



# Accurate Eigenvalues of Sign Regular Matrices

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# Recall: **Totally Positive** Means All Minors $> 0$

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- **Examples:**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \end{bmatrix}$$

**Vandermonde**

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

**Hilbert**

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

**Pascal**

- **Eigenvalues**

71.5987

3.6199

0.7168

0.0646

1.5002

0.1691

0.0067

0.0001

26.3047

2.2034

0.4538

0.0380

- **Note: Despite possible nonsymmetry:  $\lambda_i > 0$ , real!**
- **Virtually all linear algebra possible accurately (2 papers in SIMAX)**

# This Talk: Sign Regular Matrices

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- Consider

$$\text{(Totally positive)} \times \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

—called **Sign Regular**, meaning

**All minors of order  $k$  have the same sign  $\rho_k \in \{-1, +1\}$**

- In our case  $(\rho_1, \rho_2, \dots) = (+1, -1, -1, +1, \dots)$
- Generally unsymmetric; **NOT** similar to TP

# Eigenproblem for Sign Regular (SR) Matrices

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- Examples of SR matrices

$$\begin{bmatrix} 1 & 4 & 4^2 & 4^3 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1/4 & 1/5 & 1/6 & 1/7 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1 & 1/2 & 1/3 & 1/4 \end{bmatrix}$$

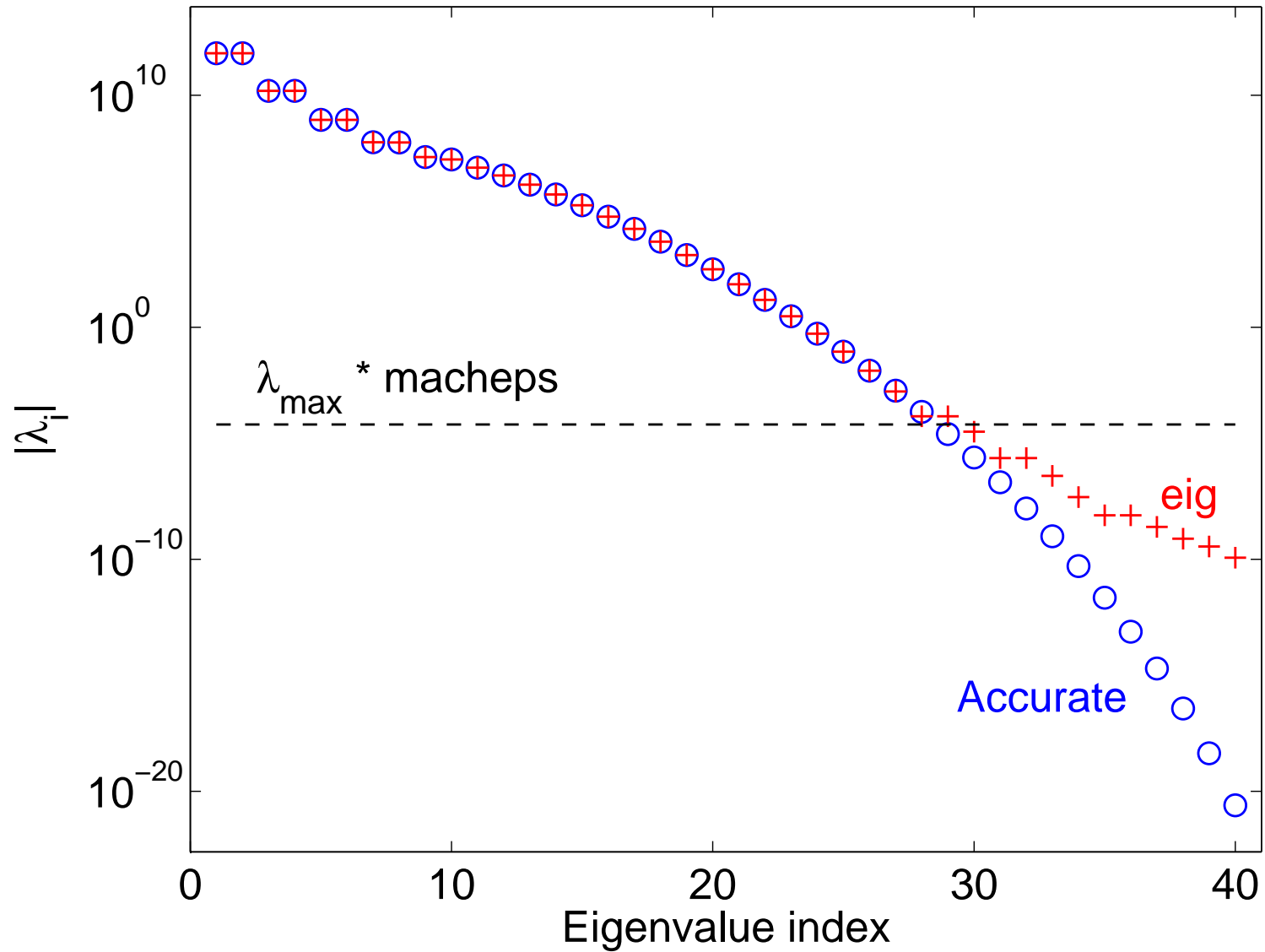
$$\begin{bmatrix} 1 & 4 & 10 & 20 \\ 1 & 3 & 6 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- Eigenvalues:

