Computer Science/Mathematics 455
Lecture Notes
Lecture # 24

Central Difference Interpolation

For central differences, we number the points

$$x_k = x_0 + kh, \quad k = 0, \pm 1, \pm 2, \ldots$$

Interpolation points are in the order

$$x_0, x_1, x_{-1}, x_2, x_{-2}, \ldots,$$

Thus we get

$$\begin{aligned} f(x) = f(x_0 + sh) &\approx f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_{-1}, x_0, x_1] + \cdots \\ &= f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_{-1} + \frac{s(s-1)(s+1)}{3!}\Delta^3 f_{-1} + \cdots \end{aligned}$$

General term *even order*

$$\frac{(s+m-1)(s+m-2)\cdots(s+1)s(s-1)\cdots(s-m)}{(2m)!}\Delta^{2m} f_{-m}$$

General term *pdd order*

$$\frac{(s+m)(s+m-1)\cdots(s+1)s(s-1)\cdots(s-m)}{(2m+1)!}\Delta^{2m+1} f_{-m}$$

Example 1 *Again do*

$$f(x) = \cos\, \pi x/2, \quad x \in [0, 1]$$

We use a spacing a 0.1. *Do* 4 *levels of differencing.*

x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	*1*	*-0.01231*	*-0.02432*	*0.00090*	*0.00058*
0.1	*0.98769*	*-0.03663*	*-0.02342*	*0.00148*	*0.00054*
0.2	*0.95106*	*-0.06005*	*-0.02194*	*0.00202*	*0.00049*
0.3	*0.89101*	*-0.08199*	*-0.01992*	*0.00251*	***0.00043***
0.4	*0.80902*	*-0.10191*	***-0.01741***	***0.00294***	*0.00036*
0.5	***0.70711***	***-0.11932***	*-0.01447*	*0.00329*	*0.00028*
0.6	*0.58779*	*-0.13379*	*-0.01118*	*0.00357*	*0.00019*
0.7	*0.45399*	*-0.14497*	*-0.00761*	*0.00376*	
0.8	*0.30902*	*-0.15258*	*-0.00385*		
0.9	*0.15643*	*-0.15643*			
1.0	*0.00000*				

The table entries in boldface are the ones that we need.

Suppose that we want to compute $f(0.51) = \cos(0.2505\pi) = 0.695912796592314$*. We have that* $s = 0.1$ *and* $h = 0.1$ *so*

$$
\begin{aligned}
&f_0 && = f[0.5] && = 0.70711 \\
&f_0 && +s\Delta f_0 && = 0.70711 - 0.1 * 0.11932 = 0.69517 \\
&f_0 && +s\Delta f_0 && +s(s-1)\Delta^2 f_{-1}/2 = 0.69596 \\
&f_0 && +s\Delta f_0 && +s(s-1)\Delta^2 f_{-1}/2 \\
& && +s(s-1)(s+1)\Delta^3 f_{-1}/6 && = 0.695909658985123 \\
&f_0 && +s\Delta f_0 && +s(s-1)\Delta^2 f_{-1}/2 \\
& && +s(s-1)(s+1)\Delta^3 f_{-1}/6 + && s(s-1)(s+1)(s-2)\Delta^4 f_{-2} = 0.695913019112432
\end{aligned}
$$

The first six digits of the last approximation are correct and that is as good as can be expected.

Splines

Definition 1 *Let* $a = x_0 < x_1 < \ldots < x_n = b$*. A function* $s\colon [a,b] \to\to \mathbf{R}$ *is a spline of degree* m *with knots* $x_0, x_1, \ldots, x_n$ *if*

1.

$$
s(x) = \begin{cases} s_0(x) & x \in [x_0, x_1] \\ s_1(x) & x \in [x_1, x_2] \\ \vdots & \vdots \\ s_{n-1}(x) & x \in [x_k, x_{k+1}] \end{cases}
$$

where $s_k(x)$ *is a polynomial of degree at most* m*.*

2. s is $m-1$ times continuously differentiable.

A function $s(x)$ that satisfies only the first specification is called a *piecewise polynomial.*

Example 2

$$s(x) = \begin{cases} x^2 + x & x \in [-1, 0] \\ x & x \in [0, 2] \\ x^2 - 3x + 4 & x \in [2, 5] \end{cases}$$

is a spline of degree 2 *on* $[-1, 5]$. *We note that*

$$s'(x) = \begin{cases} 2x + 1 & x \in (-1, 0) \\ 1 & x \in (0, 2) \\ 2x - 3 & x \in (2, 5) \end{cases}$$

The continity is verified at the interior knots 0 *and* 2.

$$\lim_{x \to 0 x<0} s(x) = \lim_{x \to 0 x>0} s(x) = 0 \; \lim_{x \to 2 x<2} s(x) = \lim_{x \to 2 x>2} s(x) = 2$$

and

$$\lim_{x \to 0 x<0} s'(x) = \lim_{x \to 0 x>0} s'(x) = 1 \; \lim_{x \to 2 x<2} s'(x) = \lim_{x \to 2 x>2} s'(x) = 1$$

Note that

$$s''(x) = \begin{cases} 2 & x \in (-1, 0) \\ 0 & x \in (0, 2) \\ 2 & x \in (2, 5) \end{cases}$$

is no longer continous.

The splines we will describe in detail are cubic splines for which each $s_k(x)$ is has degree three and s,s', and s'' are continous. It turns out that they are ideal for interpolation.

We specify that the kth function has the form

$$s_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3, \quad k = 0, \dots, n-1.$$

Thus we have $4n$ free parameters.

First, we specify the interpolation conditions. These are

$$s_k(x_k) = f(x_k), \quad s_k(x_{k+1}) = f(x_{k+1}), \quad k = 0, \ldots, n-1.$$

These also insure that the spline is at least continous. However, these are only $2n$ conditions. We do not want to specify values for $s_k(x)$ inside the interval $[x_k, x_{k+1}]$, so we need more conditions.

Continuity of the first and second derivative gives us more. We note that

$$\begin{aligned} s_k'(x) &= b_k + 2c_k(x - x_k) + 3d_k(x - x_k)^2 \\ .s_k''(x) &= 2c_k + 6d_k(x - x_k) \end{aligned}$$

and that continuity implies that

$$s_k'(x_{k+1}) = s_{k+1}'(x_{k+1}), \quad s_k''(x_{k+1}) = s_{k+1}''(x_{k+1}) \quad k = 1, \ldots, n-1$$

This is another $2n - 2$ conditions, giving us a total of $4n - 2$, we are still two short.

So we make up two more.

One of the most common, (though not always the best) are natural splines. These are specified according to

$$s''(a) = s''(b) = 0.$$

This is the only one your book talks about. If we have extra derivative information, a better set of conditions is

$$s'(a) = f'(a), \quad s'(b) = f'(b).$$

We show later than either of these two conditions leads to a very special property for splines.

Next time, we discuss how to construct the splines.