

Computer Science/Mathematics 455
Lecture Notes
Lecture # 15

Before discussing Newton's method, we need to review some important concepts from calculus.

Intermediate Value Thm

Let f be a continuous function on the interval $a \leq x \leq b$ (or $[a, b]$). Let

$$M = \max_{a \leq x \leq b} f(x), \quad m = \min_{a \leq x \leq b} f(x)$$

For any $\rho \in [m, M]$, there exists a point $\xi \in [a, b]$ such that

$$f(\xi) = \rho.$$

Taylor's Theorem (with Lagrange Remainder)

If f has n continuous derivatives on $[a, b]$ and $f^{(n+1)}$ exists on (a, b) , then for any points c and x in $[a, b]$

$$f(x) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(c)(x-c)^k + E_n(x)$$

where, for some point ξ between c and x ,

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1}$$

The error term may be new to you, but in numerical analysis it is very important to bound it.

Example 1

$$f(x) = e^x, \quad c = 0, x = 0.1$$

We have that

$$, \quad f^{(k)}(x) = e^x, \quad \text{for all } k$$

thus

$$f^{(k)}(0) = 1, \quad \text{for all } k$$

The first two terms come to

$$f(x) = 1 + x + \frac{x^2}{2!} + E_2(x)$$

$$\begin{aligned} E_2(x) &= \frac{1}{3!} f^{(3)}(\xi) x^3 \\ &= \frac{1}{3!} e^\xi \cdot x^3 \end{aligned}$$

but we have no idea what ξ is!

But we can bound it.

$$|E_2(0, 1)| \leq \left| \frac{1}{6} e^{0.1} (0.1)^3 \right| = 1.8420 \times 10^{-4}$$

This tells you that

$$p(x) = 1 + x + x^2/2$$

is within 1.8420×10^{-4} of e^x for all $x \in [0, 0.1]$.

Folk Wisdom

Note that x and c are close together in this example. Taylor series should be thought of as a LOCAL approximation.

If we need many terms of the series, we are probably doing this approximation the wrong way!

Two important corollaries of Taylor's Theorem.

Mean-Value Theorem

If f is continuous on $[a, b]$ and f' exists on (a, b) then for $x, c \in [a, b]$,

$$f(x) = f(c) + f'(\xi)(x - c)$$

where ξ is between c and x .

Example 2

$$f(x) = \sin x, \quad c = 0, x = 0.01$$

$$\begin{aligned}
f(0) &= 0 = \sin 0 \\
f(x) - f(c) &= f'(\xi)(x - c) = f'(\xi)x \\
f'(x) &= \cos x \quad \text{for all } x \\
f'(x) &\in [0, 1] \text{ for all } x \in [0, 0.01] \\
f(x) &\leq \max_{0 \leq \xi \leq 0.01} f'(\xi)x \leq 1 \cdot 0.01 = 0.01 \\
f(x) &\geq \min_{0 \leq \xi \leq 0.01} f'(\xi)x = \cos 0.01 \cdot 0.01 = 0.0099995.
\end{aligned}$$

Actual value, $f(0.01) = \sin 0.01 = 0.0099998$, so, in this case, the theorem provides good upper and lower bounds.

Rolle's Theorem If f is continuous on $[a, b]$, f' exists on (a, b) , and $f(a) = f(b) = 0$, then $f'(\xi) = 0$ for some $\xi \in (a, b)$.

Now we can discuss Newton's method.

