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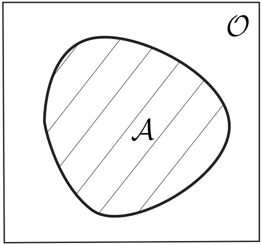
IWASEP VI, Pen State, May 21–May 24

Consider multi-scale problems, something like

$\mathbf{H}_\kappa = -\triangle + \kappa^2(1 - \chi_\mathcal{A})$
for $\kappa \to \infty$



which has a use when designing optical waveguides.
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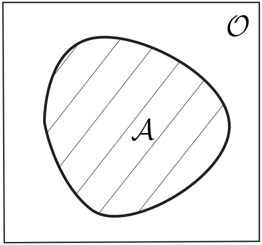

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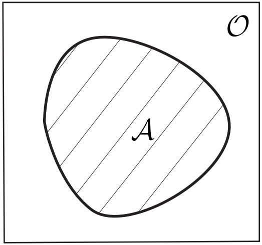

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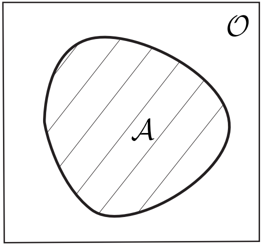

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$$H_\kappa = \begin{bmatrix} \frac{1}{101} & 0 & -\frac{1}{101} \\ 0 & \frac{1}{100} & 0 \\ -\frac{1}{101} & 0 & 1+\kappa^2 \end{bmatrix}, \text{ and } w = [1\ 0\ 0]^* \text{ as a test vector}$$

Davis–Kahan's $\tan 2\theta$

$$\sin\angle(w, v_1(H_\kappa)) \le \sin\left(\frac{1}{2}\arctan\left(\frac{2}{\frac{1}{100}-\frac{1}{101}}\frac{1}{101}\right)\right) = O(1)$$

Mathias–Veselić (and similarly from Li's $\sin\Theta$ theorems)

$$\sin\angle(w, v_1(H_\kappa)) \le \frac{\sqrt{\frac{1}{100}\frac{1}{101}}}{\frac{1}{100}-\frac{1}{101}} \frac{\sqrt{\frac{2}{101+101\kappa^2}}}{\sqrt{1-\sqrt{\frac{1}{101+101\kappa^2}}}} = O\left(\frac{1}{\kappa}\right)$$

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Relative eigenvector estimates:

- are a function of a residual which detects inaccuracies (favourable)
- but, the estimates are unsharp (?)

What, in fact, are we estimating and why do we care?

- adapt the metric which is used (relative–energy norm)
- prove that the obtained inequality is sharp

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Want $Hv = \lambda v$, $H = LL^* = H^{1/2}H^{1/2}$.
X, $\mathcal{X} = span(X)$; $\mathcal{X}$ invariant $\Leftrightarrow HX = XM$, that is,
$L^*\mathcal{X} = L^{-1}\mathcal{X}$, $\mathcal{Y} \equiv H^{1/2}\mathcal{X} = \mathcal{Z} \equiv H^{-1/2}\mathcal{X}$.
$\mathcal{X}$ not invariant,, $R \equiv HX - XM \neq 0$
$\sin \underbrace{\angle(\mathcal{Y}, \mathcal{Z})}_{\psi} = \|P_{\mathcal{Y}} - P_{\mathcal{Z}}\|$ error measure

$$\begin{pmatrix} X & X_\perp \end{pmatrix}^* H \begin{pmatrix} X & X_\perp \end{pmatrix} = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$$

$\Gamma = M^{-1/2}VW^{-1/2}$, $\|\Gamma\| = \sin\psi$, $|\lambda - \mu|/\mu \leq \sin\psi$
$|\lambda - \mu|/\mu \leq \sin^2\psi/relgap$, $relgap(\lambda, \mu) = |\lambda - \mu|/(\lambda + \mu)$
Drmač, Drmač and Hari 1997

Strang & Fix—Energy norm identity:

$H = H^*$ then (true also when H is unbounded!)

$$\|H^{1/2}(u - v)\|^2 = \lambda \|u - v\|^2 + \mu - \lambda$$

where $\mu = u^* H u$, $\|u\| = 1$ and $Hv = \lambda v$, $\|v\| = 1$

We need

Mathias–Veselić inequality

Schur–Complement for both lower and upper estimates.

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Schur–Complement for both lower and upper estimates.

Cluster robust eigenvalue estimates

Let $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$ and $\mu_1 \leq \cdots \leq \mu_m$ are $\sigma(M)$ and $Hv_i = \lambda_i v_i$

Relative residual $\Gamma := M^{-1/2} V W^{-1/2}$ + *gap* $M \leq \alpha < \beta \leq W$ *then* $\lambda_j \notin \sigma(\mathrm{W})$, $j = 1, \ldots, m$ *and*

$$0 \leq \frac{\mu_j - \lambda_j}{\mu_j} \leq \min\left\{ \|\Gamma\|, \frac{\|\Gamma\|^2}{\|(\mathbf{I} - \lambda_j W^{-1})^{-1}\|} \right\} \leq \min\left\{ \|\Gamma\|, \frac{(\lambda_{m+1} + \mu_j)\|\Gamma\|^2}{\lambda_{m+1} - \mu_j} \right\}.$$

