CSE/Math 451.1
Numerical Computation
Fall 2007
Midterm Examination I
8 October 2007

Name:
Student Number:
Open Book, Open Notes

Question	Possible	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

Please write name on this page only! Write your student number on every other page. Use your time carefully. Good luck. Show your work, please! In many cases, just the answer gets ZERO credit!

Student Number:

1. **Floating Point Arithmetic.**

   (a) Consider the expression

   $$f(x) = \frac{\sqrt{1+2x^2} - \sqrt{1-x^2}}{x^2}.$$

   Calculate $f(10^{-20})$ accurately.

   (b) Find the roots of the quadratic equation $2x^2 - 10^{20}x + 1 = 0$ accurately. It is possible to do this exercise without a calculator.

For part (a), we simply use the trick

$$\begin{aligned} f(x) &= \frac{\sqrt{1+2x^2} - \sqrt{1-x^2}}{x^2} \frac{\sqrt{1+2x^2} + \sqrt{1-x^2}}{\sqrt{1+2x^2} + \sqrt{1-x^2}} \\ &= \frac{1+2x^2-(1-x^2)}{x^2(\sqrt{1+2x^2} + \sqrt{1-x^2})} \\ &= \frac{3x^2}{x^2(\sqrt{1+2x^2} + \sqrt{1-x^2})} \\ &= \frac{3}{(\sqrt{1+2x^2} + \sqrt{1-x^2})} \end{aligned}$$

Then

$$f(10^{-20}) = \frac{3}{(\sqrt{1+2\cdot 10^{-40}} + \sqrt{1-10^{-40}})} \approx= 3/2.$$

For part (b), since $b^2 - 4ac = 10^{40} - 8 > 0$ we have two real roots. The large one may be computed by the formula

$$\begin{aligned} x_1 &= \frac{-b - \text{sign}(b)\sqrt{b^2-4ac}}{2a} \\ &= \frac{10^{20} + \sqrt{10^{40}-8}}{4} \\ &= 0.5 \cdot 10^{20} = 5 \cdot 10^{19}. \end{aligned}$$

To get the smaller root, we can use one of two formulas.

$$\begin{aligned} x_2 &= \frac{-2c}{b + \text{sign}(b)\sqrt{b^2 - 4ac}} \\ &= -2/(-10^{20} - \sqrt{10^{40} - 8}) \\ &\approx 10^{-20}. \end{aligned}$$

The formula

$$x_2 = c/(ax_1)$$

yields the same result.

Student Number:

2. **Linear Systems: Direct Methods**

   (a) Compute the PLU factorization of

$$A = \begin{pmatrix} -2.7 & -0.1 & 4.2 \\ 3 & -1 & 2 \\ -0.6 & 2.2 & -2.4 \end{pmatrix}$$

   by Gaussian elimination with partial pivoting. Display the L and U factors and the permutation.

   (b) Use your factorization to solve

$$A\mathbf{x} = \mathbf{b}$$

   with

$$\mathbf{b} = \begin{pmatrix} 1.6 \\ 6 \\ -5.2 \end{pmatrix}.$$

For part (a), we start by letting

$$\mathbf{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

track the permutations. The elements in the lower triangle will be in **boldface**. The first permutation exchanges the first and second rows yielding

$$\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad P_1 A = \begin{pmatrix} 3 & -1 & 2 \\ -2.7 & -0.1 & 4.2 \\ -0.6 & 2.2 & -2.4 \end{pmatrix}$$

Elimination on the first column yield, $\ell_{21} = -2.7/3 = -0.9$, $\ell_{31} = -0.6/3 = -0.2$. Thus

$$A^{(1)} = \begin{pmatrix} 3 & -1 & 2 \\ \mathbf{-0.9} & -1 & 6 \\ \mathbf{-0.2} & 2 & -2 \end{pmatrix}$$

To do the second elimination, we permute rows two and three, yielding

$$\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad P_2 A^{(1)} = \begin{pmatrix} 3 & -1 & 2 \\ \mathbf{-0.2} & 2 & -2 \\ \mathbf{-0.9} & -1 & 6 \end{pmatrix}$$

Then, $\ell_{32} = -1/2 = -0.5$ and we have

$$A^{(2)} = \begin{pmatrix} 3 & -1 & 2 \\ \mathbf{-0.2} & 2 & -2 \\ \mathbf{-0.9} & \mathbf{-0.5} & 5 \end{pmatrix}$$

yielding the L and U factors

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{-0.2} & 1 & 0 \\ \mathbf{-0.9} & \mathbf{-0.5} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & -1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{pmatrix}$$

The permutation P is represented by the vector

$$\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

or

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

To solve (b), we solve first

$$L\mathbf{y} = \mathbf{b}(\mathbf{p}) = \begin{pmatrix} 6 \\ -5.2 \\ 1.6 \end{pmatrix}$$

That yields

$$\begin{aligned} y_1 &= 6 \\ y_2 &= -5.2 - (-0.2)y_1 = -4 \\ y_3 &= 1.6 - (-0.9)y_1 - (-0.5)y_2 = 1.6 + 5.4 - 2 = 5 \end{aligned}$$

