Computer Science/Mathematics 455
Lecture Notes
Lecture # 23

Finite Difference Interpolation

The Newton divided difference formula is that for the points $x_0, x_1, \ldots, x_n$, if

$$p_n(x_i) = f(x_i)$$

$$p_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \ldots + (x - x_0)\cdots(x - x_{n-1})f[x_0, x_1, \ldots, x_n].$$

This is a simple elegant formula, but we can simplify it even further when the points are evenly spaced.

For that let,

$$x_k = x_0 + kh, k = 0, 1, 2, \ldots.$$

The book allows negative indicies, but for now we will not get into that.

Forward Differences

$$\begin{aligned}
\Delta f(x_k) &= f(x_{k+1}) - f(x_k) \\
\Delta^2 f(x_k) &= \Delta f(x_{k+1}) - \Delta f(x_k) = f(x_{x+2}) - 2f(x_{k+1}) + f(x_k) \\
\Delta^n f(x_k) &= \Delta^{n-1} f(x_{k+1}) - \Delta^{n-1} f(x_k)
\end{aligned}$$

For example

$$\Delta^3 f(x_0) = f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0).$$

An induction argument yields the following expression.

$$\Delta^n f(x_k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x_{k+n-j}).$$

As before we set up a table. For instance

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0.5	2	1	1	1
0.0	3	2	2	
0.5	5	4		
1.0	9			

How to construct the polynomial. Essentially,

$$f[x_k, x_{k+1}, \ldots, x_{k+n}] = \frac{\Delta^n f(x_k)}{h^n n!}..$$

We invent the shorthand

$$f_k = f(x_k), \quad \Delta^n f(x_k) = \Delta^n f_k.$$

We then let $x = x_0 + sh$,thus

$$\begin{aligned} p_n(x) &= p_n(x+sh) \stackrel{def}{=} \tilde{p}_n(s) \\ &= f_0 + sh\frac{\Delta f_0}{h} + \frac{s(s-1)h^2\Delta^2 f_0}{2!h^2} + \cdots \\ &\quad + \frac{s(s-1)\cdots(s-n+1)h^n\Delta^n f_0}{n!h^n} \\ &= f_0 + s\frac{\Delta f_0}{+}\frac{s(s-1)\Delta^2 f_0}{2!} + \cdots \\ &\quad + \frac{s(s-1)\cdots(s-n+1)\Delta^n f_0}{n!} \end{aligned}$$

This is a much easier expression to work with.

For instance, for the example above, the resulting polynomial is

$$\tilde{p}_2(s) = p_2(x_0 + 0.5 * s) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\delta^2 f_0.$$

Error in Interpolation

The error in interpolation is best expressed using the Newton formula. Let $p_n(x)$ be the polynomial such that p_n interpolates f at $x_0, x_1, \ldots, x_n$.

To find $f(t) - p_n(t)$ for a point $t \neq x_0, x_1, \ldots, x_n$, let $p_{n+1}(x)$ be the polynomial that interpolates f at $x_0, x_1, \ldots, x_n, t$, that is, add t as an extra point. Then

$$p_{n+1}(x) = p_n(x) + (x - x_0)\cdots(x - x_n)f[x_0, x_1, \ldots, x_n, t].$$

Since

$$f(t) = p_{n+1}(t),$$

we have that

$$f(t) - p_n(t) = (t - x_0)\cdots(t - x_n)f[x_0, x_1, \ldots, x_n, t].$$

If we use the fact that for some $\xi \in (a,b)$ where

$$a = \max\{\max_i x_i, t\}, \quad b = \min\{\min_i x_i, t\}$$

then

$$f[x_0, x_1, \ldots, x_n, t] = \frac{f^{(n+1)(\xi)}}{(n+1)!}$$

and we obtain the formula

$$f(t) - p_n(t) = \frac{(t-x_0)\cdots(t-x_n)f^{(n+1)}(\xi)}{n!}.$$

Third and Last Form – Lagrange Form

Problem, find the polynomial $p_n(x)$ such that

$$p_n(x_i) = y_i, \quad i = 0, \ldots, n.$$

Start with the interpolation problem

$$\ell_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

First $\ell_i(x)$ is a polynomial with n roots

$$x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n.$$

So for some constant c,

$$\begin{aligned} \ell_i(x) &= c(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n) \\ &= c\prod_{j\neq i}(x-x_j) \end{aligned}$$

Now we need only determine c. We have that

$$\ell_i(x_i) = c\prod_{j\neq i}(x_i - x_j) = 1$$

thus

$$c = 1/\prod_{j\neq i}(x_i - x_j).$$

Therefore

$$\ell_i(x) = \frac{\prod_{j\neq i}(x-x_j)}{\prod_{j\neq i}(x_i-x_j)} = \prod_{j\neq i}\left(\frac{x-x_j}{x_i-x_j}\right).$$

Then the interpolation polynomial may be written as

$$p_n(x) = \sum_{i=0} y_i \ell_i(x).$$

Since

$$p_n(x_j) = y_j \ell_j(x_j) = y_j.$$

Incidentally, this is another proof of existence and uniqueness.

A quick example.

Example 1 *The same problem again.*

$$f(x) = \cos\, x, \quad x_0 = 0, x_1 = \pi/4, x_2 = \pi/2.$$

We have that

$$\begin{aligned}
\ell_0(x) &= \frac{(x-\pi/4)(x-\pi/2)}{(0-\pi/4)(0-\pi/2)} = \frac{8}{\pi^2}(\pi/4 - x)(x - \pi/2) \\
\ell_1(x) &= \frac{x(x-\pi/2)}{\pi/4(-\pi/4)} = \frac{16}{\pi^2} x(\pi/2 - x) \\
\ell_2(x) &= \frac{8}{\pi^2} x(x - \pi/4).
\end{aligned}$$

Taking $y_0 = 1$,$y_1 = \sqrt{1/2}$, $y_2 = 0$, *we have*

$$p_2(x) = \frac{8}{\pi^2}(\pi/4 - x)(x - \pi/2) + \sqrt{1/2}\frac{16}{\pi^2} x(\pi/2 - x).$$

This is an explicit formula. If we add points, there is no easy way to do it.

On the other hand, if I want the interpolation function for $\sin\, x$ *on the same points, I can just change the* y_i *values to* $y_0 = 0$, $y_1 = \sqrt{1/2}$ *and* $y_2 = 1$. *In that case,*

$$p_2(x) = \sqrt{1/2}\frac{16}{\pi^2} x(\pi/2 - x) + \frac{8}{\pi^2} x(x - \pi/4).$$

Thus it is easier to produce another interpolation function on the same points than with the divided difference formulas.