

Computer Science/Mathematics 455
Lecture Notes
Lecture # 31

Ordinary Differential Equations

Again we consider the initial value problem (IVP)

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}$$

where t is usually time, and $\mathbf{x}(t)$ is a vector valued function.

The following theorem discusses the existence and uniqueness of a solution to the IVP (1)–(1).

Theorem 1 *Let $\mathbf{f}(t, \mathbf{x})$ be defined and continuous for all $(t, \mathbf{x}) \in \mathcal{D}$ where*

$$\mathcal{D} = \{(t, \mathbf{x}) : t \in [a, b], \|\mathbf{x}\| < \infty\}$$

for finite a and b . Let there exist a constant L such that

$$\|\mathbf{f}(t, \mathbf{x}_1) - \mathbf{f}(t, \mathbf{x}_2)\| \leq L\|\mathbf{x}_1 - \mathbf{x}_2\|_2 \tag{1}$$

for all $(t, \mathbf{x}_1), (t, \mathbf{x}_2) \in \mathcal{D}$. Then for any \mathbf{x}_0 there exists a unique solution $\mathbf{x}(t)$ which is continuous and differentiable.

Equation (1) is called a Lipschitz condition. It is stronger than continuity. The constant L is important. It determines how difficult the problem is to solve.

If \mathbf{f} is differentiable, then a bound for L is

$$L = \max_{(t, \mathbf{x}) \in \mathcal{D}} \left\| \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial \mathbf{x}} \right\|.$$

Example 1

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

Here

$$\mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus,

$$\left\| \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right\|_{\infty} = 1.$$

A more sophisticated example requires us to restrict the domain \mathcal{D} further than the theorem does.

Example 2

$$\begin{aligned} x_1' &= 0.25x_1 - 0.01x_1x_2 \\ x_2' &= -x_1 + 0.01x_1x_2 \end{aligned}$$

Here

$$\mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} 0.25x_1 - 0.01x_1x_2 \\ -x_1 + 0.01x_1x_2 \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} 0.25 - 0.01x_2 & -0.01x_1 \\ 0.01x_1 & -1 + 0.01x_1 \end{pmatrix}.$$

To bound this, we have to bound x_1 and x_2 in a certain region. Suppose that

$$0 \leq x_1, x_2 \leq 200.$$

For that region the bound is

$$\left\| \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right\|_{\infty} \leq \left\| \begin{pmatrix} 1.75 & 2 \\ 2 & 1 \end{pmatrix} \right\|_{\infty} = 3.75.$$

Now back to some methods. For now, we talk about single equations. For such single equations

$$\begin{aligned} x_n &\approx x(t_n), & t_n &= t_0 + nh \\ f_n &= f(t_n, x_n) \approx f(t_n, x(t_n)) \end{aligned}$$

Euler's method is

$$x_{n+1} = x_n + hf_n = x_n + hf(t_n, x_n).$$

This method only captures the first term of the Taylor series.

Runge–Kutta methods capture more of the Taylor series by evaluating the function f at more points. Note that

$$x(t_0 + h) = x(t_0) + hx'(t_0) + \frac{h^2}{2}x''(t_0) + O(h^3).$$

We have that

$$x'(t_0) = f(t_0, x(t_0)).$$

Using a forward difference approximation

$$\begin{aligned} x''(t_0) &= \frac{x'(t_0 + \alpha h) - x'(t_0)}{\alpha h} + O(h) \\ &= \frac{f(t_0 + \alpha h, x(t_0 + \alpha h)) - f(t_0, x_0)}{\alpha h} + O(h) \end{aligned}$$

We need to use computed values. To do that, first let

$$k_1 = f(t_0, x_0)$$

and then note that from the Lipschitz condition (yes, it really is useful!)

$$|f(t_0 + \alpha h, x(t_0 + \alpha h)) - f(t_0 + \alpha h, x_0 + \alpha h k_1)| \leq L|x(t_0 + \alpha h) - x_0 - \alpha h k_1| = O(h^2).$$

So we have that

$$x''(t_0) = \frac{f(t_0 + \alpha h, x_0 + \alpha h k_1) - f(t_0, x_0)}{\alpha h} + O(h).$$

Let

$$k_2 = f(t_0 + \alpha h, x_0 + \alpha h k_1)$$

then

$$\begin{aligned} x(t_0 + h) &= x_0 + hf(t_0, x_0) + \frac{h^2}{2} \left(\frac{f(t_0 + \alpha h, x_0 + \alpha h k_1) - f(t_0, x_0)}{\alpha h} \right) + O(h^3) \\ &= x_0 + hk_1 + \frac{h}{2}(k_2 - k_1)/(\alpha h) \\ &= x_0 + h[k_1(1 - 1/(2\alpha)) + k_2/(2\alpha)]. \end{aligned}$$

More generally, this the method

$$\begin{aligned} k_1 &= f(t_n, x_n) \\ k_2 &= f(t_n + \alpha h, x_n + \alpha h k_1) \\ x_{n+1} &= x_n + h[k_1(1 - 1/(2\alpha)) + k_2/(2\alpha)] \end{aligned}$$

This a two-stage Runge–Kutta method of order 2. Different values of α yield different methods.

$\alpha = 0.5$ **The Midpoint Rule**

$$\begin{aligned}k_2 &= f(t_n + h/2, x_n + hk_1/2) \\x_{n+1} &= x_n + hk_2\end{aligned}$$

$\alpha = 1$ **Modified Euler Method**

$$\begin{aligned}k_2 &= f(t_n + h, x_n + hk_1) \\x_{n+1} &= x_n + \frac{h}{2}(k_1 + k_2)\end{aligned}$$

$\alpha = 2/3$ **Heun's Method**

$$\begin{aligned}k_2 &= f(t_n + 2/3h, x_n + 2/3hk_1) \\x_{n+1} &= x_n + \frac{h}{4}(k_1 + 3k_2)\end{aligned}$$

This is a second order method. That essentially means that it has the same order of accuracy as taking up to the second derivative term of the Taylor series.

More function evaluations allow to increase the order further. A popular fourth order method is

$$\begin{aligned}k_1 &= f(t_n, x_n) \\k_2 &= f(t_n + h/2, x_n + hk_1/2) \\k_3 &= f(t_n + h/2, x_n + hk_2/2) \\k_4 &= f(t_n + h, x_n + hk_3) \\x_{n+1} &= x_n + \frac{h}{6}(k_1 + 2(k_2 + k_3) + k_4)\end{aligned}$$

Next time, I'll give a more sophisticated method.