Then we solve

$$U\mathbf{x} = \mathbf{y}$$

yielding

$$\begin{aligned} x_3 &= y_3/5 = 5/5 = 1 \\ x_2 &= (y_2 - (-2)x_3)/2 = (-4+2)/2 = -1 \\ x_1 &= (y_1 - (-1)x_2 - 2x_3)/3 = (6 - (-1)(-1) - 2 \cdot 1)/3 = 1 \end{aligned}$$

So the solution is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Student Number:

3. **Condition Numbers.** Let A be the matrix

$$A = \begin{pmatrix} 0.1 & 1 \\ 0 & 0.001 \end{pmatrix}.$$

(a) What is

$$\kappa_1(A) = \|A^{-1}\|_1 \|A\|_1?$$

Give a vector such that

$$\|A^{-1}\mathbf{x}\|_1 = \|A^{-1}\|_1$$

and $\|\mathbf{x}\|_1 = 1$.

(b) What is

$$\kappa_\infty(A) = \|A^{-1}\|_\infty \|A\|_\infty.$$

Give a vector such that

$$\|A^{-1}\mathbf{x}\|_\infty = \|A^{-1}\|_\infty$$

and $\|\mathbf{x}\|_\infty = 1$. [Hint: The inverse of the upper triangular matrix

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

is

$$A^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}c^{-1}b \\ 0 & c^{-1} \end{pmatrix}.$$

First, the formula yields

$$A^{-1} = \begin{pmatrix} 10 & -10000 \\ 0 & 1000 \end{pmatrix}.$$

Thus for part (a)

$$\|A\|_1 = \max\{0.1, 1.001\} = 1.001, \quad \|A^{-1}\|_1 = \max\{10, 11000\} = 11000$$

so that

$$\kappa_1(A) = \|A\|_1 \|A^{-1}\|_1 = (1.001)(11000) = 11,011.$$

For A^{-1}, the second column is the maximum column, so

$$\mathbf{x}_1^* = \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For part (b), we have that

$$\|A\|_\infty = \max\{1.1, 0.001\} = 1.1, \quad \|A^{-1}\|_\infty = \max\{10,010, 1000\} = 10,010$$

thus

$$\kappa_{\text{inf}}(A) = \|A\|_\infty \|A^{-1}\|_\infty = (1.1)(10,000) = 11,011.$$

The first row of A^{-1} is the maximum row, so

$$\mathbf{x}_{\text{inf}}^* = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are the signs of that row.

Student Number:

4. **Linear Systems: Iterative Methods.** Consider the linear system

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}.$$

(a) Using the initial guess

$$\mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

do two iterations of the Jacobi method.

(b) Compute the Jacobi iteration matrix $G = -D^{-1}(L+U)$ and give the values of $\|G\|_1$ and $\|G\|_\infty$. Should we expect convergence?

For part (a), the basic iteration step is

$$x_i^{(k+1)} = (b_i - \sum_{j \neq i} a_{ij} x_j^{(k)})/a_{ii}.$$

which for this matrix are the three steps

$$\begin{aligned} x_1^{(k+1)} &= (-2 - x_2^{(k)})/3, \\ x_2^{(k+1)} &= (4 - x_1^{(k)} - x_3^{(k)})/4, \\ x_3^{(k+1)} &= (6 - x_2^{(k)})/5. \end{aligned}$$

Thus $\mathbf{x}^{(1)}$ is given by

$$\begin{aligned} x_1^{(1)} &= (-2 - 0)/3 = -2/3 \\ x_2^{(1)} &= (4 - 0 - 0)/4 = 1 \\ x_3^{(1)} &= (6 - 0)/5 = 6/5 \end{aligned}$$

and $\mathbf{x}^{(2)}$ is given by

$$\begin{aligned} x_1^{(2)} &= (-2 - 1)/3 = -1 \\ x_2^{(2)} &= (4 - (-2/3) - 6/5)/4 = 13/15 \\ x_3^{(2)} &= (6 - 1)/5 = 1 \end{aligned}$$

Thus

$$\mathbf{x}^{(2)} = \begin{pmatrix} -1 \\ 13/15 \\ 1 \end{pmatrix}$$

FYI, the solution to this linear system is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

For part (b), the iteration matrix is

$$\begin{aligned} G = -D^{-1}(L+U) &= -\begin{pmatrix} 3 & & \\ & 4 & \\ & & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1/3 & 0 \\ -1/4 & 0 & -1/4 \\ 0 & -1/5 & 0 \end{pmatrix} \end{aligned}$$

The absolute column sums of G yield

$$\|G\|_1 = \max\{1/4, 8/15, 1/4\} = 8/15.$$

The absolute row sum of G yield

$$\|G\|_\infty = \max\{1/3, 1/2, 1/5\} = 1/2.$$

Since both norms are strictly less than 1, this iteration converges.

For your information, the asymtotic convergence of this iteration is actually a bit faster than either of these norms shows. The spectral radius or maximum absolute eigenvalue is $\rho(G) = \sqrt{2/15} \approx 0.36515$. Note that

$$\|\mathbf{x}^{(2)} - \mathbf{x}\|_\infty = 1/15$$

after a mere two iterations.