15.4897	1.1194	11.2006
-7.7063	-0.1261	-3.7655
1.2942	0.0068	0.6041
-0.0777	-0.0002	-0.0392

- Note: Despite nonsymmetry,  $\lambda_k$  are real;  $\text{sign}(\lambda_k) = (-1)^{k-1}$
- Notoriously ill conditioned, just like TP

# Eigenvalues of a $40 \times 40$ Sign Regular Vandermonde



$$V = \left[ x_i^{j-1} \right]_{i,j=1}^{40}, \quad x = (4.0, 3.9, 3.8, \dots, 0.1)$$

# New Result: Accurate Eigenvalue Algorithm for SR Matrices

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- Takes  $O(n^3)$  time
- Uses only working precision
- Accuracy unaffected by angle between left and right eigenvectors
- All eigenvalues computed to high relative accuracy:

$$|\lambda_i - \hat{\lambda}_i| = \mathcal{O}(\varepsilon) |\lambda_i|$$

as opposed to

$$|\lambda_i - \hat{\lambda}_i| = \mathcal{O}(\varepsilon) |\lambda_{\max}| \frac{1}{y_i^T x_i}$$

- First example of negative eigenvalues of a nonsymmetric matrix computed accurately

# Outline

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- (Almost) any matrix is similar to a **symmetric anti-bidiagonal** (eig of latter then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- Trick: no subtractions in above reduction
- We need a structure revealing representation—product of bidiagonals

# Outline

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- **(Almost) any matrix is similar to a symmetric anti-bidiagonal**  
(eig of latter then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- **Trick: no subtractions in above reduction**
- **We need a structure revealing representation—product of bidiagonals**

# Similarity Reduction to an Anti-Bidiagonal

- Goal:

$$\begin{array}{ccc}
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} & \longrightarrow & \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix} \\
 \text{Sign regular} & \longrightarrow & \text{anti-bidiagonal}
 \end{array}$$

- Equivalently:

$$\begin{array}{ccc}
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} & \longrightarrow & \begin{bmatrix} + & + & & \\ & + & + & \\ & & + & + \\ & & & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \text{Totally Positive} \cdot J & \longrightarrow & \text{Bidiagonal} \cdot J
 \end{array}$$

(Prefer TP  $\cdot J$  since we know how to do TP linear algebra accurately)

# Similarity Reduction to an Anti-Bidiagonal

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$$\begin{array}{ccc}
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} & \longrightarrow & \begin{bmatrix} + & + & & \\ & + & + & \\ & & + & + \\ & & & + \end{bmatrix} \cdot \begin{bmatrix} & & & 1 \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \\
 \text{Totally Positive} \cdot J & \longrightarrow & \text{Bidiagonal} \cdot J
 \end{array}$$

- Nevermind the possible instability; it won't be an issue
- Only 2 operations required; both performed on the **TP** factor:
  - Subtract a multiple of row from next to make a zero
  - Add a positive multiple of a column to next/previous
- Both operations preserve the TP structure of the left factor  
 $\Rightarrow$  SR structure preserved at every step!
- We know how to perform both operations accurately (later)

# Similarity Reduction to an Anti-Bidiagonal

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 2

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 3

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & + \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 4

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & + \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 5

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 6

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 7

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 8

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 9

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 10

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix}$$

$$\begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 11

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- First reduce to (Upper Triangular)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 12

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- Making more zeros analogous

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ \mathbf{0} & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 13

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- Making more zeros analogous

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & -1 & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 14

