Computer Science/Mathematics 455
Lecture Notes
Lecture # 30

Ordinary Differential Equations

For our purposes we consider the initial value problem (IVP)

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) &= \mathbf{x}_0 \end{aligned}$$

where t is usually time, and $\mathbf{x}(t)$ is a vector valued function.

For terminology, we let

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{f}(t, \mathbf{x}(t)) = \begin{pmatrix} f_1(t, \mathbf{x}(t)) \\ f_2(t, \mathbf{x}(t)) \\ \vdots \\ f_n(t, \mathbf{x}(t)) \end{pmatrix}$$

We will also eventually need the Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

First, let me given some examples of simple initial value problems with known solutions.

Example 1 *We think of this as the model problem.*

$$\begin{aligned} x' = \lambda x, \quad \lambda \in \mathbf{C} \\ x(0) = x_0 \end{aligned}$$

The solution is

$$x(t) = x_0 e^{\lambda t}.$$

Since we have allowed λ to be complex, this includes a wider range of functions. If we let $\lambda = \alpha + \mathbf{i}\beta$ then

$$x(t) = x_0 e^{\alpha t} e^{\mathbf{i}\beta t} = x_0 e^{\alpha t}(\cos\ \beta t + \mathbf{i} \sin\ \beta t).$$

Thus sin *and* cos *are included as solutions of this model problem.*

If $\alpha < 0$, we have a decaying solution, that is,

$$\lim_{t\to\infty} x(t) = 0.$$

If $\alpha > 0$, we have a growing solution, that is,

$$\lim_{t\to\infty} |x(t)| = \pm\infty.$$

If $\alpha = 0$, then

$$\lim_{t\to\infty} |x(t)| = |x_0|.$$

It is important that a method for solving this IVP obtain a decaying solution when it is supposed to. That will not be as easy as we might hope.

Here is an example of a system.

Example 2

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \\ x_1(0) &= 0 \\ x_2(0) &= 1 \end{aligned}$$

The solution is

$$\mathbf{x} = \begin{pmatrix} \sin\ t \\ \cos\ t \end{pmatrix}.$$

The IVPs that have known solution are ones where skillful people have managed to guess solutions over the years. The vast majority of IVPs do not have a closed form solution at all. We must use numerical methods to solve them.

Higher order derivatives are easily handled.

$$x'' = f(t, x, x')$$
$$x(t_0) = x_0$$
$$x'(t_0) = x_0'$$

This can be written as a first order system using the change of variable

$$x = x_1, \quad x' = x_2$$

thus yielding

$$x_1' = x_2, \quad x_1(t_0) = x_0$$
$$x_2' = -f(t, x_1, x_2), \quad x_2(t_0) = x_0'$$

For our purposes, this is what we always do.

In more general form, consider the problem

$$\begin{aligned} x^{(n)} &= f(t, x, x', \ldots, x^{(n-1)}) \\ x(t_0) &= x_0 \\ x'(t_0) &= x_0' \\ &\vdots \\ x^{(n-1)}(t_0) &= x_0^{(n-1)}. \end{aligned}$$

For that, just let

$$x_1 = x, \quad x_2 = x', \quad , \ldots, x_n = x^{(n-1)}.$$

Then this becomes

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ &\vdots \\ x_{n-1}' &= x_n \\ x_n' &= f(t, x_1, x_2, \ldots, x_n) \\ x_1(t_0) &= x_0 \\ x_2(t_0) &= x_0' \\ &\vdots \\ x_n(t_0) &= x_0^{(n-1)}. \end{aligned}$$

Example 3

$$\begin{aligned} x'' &= -x \\ x(0) &= 0, \quad x'(0) = 1 \end{aligned}$$

This leads to the system

$$\begin{aligned} x_1' = x_2, &\quad x_1(0) = 0 \\ x_2' = -x_1, &\quad x_2(0) = 1 \end{aligned}$$

which is the system we discussed earlier.

Although all higher order equation can be broken down into systems using this technique, not all systems can be written as single higher order equations.

Example 4 ***The Preditor–Prey Equation***

$$\begin{aligned} x_1' &= -\alpha x_1 + \beta x_1 x_2 \\ x_2' &= \gamma x_2 - \delta x_1 x_2 \end{aligned}$$

for parameters $\alpha, \beta, \gamma, \delta > 0$. *With initial values*

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}.$$

For certain positive initial values of x_{10} *and* x_{20} *the solution to this equation should form a cycle.*

Numerical Methods

The simplest numerical method is Euler's method. Essentially, it is what we want to improve. It would be silly to stop there.

The idea starts with Taylor series (what doesn't ?). If we expand $x(t)$ around t_0 to get the solution at $t_0 + h$ we have

$$x(t_0 + h) = x(t_0) + hx'(t_0) + \frac{h^2}{2}x''(\xi).$$

We throw away the second order term, to get

$$x(t_0 + h) \approx x_1 = x(t_0) + hx'(t_0) = x_0 + hf(t_0, x_0).$$

Let $t_k = t_0 + kh$ for some spacing h. Then

$$x(t_{k+1}) = x(t_k + h) \approx x_{k+1} = x_k + hf(t_k, x_k).$$

This is Euler's method. It is a bit crude. But if h is small enough and $f(t, x)$ is well behaved and you have enough computer time, it will give you a half way decent solution. I hope my praise was faint enough, because, of course, we want to improve this later!

Let us try this on the model problem

$$x' = \lambda x, \quad x(0) = 1.$$

It's solution is $x(t) = e^{\lambda t}$. We suppose that λ is real and negative.

Then

$$x_{k+1} = x_k + hf(t_k, x_k) = x_k + h\lambda x_k = (1 + h\lambda)x_k.$$

An induction argument yields the expression

$$x_n = (1 + \lambda h)^n x_0 = (1 + \lambda h)^n.$$

Let $t_n = t$ be fixed and let $h = t/n$. Then

$$x_n = (1 + \lambda t/n)^n.$$

We can easily see that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} (1 + \lambda t/n)^n = e^{\lambda t}.$$

Thus the method converges to the right value. However, this is not enough.

Now, fix h. We know that

$$\lim_{n\to\infty} x(t_n) = \lim_{n\to\infty} e^{n\lambda ht} = 0.$$

We would like for x_n to have this property as well. However,

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |1 + \lambda h|^n = \begin{cases} 0 & |1 + \lambda h| < 1 \\ 1 & |1 + \lambda h| = 1 \\ \infty & |1 + \lambda h| > 1 \end{cases}$$

The restriction

$$|1 + \lambda h| < 1$$

is the same as the restriction

$$-2 < \lambda h < 0.$$

For large negative values of λ this can be a severe restriction on the step size.