Computer Science/Mathematics 455
Lecture Notes
Lecture # 22

Finite Differences and Interpolation

Your book gives a simplified version of Newton divided differences for evenly spaced points. I prefer not to.

We have covered the interpolation problem

$$p_n(x_i) = y_i = f(x_i), \quad i = 0, \ldots, n$$

where

$$p_n(x) = a_0 + a_1 x + \ldots + a_n x^n.$$

This leads to the Vandermonde system and we showed the solution exists and is unique if the points $x_0, x_1, \ldots, x_n$ are distinct. To get a better algorithm we write the polynomial differently.

$$p_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_n(x - x_0)\cdots(x - x_{n-1}).$$

We compute the coefficients

$$c_i, i = 0, \ldots, n.$$

Again, let us start with $n = 2$ and the points x_0, x_1, x_2. The interpolation problem may be written as

$$\begin{aligned} p_2(x_0) = c_0 \quad &= y_0 = f(x_0) \\ p_2(x_1) = c_0 + c_1(x_1 - x_0) \quad &= y_1 = f(x_1) \\ p_2(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) \quad &= y_2 = f(x_2) \end{aligned}$$

This is the same as solving the lower triangular system

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & x_1 - x_0 & 0 \\ 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}.$$

In fact, it simplifies much more than this! We get

$$
\begin{aligned}
c_0 &= f(x_0) \stackrel{def}{=} f[x_0] \\
c_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \stackrel{def}{=} f[x_0, x_1] \\
c_2 &= \frac{f(x_2) - f(x_0)}{(x_2 - x_1)(x_2 - x_0)} - \frac{f[x_0, x_1]}{x_2 - x_1} \\
&= \frac{f[x_2, x_0] - f[x_0, x_1]}{x_2 - x_1} \stackrel{def}{=} f[x_0, x_1, x_2].
\end{aligned}
$$

The ordering of the points does not matter. So a more useful definition for the second order divided difference is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

We then have the following expression for the quadratic interpolation polynomial.

$$p_2(x) = f[x_0]] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2].$$

Example 1

$$f(x) = \cos\ x, \quad x_0 = 0, x_1 = \pi/4, x_2 = \pi/2.$$

$$f[x_0] = f(x_0) = 1, \quad f[x_1] = f(x_1) = \sqrt{2}/2, \quad f[x_2] = f(x_2) = 0.$$

The first order divided differences are

$$f[x_1, x_0] = \frac{\sqrt{2}/2 - 1}{(\pi/4)} = -0.37292$$

$$f[x_1, x_2] = \frac{0 - \sqrt{2}/2}{(\pi/4)} = -0.90032.$$

The second order divided difference is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = -0.33575.$$

Thus the expression for the polynomial is

$$p_2(x) = 1 - 0.37282x - 0.33575x(x - \pi/4).$$

The Vandermonde solution is

$$p_2(x) = 1 - 0.10923x - 0.33575x^2.$$

Except for rounding errors these are both the same polynomial. In fact, we can construct the interpolation polynomial in a number of ways from these divided differences.

We set up a divided difference table.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
0	1	-0.37292	-0.33575
$\pi/4$	$\sqrt{1/2}$	-0.090032	
$\pi/2$	0		

Any route through the table yields an expression for the polynomial. For the first case, we took the route x_0,x_1,x_2.

Here are two other paths.

x_1, x_0, x_2.

$$\begin{aligned} p_2(x) &= f[x_1] + (x - x_1)f[x_0, x_1] + (x - x_1)(x - x_0)f[x_0, x_1, x_2] \\ &= \sqrt{1/2} - 0.37292(x - \pi/4) - 0.33575(x - \pi/4)x \end{aligned}$$

x_1, x_2, x_0

$$\begin{aligned} p_2(x) &= f[x_1] + (x - x_1)f[x_1, x_2] + (x - x_1)(x - x_2)f[x_0, x_1, x_2] \\ &= \sqrt{1/2} - 0.90032(x - \pi/4) - 0.33575(x - \pi/4)(x - \pi/2) \end{aligned}$$

In general

$$p_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots + (x - x_0)\cdots(x - x_{n-1})f[x_0, x_1, \ldots, x$$

Coefficients are computed by the recurrance

$$\begin{aligned} f[x_i] &= f(x_i) \\ f[x_i, x_{i+1}] &= \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} \\ f[x_0, \ldots, x_n] &= \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}. \end{aligned}$$

If we have, say, the interpolation polynomial

$$p_2(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

it is fairly easy to "add a point" x_3 and produce the interpolation polynomial

$$p_3(x) = p_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

provided that we have the divided difference table

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$
x_1	$f[x_1]$	$f[x_1, x_2]$	
x_2	$f[x_2]$		

We simply use $f[x_3]$ to produce $f[x_2, x_3]$, $f[x_1, x_2, x_3]$ and $f[x_0, x_1, x_2, x_3]$ by the recurrances. Then we have the new table

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	
x_0	$f[x_0]$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$\mathbf{f[x_0, x_1, x_2, x_3]}$
x_1	$f[x_1]$	$f[x_1, x_2]$	$\mathbf{f[x_1, x_2, x_3]}$	
x_2	$f[x_2]$	$\mathbf{f[x_2, x_3]}$		
$\mathbf{x_3}$	$\mathbf{f[x_3]}$			

The new diagonal in boldface is all that needs to be stored if we wish to add more points.

Lastly, an important connection between divided differences and derivatives.

Theorem 1 *If $f(x_i) = y_i$, for $i = 0, \ldots, n$ is n times continuously differentiable and*

$$a = \min_{0 \le i \le n} x_i, \quad b = \max_{0 \le i \le n} x_i$$

then for some $\xi \in (a, b)$

$$f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}}{n!}.$$

Proof: Let

$$r(x) = f(x) - p_n(x).$$

Note that $r(x_i) = 0, i = 0, \ldots, n$ at $n+1$ points. Thus if we take n derivatives

$$r^{(n)}(x) = f^{(n)}(x) - p_n^{(n)}(x) = 0$$

for at least one point $x = \xi$ that is strictly between a and b above. It is straightforward to show that

$$p_n^{(n)}(x) = \frac{f[x_0, \ldots, x_n]}{n!},$$

so

$$r^{(n)}(\xi) = f^{(n)}(\xi) - \frac{f[x_0, \ldots, x_n]}{n!} = 0.$$

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