CSE/Math 455
Lecture # 11

Vector Norms and Matrix Norms

Last time (Lecture # 10) we considered vector norms. For the most part, we use these three

$$
\begin{aligned}
\|\mathbf{x}\|_2 &= \left(\mathbf{x}^T\mathbf{x}\right)^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \\
\|\mathbf{x}\|_1 &= \sum_{i=1}^{n} |x_i|, \\
\|\mathbf{x}\|_\infty &= \max_{1\le i\le n} |x_i|.
\end{aligned}
$$

They are called the two-norm, the one-norm, and the infinity-norm. In MATLAB, these are **norm**(x),**norm**$(x,1)$ and **norm**$(x,'inf')$ provided that $\mathbf{x}$ is a column vector!

They satisfy the inequality

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1.$$

Equality occurs here when

$$\mathbf{x} = \mathbf{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is in the jth position. This is the jth column of the identity matrix.

Other inequalities are

$$
\begin{aligned}
\|\mathbf{x}\|_1 &\le n\|\mathbf{x}\|_\infty, \\
\|\mathbf{x}\|_2 &\le \sqrt{n}\|\mathbf{x}\|_\infty, \\
\|\mathbf{x}\|_1 &\le \sqrt{n}\|\mathbf{x}\|_2.
\end{aligned}
$$

These inequalities are equality whenever

$$\mathbf{x} = \begin{pmatrix} \pm 1 \\ \pm 1 \\ \vdots \\ \pm 1 \end{pmatrix}.$$

To see a geometric version of the relationship among these three norms, look at www.cse.psu.edu/∼barlow/cse455/unitcircles.pdf.

We also attach norms to matrices. The fundamental rules are the same.

Definition 1 *A norm in* $\mathrm{R}^{m\times n}$ *is a function* $\|\cdot\|$ *mapping* $\mathrm{R}^{m\times n}$ *into* R *satisfying the following three axioms*

1. $\|A\| \geq 0$;
   $\|A\| = 0$ *if and only if* $A = 0, \quad A \in \mathrm{R}^{m\times n}$
2. $\|\alpha A\| = |\alpha|\|A\| \quad A \in \mathrm{R}^{m\times n}, \alpha \in \mathrm{R}$
3. $\|A + B\| \leq \|A\| + \|B\| \quad A, B \in \mathrm{R}^{m\times n}$.

This definition is isomorphic to the definition of a vector norm on R^{mn}. For example, the Frobenius norm defined by

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2} \tag{1}$$

is isomorphic to the two-norm on R^{mn}. It is the MATLAB command $norm(A,' fro')$.

Example 1

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -9 \end{pmatrix}.$$

We have $\|A\|_F = 16.88819$.

This norm treats A as a "box of numbers." (*Of course, that is all a matrix is, but never mind that.*)

We want to think of A as an operator.

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}.$$

If we know how "large" $\mathbf{x}$ is, can we get a bound on how "large" $A\mathbf{x}$ is?

Operator Norms (or Induced Norms)

Begin with a vector norm $\|\cdot\|_\alpha$. Define the induced norm or operator norm associated with it as

$$\|A\|_\alpha = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|_\alpha}{\|\mathbf{x}\|_\alpha}.$$

This is the so-called "maximum stretch." The term "sup" is a generalization of "maximum," it is also called a least upper bound. For instance, the set $x < 3$ has no maximum, but its "sup" is 3.

An easier way to look at this is to take

$$\mathbf{y} = \mathbf{x}/\|\mathbf{x}\|_\alpha,$$

or

$$\mathbf{x} = \|\mathbf{x}\|_\alpha \mathbf{y}, \quad \|\mathbf{y}\|_\alpha = 1.$$

Then

$$\begin{aligned} \frac{\|A\mathbf{x}\|_\alpha}{\|\mathbf{x}\|_\alpha} &= \frac{\|A\|\mathbf{x}\|_\alpha \mathbf{y}\|}{\|\mathbf{x}\|_\alpha} \\ &= \frac{\|\mathbf{x}\|_\alpha}{\|\mathbf{x}\|_\alpha}\|A\mathbf{y}\|_\alpha = \|A\mathbf{y}\|_\alpha. \end{aligned}$$

