Computer Science/Mathematics 455
Lecture Notes
Lecture # 20

The Problem

Find a polynomial

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n \tag{1}$$

such that

$$p_n(x) \approx f(x), \quad x \in [a, b]$$

$f(x)$ continous function, preferably with a few continuous derivatives.

In theory, we can solve this problem! Theorem 1, p.75.

Theorem 1 (Weierstrass) *Let f be a continuous function on $[a, b]$. Given $h > 0$, there exists a polynomial $p_{N(h)}$ of degree $N(h)$ such that*

$$\max_{x \in [a,b]} |f(x) - p_{N(h)}(x)| < h. \tag{2}$$

Therefore there exists a sequence of polynomials $\{p_n(x)\}$ such that

$$\lim_{n \to \infty} \max_{x \in [a,b]} |f(x) - p_{N(h)}(x)| = 0.$$

The proof of this theorem is constructive, that is, it actually produces a polynomial that satisfies (2), but sequence produced this way can converge arbitrarily slowly.

Sidebar – Horner's Rule

To evaluate the polynomial (1) efficiently use the algorithm.

$p = a_n * x + a_{n-1}$;
for $k = n - 2 : -1 : 0$
  $p = p * x + a_k$;
end;

MATLAB code on p.76.

Example 1

$$\begin{aligned} p_5(x) &= 4x^5 - 3x^4 + 2x^3 - 10x^2 + 5x - 6. \\ &= ((((4x-3)x+2)-10)x+5)x-6. \end{aligned}$$

Interpolation – not quite approximation Another name for interpolation is collocation (usually used in the context of solving partial differential equations).

Your book derives interpolation two different ways which we will discuss later. I will give a third.

Start with $n = 2$.

$$p_2(x) = a_0 + a_1 x + a_2 x^2, \quad x_0, x_1, x_2$$

Such that

$$p_2(x_i) = f(x_i), \quad i = 1, 2, 3.$$

This leads to three equations in three unknowns a_0,a_1,a_2

$$\begin{aligned} a_0 + a_1 x_0 + a_2 x_0^2 &= f(x_0) \\ a_0 + a_1 x_1 + a_2 x_1^2 &= f(x_1) \\ a_0 + a_1 x_2 + a_2 x_2^2 &= f(x_2) \end{aligned}$$

In matrix form this is

$$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{pmatrix}.$$

The matrix is a *Vandermonde* matrix. Now to an example.

Example 2

$$f(x) = \cos x$$

Take

$$x_0 = 0, \quad x_1 = \pi/4, \quad x_2 = \pi/2.$$

That leads to the linear system

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & \pi/4 & \pi^2/16 \\ 1 & \pi/2 & \pi^2/4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{1/2} \\ 0 \end{pmatrix}.$$

The solution is

$$\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.1092 \\ -0.3352 \end{pmatrix}$$

which gives us the polynomial

$$p_2(x) = 1 - 0.1092x - 0.3357x^2.$$

This can be produced with the following MATLAB code.

$x = [0; pi/4; pi/2];$
$V = vander(x);$ (columns will be in reverse order)
$[L, U] = lu(V);$
$a = L\backslash b;$
$a = U\backslash a;$

The coefficients of a are in reverse order. You can evaluate the polynomial using the MATLAB command $y = polyval(a, x)$.

For instance the commands

$xx = 0{:}\, pi/20{:}\, pi/2;$
$yy = polyval(a, xx);$
$yexact = cos(xx);$
$err = yexact - yy;$
plot(xx,err);

produces a plot of the error in interpolation.

More generally, to obtain $p_n(x)$ of the form (1) such that

$$p_n(x_i) = y_i = f(x_i), \quad i = 0, \ldots, n$$

we obtain the $(n+1) \times (n+1)$ linear system

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$V_n \mathbf{a} = \mathbf{y}$$

This the Vandermonde system of equations. The following can be shown about it.

Theorem 2 *If* $x_0, x_1, \ldots, x_n$ *are distinct then*

- V_n *is nonsingular*
- *The vector* **a** *is unique.*
- *The polynomial* $p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$ *solving the interpolation problem exists and is unique.*

Later, we will prove this theorem.