---

- Making more zeros analogous

$$\begin{array}{ccccccc}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} \\
 & \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} & & \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix} & & \begin{bmatrix} & & & 1 \\ & & & \\ & & & \\ 1 & & & \end{bmatrix} & & 
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 15

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- Making more zeros analogous

$$\begin{array}{c}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 16

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- Making more zeros analogous

$$\begin{array}{c}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 17

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- Making more zeros analogous

$$\begin{array}{c}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 18

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- Making more zeros analogous

$$\begin{array}{c}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 19

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- Making more zeros analogous

$$\begin{array}{c}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & & \\ & & 1 & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 20

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- Making more zeros analogous

$$\begin{array}{c}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -1 & & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & + & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ + & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & + & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 21

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- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 22

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- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 23

---

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$
  

$$\begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

# Similarity Reduction to an Anti-Bidiagonal 24

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{c}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & + & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 25

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{cccccc}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 & \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} & & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} & & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & - & 1 \end{bmatrix} & \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} & \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$



# Similarity Reduction to an Anti-Bidiagonal 27

---

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{c}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & + & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & \mathbf{0} & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 28

---

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{c}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & + & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 29

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{c}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & + & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 30

---

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{c}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & \mathbf{0} & \mathbf{0} \\ 0 & + & + & \mathbf{0} \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix}
 \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 31

- (Upper Triangular)  $\times J \rightarrow$  (Upper bidiagonal)  $\times J$

$$\begin{array}{c}
 \begin{bmatrix} 1 & - & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} + & + & + & + \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} 1 & + & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & + & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \\
 \\
 \begin{bmatrix} + & + & 0 & 0 \\ 0 & + & + & 0 \\ 0 & 0 & + & + \\ 0 & 0 & 0 & + \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}
 \end{array}$$

# Similarity Reduction to an Anti-Bidiagonal 32

---

- Finally, symmetrizing

$$\begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} & & b_1 & a_1 \\ & & b_2 & a_2 \\ b_3 & & a_3 & \\ a_4 & & & \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1}$$

# Similarity Reduction to an Anti-Bidiagonal 32

---

- Finally, symmetrizing

$$\begin{aligned} & \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix} \begin{bmatrix} & & b_1 & a_1 \\ & & b_2 & a_2 \\ b_3 & & a_3 & \\ a_4 & & & \end{bmatrix} \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & d_3 & \\ & & & d_4 \end{bmatrix}^{-1} \\ & = \begin{bmatrix} & & \sqrt{b_1 b_3} & \sqrt{a_1 a_4} \\ & b_2 & \sqrt{a_2 a_3} & \\ \sqrt{b_1 b_3} & \sqrt{a_2 a_3} & & \\ \sqrt{a_1 a_4} & & & \end{bmatrix} \end{aligned}$$

# Eigenvalues of a Sign Regular Matrix

---

$$\text{eig}(\text{Sign Regular}) = \text{eig} \left( \begin{bmatrix} & & & b_1 & a_1 \\ & & & b_2 & a_2 \\ & & & b_2 & a_3 \\ b_1 & a_2 & & & \\ a_1 & & & & \end{bmatrix} \right)$$

- The latter is symmetric, thus  $|\lambda_i| = \sigma_i^2$
- $\text{sign } \lambda_i = (-1)^{i-1}$  (from theory)
- Thus suffices to compute

$$\text{svd} \left( \begin{bmatrix} & & & b_1 & a_1 \\ & & & b_2 & a_2 \\ & & & b_2 & a_3 \\ b_1 & a_2 & & & \\ a_1 & & & & \end{bmatrix} \right) = \text{svd} \left( \begin{bmatrix} a_1 & b_1 & & & \\ & a_1 & b_2 & & \\ & & a_3 & b_2 & \\ & & & a_2 & b_1 \\ & & & & a_1 \end{bmatrix} \right)$$

—the latter solved by Demmel and Kahan 1990

# Outline

---

- (Almost) any matrix is similar to a **symmetric anti-bidiagonal**  
(eig of latter then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- **Trick: no subtractions in above reduction**
- We need a structure revealing representation—product of bidiagonals

# No Subtractive Cancellation

---

- Relative accuracy preserved in  $\times, +, /$   
Proof:  $(1 + \delta)$  factors accumulate multiplicatively
- Subtractions of approximate quantities dangerous:

$$\begin{array}{r} .123456789xxx \\ - .123456789yyy \\ \hline .000000000zzz \end{array}$$