Thus

$$\|A\|_\alpha = \max_{\|\mathbf{x}\|_\alpha = 1} \|A\mathbf{x}\|_\alpha.$$

Using an argument from classical analysis (thus way beyond the scope of this course), one can show that for all matrices A and induced norms $\|\cdot\|_\alpha$ there exists a vector $\mathbf{x}^*$ such that

$$\|A\mathbf{x}^*\| = \|A\|_\alpha = \max_{\|\mathbf{x}\|_\alpha = 1} \|A\mathbf{x}\|_\alpha.$$

with $\|\mathbf{x}^*\| = 1$. This vector has no name in the literature, I call it the "magic" vector.

Look at $\|\mathbf{x}\|_1$,$\|\mathbf{x}\|_\infty$, and $\|\mathbf{x}\|_2$ and the operator norms they generate.

I will derive an expression for $\|A\|_\infty$ and its "magic" vector. I did it for $\|A\|_1$ in class. The derivations are very similar. We have

$$\|A\|_\infty = \max_{\|\mathbf{x}\|_\infty = 1} \|A\mathbf{x}\|_\infty.$$

We have that

$$(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij} x_j.$$

Thus

$$\begin{aligned}
\|A\mathbf{x}\|_\infty &= \max_{1\le i\le m} |(A\mathbf{x})_i| = \max_{1\le i\le m} |\sum_{j=1}^{n} a_{ij}x_j| \\
&\le \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}x_j| \\
&= \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}||x_j| \\
&\le \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| \max_{1\le k\le n} |x_k| \\
&= \max_{1\le k\le n} |x_k| \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| \\
&= \|\mathbf{x}\|_\infty \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.
\end{aligned}$$

Since $\|\mathbf{x}\|_\infty = 1$, we have

$$\|A\mathbf{x}\|_\infty \le \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$

for all $\mathbf{x}$ such that $\|\mathbf{x}\|_\infty = 1$. Thus

$$\|A\|_\infty \le \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|. \tag{2}$$

To show that (2) is, in fact, an equality, choose an $\mathbf{x}_\infty^*$ such that

$$\|A\mathbf{x}_\infty^*\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$$

and $\|\mathbf{x}_\infty^*\|_\infty = 1$. For that, we find a row, i_{max}, such that

$$\sum_{j=1}^n |a_{i_{max},j}| = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|.$$

Then let

$$\mathbf{x}_\infty^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \tag{3}$$

where

$$x_j^* = \text{sign}(a_{i_{max},j}). \tag{4}$$

Thus $\mathbf{x}_\infty^*$ is a vector of ± 1's. Then

$$(A\mathbf{x}^*)_{i_{max}} = \sum_{j=1}^n |a_{i_{max},j}| = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|$$

and so

$$\|A\mathbf{x}^*\|_\infty \ge |(A\mathbf{x}^*)_{i_{max}}| = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|.$$

Thus

$$\|A\|_\infty \ge \|A\mathbf{x}^*\|_\infty \ge \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}|.$$

Combining this with (2), we have

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^n |a_{ij}| \tag{5}$$

with the "magic" vector $\mathbf{x}_\infty^*$ in (3)–(4). In MATLAB, the expression (5) is $norm(A,'inf')$ or $norm(A, inf)$.

As we showed in class,

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^m |a_{ij}| = \|A^T\|_\infty. \tag{6}$$

In MATLAB, it is $\mathbf{norm}(A,1)$. Its "magic" vector is

$$\mathbf{x}_1^* = \mathbf{e}_{j_{max}}$$

where j_{max} denotes a column such that

$$\sum_{i=1}^{m} |a_{i,j_{max}}| = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$

Take the matrix A given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & -9 \end{pmatrix}.$$

Its absolute row sums are 6,15, and 24. Thus

$$\|A\|_\infty = 24, \quad \mathbf{x}_\infty^* = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

since the third row is the maximum row.

Its absolute column sums are 12,15, and 18. Thus

$$\|A\|_1 = 18, \quad \mathbf{x}_1^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

since the third column is the maximum column.

What is the induced norm for the two-norm? Next time.