] = 1 \times 1/4$$

$$= 1/4$$

$$E[\text{Loss}] = 1 \times 3/4 \times 1/3$$

$$+ 1 \times 1/4 \times 1/3$$

$$= 1/3$$

- Mechanism $\mathcal{M}_2$ has lower expected loss
- Yet contains no information
- $\mathcal{M}_2(\text{true answer}) = \mathcal{A}(\mathcal{M}_1(\text{true answer}))$
Example: Expected Utility

- User has a prior distribution over the input space $\mathcal{I}$.
- Output space $\mathcal{O} = \mathcal{I}$.
- User has a loss function $L(i,j)$.
- Create mechanism with smallest expected loss.

Theorem ([GRS09])

*Under suitable conditions on $\mathcal{I}$ and $L$, the geometric mechanism is universal – for any prior, the optimal mechanism is achieved by applying a many-to-one deterministic function to the output of geometric mechanism.*

- In general, cannot recover geometric mechanism from “optimal” mechanism.
- $\therefore$ “Optimal” mechanism contains less information than geometric mechanism.
  - “Optimal” mechanism should not be considered optimal.
  - Expected utility may not be an appropriate measure of utility.
How to measure utility

- We should take a step back and think about what properties our utility measures should have.

**Definition (Sufficiency partial order)**
Privacy mechanism $M_2$ is sufficient for $M_1$ ($M_2 \prec M_1$) if there exists a randomized algorithm $A$ such that $M_2 = A \circ M_1$.

**Axiom (Sufficiency)**

If $M_2 \prec M_1$ then $\mu(M_2) \leq \mu(M_1)$

**Definition (Sufficient Covering Set)**

A set $S$ of privacy mechanisms is a covering set if every mechanism in $S$ is maximally sufficient and: $\forall M, \exists M^* \in S$ such that $M \prec M^*$

- Utility metric $\mu$ should choose some $M^* \in S$. 
Examles - finite input/output spaces

\[
\mathcal{M} = \begin{pmatrix}
P(O_1 | *) \\
P(O_2 | *) \\
P(O_3 | *) \\
P(O_4 | *)
\end{pmatrix} = \begin{pmatrix}
P(O_1 | i_1) & P(O_1 | i_2) & P(O_1 | i_3) \\
P(O_2 | i_1) & P(O_2 | i_2) & P(O_2 | i_3) \\
P(O_3 | i_1) & P(O_3 | i_2) & P(O_3 | i_3) \\
P(O_4 | i_1) & P(O_4 | i_2) & P(O_4 | i_3)
\end{pmatrix}
\]
Examples

- $|\text{det } \mathcal{M}|$
  - For finite input space and output space of the same size.
  - Measures how much $\mathcal{M}$ shrinks the unit hypercube (identity matrix).
  - Piecewise multilinear.

- Negative Dobrushin’s coefficient of ergodicity.
  - $-\min_{j,k} \sum \min(m_{i,j}, m_{i,k})$
  - Finds the two columns that are hardest to distinguish.
  - Finds the two inputs hardest to distinguish.
  - Another measure of how the matrix contracts the input space [CDZ93].

- Branching Measures.
  - $\sum_i F(r_i)$
  - $r_i$ are the rows
  - $F$ is convex and $F(cx) = cF(x)$.
  - Example:

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \log \frac{x_i}{x_1 + \cdots + x_n}$$
Maximally Sufficient Mechanisms

Definition (Sufficient Covering Set)

A set $S$ of privacy mechanisms is a covering set if every mechanism in $S$ is maximally sufficient and:

$$\forall M, \exists M^* \in S \text{ such that } M \prec M^*$$

- What do they look like?
- For finite input spaces, output space is finite but larger.
- Neighboring databases form a connected graph of input space.
- For each output $o_1$, its row subgraph must be a spanning tree*.
- Output space can be identified with a set of graphs.
  - Output space is a set of spanning trees* of input space.
  - Edges correspond to equality constraints in differential privacy.
  - Can also be interpreted as a restricted set of likelihood functions.
Output Space

\[ P(O_1 \mid \ast) \quad P(O_2 \mid \ast) \quad P(O_3 \mid \ast) \]
Output of a privacy mechanism may not correspond to a query answer.

- Input: heads or tails
- Output: red or blue or green

Output of a privacy mechanism may not correspond to synthetic data.

- May not have “attributes”
- May not have “rows”

You will need to postprocess the output for what you want to do.

Use the likelihood principle.

Goal: find a mechanism that allows greatest flexibility for postprocessing.
Take home message

- Axioms are our building blocks.
  - Easier to understand and argue about than privacy definitions and utility measures.
  - Abstraction allows for generality.
  - Allows for comparison of privacy definitions.

- Shouldn’t specify privacy definition directly, let axioms disqualify sets of randomized algorithms.

- Use axioms to choose the best mechanisms via utility.

- Output space may not correspond to query answers or synthetic data.
  - Because of potentially many different uses for the data.

- Need statistical postprocessing tools to work with resulting data.
Joel E. Cohen, Yves Derriennic, and Gh. Zbaganu.  
Majorization, monotonicity of relative entropy and stochastic matrices.  

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith.  
Calibrating noise to sensitivity in private data analysis.  

Cynthia Dwork.  
Differential privacy.  
*In ICALP*, 2006.

Arpita Ghosh, Tim Roughgarden, and Mukund Sundararajan.  
Universally utility-maximizing privacy mechanisms.  
*In STOC*, 2009.

S. L. Warner.  
Randomized response: A survey technique for eliminating evasive answer bias.