Computer Science/Mathematics 555
Final Exam
Due 17 December 2003
No Extensions Please!

1. 16.2

2. 16.8

3. Show that the matrix
$$M = \begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix}$$
with $G \in \mathrm{R}^{n \times n}$ and $A \in \mathrm{R}^{m \times n}, m \leq n$ is nonsingular if and only if $\mathrm{Rank}(A) = m$ and for a null basis matrix $Z \in \mathrm{R}^{n \times (n-m)}$ we have that $Z^T G Z$ is nonsingular.

   [Hint: Consider the case where
$$M\mathbf{z} = 0, \quad \mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{pmatrix}. \tag{1}$$
Note that M is singular if and only this holds for some $\mathbf{z} \neq 0$. For one direction, show that if $Z^T G Z$ is singular, the problem (1) a nonzero solution with a nonzero $\mathbf{x}$ of the form $\mathbf{x} = Z\mathbf{w}$ (also use the fact the null(A) is orthogonal to range(A^T), and that if $\mathrm{Rank}(A) < m$, (1) will have a solution with nonzero multipliers $\boldsymbol{\lambda}$. For the other direction, suppose (1) has a nonzero solution, if $\mathbf{x}$ is nonzero then $Z^T G Z$ is singular, and if $\boldsymbol{\lambda} \neq 0$, then $\mathrm{Rank}(A) < m$.]

4. Consider the problem where
$$\min \frac{1}{2}\mathbf{x}^T G \mathbf{x} + \mathbf{x}^T \mathbf{c} \tag{2}$$
subject to
$$A\mathbf{x} = b. \tag{3}$$
Assume that G is positive definite and A has full row rank. Suppose we use the penalty method
$$\min_{\mathbf{x} \in \mathbf{R}^n} f(\mathbf{x}, \mu)$$

$$f(\mathbf{x},\mu) = \frac{1}{2}\mathbf{x}^T G\mathbf{x} + \mathbf{x}^T\mathbf{c} + \frac{1}{2\mu}\|A\mathbf{x}-\mathbf{b}\|_2^2.$$

Show that the first order condition $\nabla_x f(\mathbf{x},\mu) = 0$ becomes

$$\begin{pmatrix} G & -\mu^{-1}A^T \\ A & I \end{pmatrix}\begin{pmatrix} \mathbf{x}(\mu) \\ \mathbf{r}(\mu) \end{pmatrix} = \begin{pmatrix} -\mathbf{c} \\ \mathbf{b} \end{pmatrix}.$$

Use this fact to show that as $\mu \to 0$, $\mathbf{x}(\mu)$ approaches the solution of (2)–(3).

5. Starting with $\mathbf{x} = (x_1, x_2)^T = (0,0)^T$, use the active set method to solve

$$\min\ x_1^2 - x_1x_2 + x_2^2 - 3x_1$$

subject to

$$x_1 \geq 0, \quad x_2 \geq 0, \quad -x_1 - x_2 \geq -2.$$

Try to solve the problem graphically first, does that tell you which constraints will be active at the solution. [Hint: Formulate this problem in the standard form with (2) with constraints $A\mathbf{x} \geq \mathbf{b}$.]