CSE/Math 451.1
Numerical Computation
Fall 2007
Midterm Examination II
14 November 2007

Name:
Student Number:
Open Book, Open Notes

Question	Possible	Score
1	25	
2	25	
3	25	
4	25	
Total	100	

Please write name on this page only! Write your student number on every other page. Use your time carefully. Good luck. Show your work, please! In many cases, just the answer gets ZERO credit!

Student Number:

1. **Linear Equations – Iterative Methods.**
   Consider the linear system

$$\begin{pmatrix} 2 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 4 & 1 \\ & & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -2 \\ 1 \end{pmatrix}$$

   (a) Do one iteration of Gauss–Seidel with $\mathbf{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

   (b) Construct the **Jacobi** iteration matrix

$$G = -D^{-1}(L+U).$$

   Compute $\|G\|_1$ and determine if the Jacobi iteration converges.

(a) A Gauss–Seidel iteration goes as follows

$$\begin{aligned}
x_1^{(1)} &= (b_1 - x_2^{(0)})/2 = (-1-0)/2 = -1/2 \\
x_2^{(1)} &= (b_2 - x_1^{(1)} - x_3^{(0)})/4 = (2-(-1/2)-0)/4 = 5/8 \\
x_3^{(1)} &= (b_3 - x_2^{(1)} - x_4^{(0)})/4 = (-2-5/8-0)/4 = -21/32 \\
x_4^{(1)} &= (b_4 - x_3^{(1)})/2 = (1-(-21/32))/2 = 53/64
\end{aligned}$$

For your information, the solution to the linear system is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

(b) The iteration matrix G for the Jacobi iteration is

$$G = -D^{-1}(L+U) = -\begin{pmatrix} 2 & & & \\ & 4 & & \\ & & 4 & \\ & & & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1/2 & & \\ -1/4 & 0 & -1/4 & \\ & -1/4 & 0 & -1/4 \\ & & -1/4 & 0 \end{pmatrix}$$

The absolute column sums of G are

$$\{1/4, 3/4, 3/4, 1/4\}$$

so

$$\|G\|_1 = \max_{1 \le j \le n} \|G\mathbf{e}_j\|_1 = 3/4.$$

Since $\|G\|_1 = 3/4 < 1$. the Jacobi iteration converges.

Student Number:

2. **Nonlinear Equations.** Consider the nonlinear equation

$$f(x) = x^2 - 2 = 0 \tag{1}$$

with roots $x = \pm\sqrt{2}$.

(a) Show that Newton's method for the finding the roots of (1) yields the iteration

$$x_{n+1} = \frac{1}{2}(x_n + 2/x_n).$$

(b) If $\alpha = \sqrt{2}$ show that

$$x_{n+1} - \alpha = \frac{1}{2x_n}(x_n - \alpha)^2.$$

[Hint: Use the formulation $x_{n+1} = x_n - f(x_n)/f'(x_n)$, note that

$$f(x) = x^2 - 2 = (x - \alpha)(x + \alpha),$$

and use some algebra.]

(a) Since $f'(x) = 2$, simply use

$$\begin{aligned} x_{n+1} &= x_n - f(x_n)/f'(x_n) \\ &= x_n - (x_n^2 - 2)/(2x_n) \\ &= x_n - 1/2x_n + 1/x_n = \frac{1}{2}(x_n + 2/x_n) \end{aligned}$$

(b) I'll give a different derivation from the one I gave in class, more like the one most folks did. We have that

$$\begin{aligned} x_{n+1} - \alpha &= \frac{1}{2}(x_n + 2/x_n) - \alpha \\ &= \frac{1}{2x_n}(x_n^2 + 2 - 2x_n\alpha) \\ &= \frac{1}{2x_n}(x_n^2 + \alpha^2 - 2x_n\alpha) \\ &= \frac{1}{2x_n}(x_n - \alpha)^2 \end{aligned}$$

Student Number:

3. **Numerical Integration.** !Consider the integral

$$I = \int_{-1}^{1} f(x)dx.$$

Let

$$I_2(f) = w_0 f(-3/4) + w_1 f(0) + w_2 f(3/4).$$

Find the weights w_0, w_1, w_2 such that $I_2(f)$ is exact for polynomials of degree two or less.