- Initial data is OK to subtract:  $(x_i - y_j)$
- Reduction to anti-bidiagonal involves no subtractions

# Outline

---

- (Almost) any matrix is similar to a **symmetric anti-bidiagonal** (eig of latter then easy)

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix} \longrightarrow \begin{bmatrix} & & + & + \\ & + & + & \\ + & + & & \\ + & & & \end{bmatrix}$$

- Trick: no subtractions in above reduction
- **We need a structure revealing representation—product of bidiagonals**

# The Need for a Structure Revealing Representation

---

- Matrix entries are poor choice of parameters
- An  $\varepsilon$  perturbation in (1, 2) entry of the SR matrix

$$\begin{bmatrix} 1 & 1 + \varepsilon \\ 1 & 1 \end{bmatrix} \quad \lambda_1 \approx 1, \quad \lambda_2 \approx -\varepsilon$$

↓

$$\begin{bmatrix} 1 & 1 + 2\varepsilon \\ 1 & 1 \end{bmatrix} \quad \lambda_1 \approx 1, \quad \lambda_2 \approx -2\varepsilon$$

results in a 100% perturbation in  $\lambda_2$

- Better idea: product of bidiagonals

# Bidiagonal Decomposition of Totally Positive Matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 2 & \\ & & & 6 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 1 & & \\ & 1 & 2 & \\ & & 1 & 3 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_U$$

• Any TP matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & b_{21} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_U$$

•  $b_{ij}$  = multiplier used to make zero in position  $(i, j)$

# Representation of Sign Regular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & b_{41} 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & b_{31} 1 & \\ & & & b_{42} 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & b_{21} 1 & \\ & & & b_{32} 1 \\ & & & & b_{43} 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Any  $b_{ij} > 0$  yield a SR matrix  $\Rightarrow$  SR = octant in  $n^2$  space
- For most famous SR matrices,  $b_{ij}$  computable accurately
- We choose  $b_{ij}$  as inputs:
  - $b_{ij}$  reveal SR structure
  - $b_{ij}$  determine  $\lambda_i$  accurately

# Representation of Sign Regular Matrices

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{43} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{43} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & b_{41} 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & b_{31} 1 & \\ & & & b_{42} 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & b_{21} 1 & \\ & & & b_{32} 1 \\ & & & & b_{43} 1 \end{bmatrix} \begin{bmatrix} b_{11} & & & \\ & b_{22} & & \\ & & b_{33} & \\ & & & b_{44} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & & \\ & 1 & b_{23} & \\ & & 1 & b_{34} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & b_{13} & \\ & & 1 & b_{24} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & b_{14} \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}$$

- Any  $b_{ij} > 0$  yield a SR matrix  $\Rightarrow$  SR = octant in  $n^2$  space
- For most famous SR matrices,  $b_{ij}$  computable accurately
- We choose  $b_{ij}$  as inputs:
  - $b_{ij}$  reveal SR structure
  - $b_{ij}$  determine  $\lambda_i$  accurately

Rationale:  $\hat{\lambda}_i$ , as rational functions of  $b_{ij}$ , likely contain no subtractions

# How we avoid subtractions

---

- Recall: We only need to apply 2 operations to TP:
  - Subtract a multiple of row from next to make a zero
  - Add a positive multiple of a row to next/previous
- Both preserve TP structure
- When applied implicitly, no subtractions are required

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & b_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & b_{32} & 1 \end{bmatrix} \begin{bmatrix} b_{11} & & \\ & b_{22} & \\ & & b_{33} \end{bmatrix} \begin{bmatrix} 1 & b_{12} & \\ & 1 & b_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & b_{13} \\ & & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & & \\ & 1 & \\ & b'_{31} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & b'_{32} & 1 \end{bmatrix} \begin{bmatrix} b'_{11} & & \\ & b'_{22} & \\ & & b'_{33} \end{bmatrix} \begin{bmatrix} 1 & b'_{12} & \\ & 1 & b'_{23} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & b'_{13} \\ & & 1 \end{bmatrix}$$

# Conclusions

---

- New  $O(n^3)$  algorithm
- Computes all eigenvalues of a SR matrix to high relative accuracy
- First example of negative eigenvalues of a nonsymmetric matrix computed to high relative accuracy

## Open problems

- Sign regular matrices with other signatures—no parameterization known, let alone algorithms
- Eigenvectors: Products of accurate factors, but what does accuracy mean?