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If TK + $\mu_i := u_i^* H u_i, \|u_i\| = 1$ *then*

$$\frac{\|H^{1/2}(u_i - v_i)\|^2}{\|H^{1/2} u_i\|^2} \leq \|u_i - v_i\|^2 + \frac{(\lambda_{m+1} + \mu_i)\|\Gamma\|^2}{\lambda_{m+1} - \mu_i}$$

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$$\frac{\|H^{1/2}(u_i - v_i)\|^2}{\|H^{1/2} u_i\|^2} \leq \max_{\nu \in \sigma(H) \setminus \{\lambda_i\}} \frac{\nu \mu_i}{(\nu - \mu_i)^2} \frac{\|\Gamma\|^2}{1 - \|\Gamma\|} + \frac{(\lambda_{m+1} + \mu_i)\|\Gamma\|^2}{\lambda_{m+1} - \mu_i}$$

Cluster robust eigenvalue estimates II

Chose λ_j, $1 \leq j \leq m$ and make a shift $H - \lambda_j I$.

Set

$$H_{cong} = \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix} \left\{ \mathbf{I} - \begin{bmatrix} \Gamma^*(\mathbf{I} - \lambda_j W^{-1})^{-1}\Gamma & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix}$$

and use Schur-complement congruence to conclude $\lambda_j(H_{cong}) = \lambda_j(H)$.

Use $\|\Gamma^*(\mathbf{I} - \lambda_j W^{-1})^{-1}\Gamma\| < 1$ and the standard argument to complete the proof.

Similar techniques yield estimates for other error functions too.

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Assume $\lambda_1 = \lambda_m$ and go back to $\mathbf{I} - \lambda M^{-1} = \Gamma^*(\mathbf{I} - \lambda W^{-1})^{-1}\Gamma$
then $\lambda \leq W$ implies

$$\mathbf{I} - \lambda M^{-1} \geq \Gamma^*\Gamma \qquad (*)$$

This is significant for FEM refinement since

- estimates on SVD of Γ are computable at the price of one DDOT in PINVIT (Neymeyr 2002)
- Temple–Kato + (*) infers

$$F(h_{\mathcal{T}_d}) \geq \frac{|\lambda - \mu|}{\mu} \geq f(h_{\mathcal{T}_d})$$

where $h_{\mathcal{T}_d}$ describes the local mesh size

- Γ is a basis for an efficient estimator

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Eigenvalues: lower estimate

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and upper estimate $\frac{\mu-\lambda_1(H_\kappa)}{\mu} \le \frac{\frac{1}{100}+\frac{1}{101}}{\frac{1}{100}-\frac{1}{101}}\frac{1}{101+101\kappa^2} \le 2\frac{1}{\kappa^2}$

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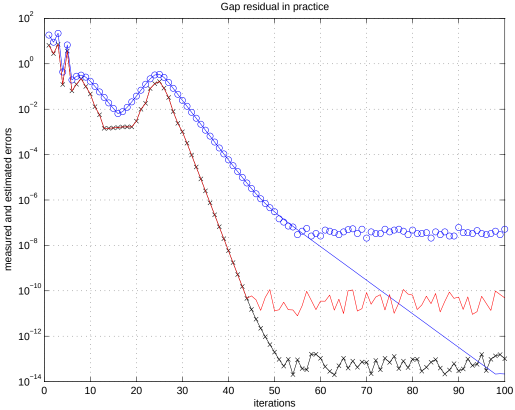
A history for $\sum_{i=1}^{m} \lambda_i = \min_{X^*X=I_m} \mathrm{Trace}(X^* H X)$



○ and × ... values of $\|\Gamma\|$

–, – and – ... relative distances

Let $P \oplus P_\perp = \mathbf{I}$ and $0 < M \leq \alpha < \beta \leq W$.
To diagonalise $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$
search for the weak solution of

$$WX - XM - XVX + V^* = 0$$

and compute

$$\tan 2\Theta(P, E(\alpha)) \leq \frac{2}{\frac{\beta-\alpha}{\sqrt{\beta\alpha}}} \|M^{-1/2} V W^{-1/2}\|$$

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$$\sin\Theta(P, E(\alpha)) \leq \sin\frac{1}{2}\arctan\frac{2\sqrt{\beta\alpha}}{\beta-\alpha}\|M^{-1/2}VW^{-1/2}\|$$

is sharper than any of the $\sin\Theta$ *theorems!*

Works also when $M = 0$ *and no assumption on* $\|VW^{-1/2}\|$ *e.g.*

$$S = \begin{pmatrix} 0 & div \\ -grad & -\triangle \end{pmatrix}.$$

the Stokes operator

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Works also when $M = 0$ *and no assumption on* $\|VW^{-1/2}\|$ *e.g.*

$$S = \begin{pmatrix} 0 & div \\ -grad & -\triangle \end{pmatrix}.$$

the Stokes operator

Since $\sin\frac{1}{2}\arctan 2x \le x \le \frac{x}{\sqrt{1-x}}$, *estimate*

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- equally sharp energy norm estimates for eigenvectors/invariant subspaces
- Also: avoidance of locking when modelling thin elastic structures e.g. Arch model

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