CSE/Math 455
Lecture # 7

We are going to learn LU decomposition by Gaussian elimination. That is, to solve

$$A\mathbf{x} = \mathbf{b}$$

we factor A into

$$\begin{aligned} A &= LU, \\ L &\quad \text{Lower Triangular }, \\ U &\quad \text{Upper Triangular }. \end{aligned}$$

We then do the two solution steps

$$L\mathbf{y} = \mathbf{b}, \tag{1}$$
$$U\mathbf{x} = \mathbf{y}. \tag{2}$$

Start with a 3×3 example.

$$A = \begin{pmatrix} 1 & 1/2 & 3 \\ 1/3 & 2\,1/6 & 2 \\ -1/2 & -2\,1/4 & 1\,1/2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3.5 \\ 1/6 \\ 3\,1/4 \end{pmatrix}.$$

A first elimination step produces

$$A^{(1)} = \begin{pmatrix} 1 & 1/2 & 3 \\ \mathit{1/3} & 2 & 1 \\ \mathit{-1/2} & -2 & 3 \end{pmatrix}$$

where the elements in italics are the multipliers.

The second elimination step produces

$$A^{(2)} = \begin{pmatrix} 1 & 1/2 & 3 \\ \mathit{1/3} & 2 & 1 \\ \mathit{-1/2} & \mathit{-1} & 4 \end{pmatrix}.$$

Thus,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/2 & -1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1/2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

The solution of (1) yields

$$\begin{aligned} y_1 &= 3.5 \\ y_2 &= \frac{1}{6} - \frac{1}{3} \cdot 3.5 = -1 \\ y_3 &= 3.25 - (-1/2) \cdot 3.5 - (-1)(-1) = 4 \end{aligned}$$

Thus,

$$\mathbf{y} = \begin{pmatrix} 3.5 \\ -1 \\ 4 \end{pmatrix}.$$

If we then solve (2) then we get

$$\begin{aligned} x_3 &= 4/4 = 1 \\ x_2 &= (-1 - 1 \cdot 1)/2 = -1 \\ x_1 &= (3.5 - 1/2 \cdot (-1) - 3 \cdot 1)/1 = 1 \end{aligned}$$

Verify for yourself $\underline{A = LU}$.
Why preserve L and U this way?
Suppose we want to solve

$$\begin{aligned} A\mathbf{x}_1 &= \mathbf{b}_1 \\ &\vdots \qquad\qquad p \text{ separate systems} \\ A\mathbf{x}_p &= \mathbf{b}_p \end{aligned}$$

We just solve

$$\begin{aligned} L\mathbf{y}_i &= \mathbf{b}_i, \quad i = 1, \ldots, p \\ U\mathbf{x}_i &= \mathbf{y}_i \end{aligned}$$

without recomputing L and U. It is perhaps better to set

$$B = (\mathbf{b}_1, \ldots, \mathbf{b}_p), \quad X = (\mathbf{x}_1, \ldots, \mathbf{x}_p)$$

and solve

$$\begin{aligned} LY &= B \\ UX &= Y \end{aligned}$$

For instance, if we make

$$B = I, \qquad \text{The Identity Matrix}$$

then $X = A^{-1}$ yields the inverse.

Now consider the computation of a general 3×3 system.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_2 \end{pmatrix}$$

Eliminate first column

Assume $\quad a_{11} \neq 0$ and compute the multipliers

$$\begin{aligned} \ell_{21} &= a_{21}/a_{11} \\ \ell_{31} &= a_{31}/a_{11} \end{aligned}$$

Then modify the second and third row and column

$$\begin{aligned} a_{22}^{(1)} &= a_{22} - \ell_{21}a_{12}. \quad a_{23}^{(1)} = a_{23} - \ell_{21}a_{13} \\ a_{32}^{(1)} &= a_{32} - \ell_{31}a_{12}, \quad a_{33}^{(1)} = a_{33} - \ell_{31}a_{13} \end{aligned}$$

Thus

$$A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \ell_{21} & a_{22}^{(1)} & a_{23}^{(1)} \\ \ell_{31} & a_{32}^{(1)} & a_{33}^{(1)} \end{pmatrix}$$

Eliminate second column

Assume that $a_{22}^{(1)} \neq 0$. Then we do the computations

$$\begin{aligned} \ell_{32} &= a_{32}^{(1)}/a_{22}^{(1)}, \\ a_{33}^{(2)} &= a_{33}^{(1)} - \ell_{32}a_{23}^{(1)} \end{aligned}$$

Thus finally,

$$A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \ell_{21} & a_{22}^{(1)} & a_{23}^{(1)} \\ \ell_{31} & \ell_{32} & a_{33}^{(2)} \end{pmatrix}.$$

Thus obtaining

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{pmatrix}.$$

The code for what we have just done looks like this.

for$k = 1\colon n-1$
  for $i = k+1\colon n$
    $\ell_{ik} = a_{ik}/a_{kk}$; % Stored in a_{ik}.
    for $j = k+1\colon n$
      $a_{ij} = a_{ij} - \ell_{ik}a_{kj}$;
    end;
  end ;
end;

As I showed in class, a columnwise version of this is easily produced.

for$k = 1\colon n-1$
  $A(k+1\colon n, k) = A(k+1\colon n, k)/A(k,k)$; % Compute Multipliers
  for $j = k+1\colon n$
    $A(k+1\colon n, j) = A(k+1\colon n, j) - A(k,j) * A(k+1\colon n, k)$;
  end;
end;

This is how it is coded in LINPACK, a software package from the late 70's. (It is not quite a simple as this, but it was coded columnwise to comform with FORTRAN 77.)

The code I used in class is at www.cse.psu.edu/∼barlow/cse455/badfact.m .

There will be a link to it from the classnotes.html web page.

Next time, partial pivoting.