Answer: The weight are the solution to the Vandermonde system

$$\begin{pmatrix} 1 & 1 & 1 \\ -3/4 & 0 & 3/4 \\ 9/16 & 0 & 9/16 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} 1dx \\ \int_{-1}^{1} xdx \\ \int_{-1}^{1} x^2dx \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2/3 \end{pmatrix}$$

One can straightforwardly do Gaussian elimination (which I will demonstrate later) or observe that the middle equation is

$$-3/4w_0 + 3/4w_2 = 0$$

implying that $w_0 = w_2$. Substituting this into the third equation yields

$$18/16w_0 = 2/3$$

yielding

$$w_0 = w_2 = 16/27.$$

The third weight w_1 is recovered from the first equation by

$$w_1 = 2 - w_0 - w_2 = 22/27.$$

If, however, you are in a more virtuous mood, you can do Gaussian elimination with partial pivoting. Here goes. It turns out that you never pivot– so I won't bother with the permutation vector $\mathbf{p} = (1, 2, 3)^T$. The first elimination (multipliers in bold face) is

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{-3/4} & 3/4 & 3/2 \\ \mathbf{9/16} & -9/16 & 0 \end{pmatrix}$$

The second one yields

$$\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{-3/4} & 3/4 & 3/2 \\ \mathbf{9/16} & \mathbf{-3/4} & 9/8 \end{pmatrix}$$

Thus $A = LU$ where

$$L = \begin{pmatrix} 1 & & \\ -3/4 & 1 & \\ 9/16 & -3/4 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 \\ & 3/4 & 3/2 \\ & & 9/8 \end{pmatrix}.$$

Forward substitution

$$L\mathbf{y} = \mathbf{b}$$

yields

$$\begin{aligned} y_1 &= 2 \\ y_2 &= 0 - (-3/4)y_1 = 3/2 \\ y_3 &= 2/3 - (9/16)y_1 - (-3/4)y_2 = 2/3 - (9/16)2 - (-3/4)(3/2) = 2/3 \end{aligned}$$

Back substituion

$$U\mathbf{w} = \mathbf{y}$$

yields

$$\begin{aligned} w_2 &= y_3/(9/8) = 16/27 \\ w_1 &= (y_2 - 3/2w_1)/(3/4) = 22/27 \\ w_0 &= y_1 - w_1 - w_2 = 16/27. \end{aligned}$$

Student Number:

4. **Splines.** Determine whether the following function is a quadratic spline

$$s(x) = \begin{cases} x^2+2 & -1 \le x \le 0 \\ -x^2+2 & 0 \le x \le 1 \\ x^2/2 - 2x + 5/2 & 1 \le x \le 2. \end{cases}$$

Is it a cubic spline? Explain briefly.

Answer: You need the two derivatives

$$s'(x) = \begin{cases} 2x & -1 < x < 0 \\ -2x & 0 < x < 1 \\ x-2 & 1 < x < 2. \end{cases}$$

and

$$s''(x) = \begin{cases} 2 & -1 < x < 0 \\ -2 & 0 < x < 1 \\ 1 & 1 < x < 2. \end{cases}$$

It is clearly not a cubic spline, since the second derviative is not continuous. To see if it is a quadratic spline, we check continuity of the function and of the first derivative at the knots.

We have that

$$\begin{aligned} s_0(0) &= s_1(0) = 2 \\ s_1(1) &= s_2(1) = 1 \end{aligned}$$

so $s(x)$ is continuous. However, although

$$s_0'(0) = s_1'(0) = 0$$

we have that

$$s_1'(1) = -2 \neq s_2'(1) = -1.$$