Computer Science/Mathematics 550
Numerical Algebra
Assignment One
Due 10 September 2007

1. Let $\|\cdot\|_\alpha$ be a norm on R^n such that $\|\mathbf{e}_i\|_\alpha = 1, i = 1, 2, \ldots, n$ where e_i is the i^{th} column of the identity matrix. Show the following:

   (a) $\|\mathbf{x}\|_\alpha \leq \|\mathbf{x}\|_1$ for all $\mathbf{x} \in \mathrm{R}^n$.

   (b) Assume also that if $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{y} = (y_1, \ldots, y_n)^T$ and $|y_i| \geq |x_i|$ $i = 1, 2, \ldots, n$ then $\|\mathbf{y}\|_\alpha \geq \|\mathbf{x}\|_\alpha$. Show that $\|\mathbf{x}\|_\alpha \geq \|\mathbf{x}\|_\infty$ for all $\mathbf{x} \in \mathrm{R}^n$.

2. Let $A \in \mathrm{R}^{m\times n}$, $\mathbf{w} \in \mathrm{R}^n$ and define

$$B = \begin{pmatrix} A \\ \mathbf{w}^T \end{pmatrix}.$$

   Show that

$$\|A\|_2 \leq \|B\|_2 \leq \sqrt{\|A\|_2^2 + \|\mathbf{w}\|_2^2}.$$

3. Let $\mathbf{x}, \mathbf{y} \in \mathrm{R}^n$ and define $\psi : R \to R$ by $\psi(\alpha) = \|\mathbf{x} - \alpha\mathbf{y}\|_2$. Show that ψ is minimized when $\alpha = \mathbf{x}^T\mathbf{y}/(\mathbf{y}^T\mathbf{y})$.

4. Suppose that for $A \in \mathrm{R}^{m\times n}$, we have

$$A = \begin{array}{c} \\ \ell \\ m-\ell \end{array} \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} B & C \\ D & E \end{pmatrix} \end{array}.$$

   Show that

$$\|B\|_2 \leq \|A\|_2.$$

5. Let $\mathbf{s}$ be such that $\|\mathbf{s}\|_2 = 1$. Show that

$$\|E(I - \mathbf{s}\mathbf{s}^T)\|_F^2 = \|E\|_F^2 - \|E\mathbf{s}\|_2^2.$$

   This is one of the interesting and useful properties of the Frobenius norm.