Once again, we want to solve

$$f(\alpha) = 0, \quad \alpha \in \mathbf{R}. \tag{1}$$

We assume that f has at least two continuous derivatives. We also assume that we have an initial guess x_0 that is a "pretty good" guess. Expanding f in a Taylor series about x_0 , we obtain

$$f(\alpha) = f(x_0) + (\alpha - x_0)f'(x_0) + (\alpha - x_0)^2 f''(\xi)/2$$

for some ξ between α and x_0 . We ignore the last term and solve

$$f(\alpha) \approx f(x_0) + (\alpha - x_0)f'(x_0) = 0.$$

Thus obtaining

$$\alpha \approx x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We then keep doing this over and over again. Thus the sequence $\{x_n\}$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We hope that

$$\lim_{n \rightarrow \infty} x_n = \alpha$$

and that convergence is rapid. Under certain circumstances, we have all of that. Define the following parameters

$$\begin{aligned}\beta &= \frac{1}{|f'(x_0)|}, \\ \eta &= \frac{|f(x_0)|}{|f'(x_0)|}, \\ K &= \max_{x \in [a,b]} |f''(x)|, \\ \gamma &= K\beta\eta, \\ r &= 2\gamma / (K\beta * (1 + \sqrt{1 - 2 * \gamma})).\end{aligned}$$

Theorem 1 *If $\gamma < 0.5$, $x_0 - r \geq a$, $x_0 + r \leq b$, then the Newton iterates $\{x_n\}$ satisfy*

$$\lim_{n \rightarrow \infty} x_n = \alpha, \quad f(\alpha) = 0$$

for a unique value α . Moreover,

$$\lim_{n \rightarrow \infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = \frac{f''(\alpha)}{2f'(\alpha)}.$$

Note that the requirements above are complex. The value of $f'(x_0)$ must be bounded away from zero, $f''(x)$ must be bounded throughout the region, and the further restrictions on them are strict. When these conditions are, we have quadratic convergence near the root. Thus, we hope and expect that Newton's method will converge rapidly.

Example 3

$$f(x) = \cos x - x = 0.$$

The first two derivatives of f are

$$f'(x) = -(\sin x + 1)$$

$f''(x) = -\cos x$. Let $a = 0$ and $b = \pi/2$, so

$$K = \max_{x \in [a,b]} |f''(x)| = \max_{x \in [0, \pi/2]} |\cos x| = 1.$$

We choose $x_0 = \pi/4$. Thus

$$\beta = \frac{1}{|f'(x_0)|} = 1/|1 + \sin \pi/4| \approx 0.5858,$$

$$\eta = \frac{|f(x_0)|}{|f'(x_0)|} = 4.5862 \times 10^{-2},$$

$$r = 2\gamma/(K\beta * (1 + \sqrt{1 - 2 * \gamma})) = 4.6496 \times 10^{-2}.$$

This shows that Newton's method will converge and that the value α is very close to x_0 . Indeed, it obtains the correct answer to 14 significant digits in 4 iterations. The initial guess has at least one digit right (perhaps more).

There are some other sufficient conditions for convergence. These are sometimes much easier to verify than the complex set of conditions given above.

In the following example, Newton's method converges for every initial guess except $x_0 = 0$.

Example 4

$$f(x) = x^2 - a.$$

We have

$$f'(x) = 2x,$$

$$f''(x) = 2.$$

For all $x_0 >$, Newton's method converges to $\alpha = \sqrt{a}$. If $x_0 < 0$, it converges to $\alpha = -\sqrt{a}$.

Moreover, the iteration simplifies to a really nice algebraic form.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right).$$

This is very nice, useful formula. A good guess is important. Suppose I guess $x_0 = 50$. Then I get

$$x_1 = 25.02, \quad x_2 = 12.55, \dots, x_5 = 1.9672,$$

$$x_6 = 1.4919, \quad x_7 = 1.4162, \quad x_8 = 1.4142(\text{six digits})$$

The value x_9 has 13 digits right. However, the iterations before x_6 are a waste of time. Quadratic convergence is not visible before that.

If we let $x_0 = 1.5$, x_2 is accurate to six digits, x_3 is accurate to 12 digits, and x_4 is accurate to IEEE double precision. We would like for Newton's method to behave in this fashion as often as possible.

However, with a sensible initial guess, this is an excellent way to compute the square root of a number quickly.

We give a theorem next time which shows why this iteration always converges.