## Other work

- Elliptic PDEs with tetrahedral finite elements (with Demmel and Vavasis)
- This talk, papers, software: <http://math.mit.edu/~plamen>



# Accurate $\mathcal{BD}$ of Vandermonde and Cauchy

---

- $V = \left[ x_i^{j-1} \right]_{i,j=1}^n$  (TP if  $0 < x_1 < x_2 < \dots < x_n$ )

$$D_{ii} = \prod_{j=1}^{i-1} (x_i - x_j), \quad L_{i+1,i}^{(k)} = \prod_{j=n-k}^{i-1} \frac{x_{i+1} - x_{j+1}}{x_i - x_j}, \quad U_{i,i+1}^{(k)} = x_{i+n-k}$$

- $C = \left[ \frac{1}{x_i + y_j} \right]_{i,j=1}^n$  (TP if  $0 < x_1 < \dots < x_n, 0 < y_1 < \dots < y_n$ )

$$D_{ii} = \prod_{k=1}^{i-1} \frac{(x_i - x_k)(y_i - y_k)}{(x_i + y_k)(y_i + x_k)}$$

$$L_{i,i+1}^{(k)} = \frac{x_{n-k} + y_{i-n+k+1}}{x_i + y_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{x_{i+1} - x_{l+1}}{x_i - x_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{x_i + y_l}{x_{i+1} + y_l}$$

$$U_{i+1,i}^{(k)} = \frac{y_{n-k} + x_{i-n+k+1}}{y_i + x_{i-n+k+1}} \prod_{l=n-k}^{i-1} \frac{y_{i+1} - y_{l+1}}{y_i - y_l} \cdot \prod_{l=1}^{i-n+k-1} \frac{y_i + x_l}{y_{i+1} + x_l}$$

- Similar formulas for Cauchy–Vandermonde, confluent (José Javier Martínez)
- No subtractive cancellation  $\Rightarrow$  accurate



## Applying EETs on $\mathcal{BD}(A)$ – II

---

- Subtracting a row from next to make a zero

$$\begin{aligned} & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -7 & 1 & \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix} \\ = & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -7 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 7 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ 3 & 1 & & \\ & 8 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 4 & & \\ & & 9 & \end{bmatrix} \begin{bmatrix} 1 & 2 & & \\ & 1 & 5 & \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 3 & \\ & & 1 & \end{bmatrix} \end{aligned}$$

## Applying EETs on $\mathcal{BD}(A)$ – III

---

- Subtracting a row from next to make a zero

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & -7 & 1 & \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 3 & 10 & 50 \\ 21 & 102 & 615 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -7 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 7 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 3 & 1 & \\ & 8 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 4 & & \\ & & 9 & \end{bmatrix} \begin{bmatrix} 1 & 2 & & \\ & 1 & 5 & \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 3 & \\ & & 1 & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 0 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 3 & 1 & \\ & 8 & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 4 & & \\ & & 9 & \end{bmatrix} \begin{bmatrix} 1 & 2 & & \\ & 1 & 5 & \\ & & 1 & \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & 3 & \\ & & 1 & \end{bmatrix}$$

- Is equivalent to setting an entry of  $\mathcal{BD}(A)$  to zero and performing **no arithmetic**.

## Adding a multiple of a column to previous – I

---

$$\begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ b & 1 & \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \underline{\begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix}}$$

## Adding a multiple of a column to previous – II

---

$$\begin{aligned}
 & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix} \\
 = & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}
 \end{aligned}$$

$$x' = x$$

$$y' = y + kx$$

$$z' = 1/y'$$

$$k' = kz/y_1$$

... it's all qd recurrences





## Adding a multiple of a column to previous – V

$$\begin{aligned}
 & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y & \\ & x & z \end{bmatrix} \\
 = & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & g & \\ & 1 & h \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & y' & \\ & x' & z' \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix} \\
 = & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e & \\ & & f \end{bmatrix} \begin{bmatrix} 1 & & \\ & y'' & \\ & x'' & z'' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix} \\
 = & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix} \\
 = & \begin{bmatrix} 1 & & \\ & 1 & \\ a & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & b & 1 \\ & c + x''' & 1 \end{bmatrix} \begin{bmatrix} d & & \\ & e' & \\ & & f' \end{bmatrix} \begin{bmatrix} 1 & g' & \\ & 1 & h' \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & k' \\ & & 1 \end{bmatrix}
 \end{aligned}$$

Done.