Z. Drmač\textsuperscript{1} L. Grubišić\textsuperscript{2} V. Kostrykin\textsuperscript{3} K. Veselić\textsuperscript{4}

\textsuperscript{1}Department of Mathematics, University of Zagreb

\textsuperscript{2}Institut fuer Reine und Angewandte Mathematik, RWTH Aachen

\textsuperscript{3}Fraunhofer Institut fuer Lasertechnik, Aachen

\textsuperscript{4}Lehrgebiet Anwendungsorientierte Analysis, Fernuniversitaet in Hagen

IWASEP VI, Pen State, May 21–May 24
Consider multi-scale problems, something like

$$H_\kappa = -\Delta + \kappa^2 (1 - \chi A)$$

for $\kappa \to \infty$

which has a use when designing optical waveguides.

Numerical methods are influenced by:

- singularity when $\kappa \to \infty$
- clustering of eigenvalues ...
Consider multi-scale problems, something like

\[ H_\kappa = -\Delta + \kappa^2(1 - \chi_A) \]
for \( \kappa \to \infty \)

which has a use when designing optical waveguides.

Numerical methods are influenced by:

- singularity when \( \kappa \to \infty \)
- clustering of eigenvalues...
Consider multi-scale problems, something like

\[ H_\kappa = -\Delta + \kappa^2 (1 - \chi_A) \]

for \( \kappa \to \infty \)

which has a use when designing optical waveguides. Numerical methods are influenced by:

- singularity when \( \kappa \to \infty \)
- clustering of eigenvalues ...
Consider multi-scale problems, something like

\[ H_{\kappa} = -\triangle + \kappa^2 (1 - \chi_A) \]

for \( \kappa \to \infty \)

which has a use when designing optical waveguides. Numerical methods are influenced by:

- singularity when \( \kappa \to \infty \)
- clustering of eigenvalues ...
H_κ = \begin{bmatrix}
\frac{1}{101} & 0 & -\frac{1}{101} \\
0 & \frac{1}{100} & 0 \\
-\frac{1}{101} & 0 & 1 + \kappa^2
\end{bmatrix}, \text{ and } w = [1 0 0]^* \text{ as a test vector}

Davis–Kahan’s \tan 2\theta
\sin \angle(w, v_1(H_κ)) \leq \sin \left(\frac{1}{2} \arctan \left(\frac{2}{100 - \frac{1}{101}}\right)\right) = O(1)

Mathias–Veselić (and similarly from Li’s \sin \Theta theorems)
\sin \angle(w, v_1(H_κ)) \leq \frac{\sqrt{\frac{1}{100} - \frac{1}{101}}}{\sqrt{1 - \sqrt{\frac{1}{101 + 101\kappa^2}}}} = O\left(\frac{1}{\kappa}\right)

But: \sin \angle(w, v_1(H_κ)) = \frac{1}{101\kappa^2} - \frac{100}{10201\kappa^4} + O\left(\frac{1}{\kappa^6}\right)
\[ H_\kappa = \begin{bmatrix} \frac{1}{101} & 0 & -\frac{1}{101} \\ 0 & \frac{1}{100} & 0 \\ -\frac{1}{101} & 0 & 1 + \kappa^2 \end{bmatrix}, \text{ and } w = [1 \ 0 \ 0]^* \text{ as a test vector} \]

Davis–Kahan’s \( \tan 2\theta \)
\[
\sin \angle(w, v_1(H_\kappa)) \leq \sin \left(\frac{1}{2} \arctan \left( \frac{2}{100} - \frac{1}{101} \right) \right) = O(1)
\]

Mathias–Veselić (and similarly from Li’s \( \sin \Theta \) theorems)
\[
\sin \angle(w, v_1(H_\kappa)) \leq \frac{\sqrt{\frac{1}{100} + \frac{1}{101} \kappa^2}}{\sqrt{1 - \sqrt{\frac{1}{101 + 101 \kappa^2}}}} = O\left(\frac{1}{\kappa}\right)
\]

But: \( \sin \angle(w, v_1(H_\kappa)) = \frac{1}{101 \kappa^2} - \frac{100}{10201 \kappa^4} + O\left(\frac{1}{\kappa^6}\right) \)
\[ H_\kappa = \begin{bmatrix}
\frac{1}{101} & 0 & -\frac{1}{101} \\
0 & \frac{1}{100} & 0 \\
-\frac{1}{101} & 0 & 1 + \kappa^2
\end{bmatrix}, \text{ and } w = [1 \ 0 \ 0]^* \text{ as a test vector}
\]

Davis–Kahan’s \( \tan 2\theta \)

\[
\sin \angle(w, v_1(H_\kappa)) \leq \sin \left( \frac{1}{2} \arctan \left( \frac{2}{100 - 101} \right) \right) = O(1)
\]

Mathias–Veselić (and similarly from Li’s \( \sin \Theta \) theorems)

\[
\sin \angle(w, v_1(H_\kappa)) \leq \sqrt{\frac{1}{100} \frac{1}{101}} \sqrt{\frac{2}{101 + 101\kappa^2}} \sqrt{1 - \sqrt{\frac{1}{101 + 101\kappa^2}}} = O\left( \frac{1}{\kappa} \right)
\]

But:

\[
\sin \angle(w, v_1(H_\kappa)) = \frac{1}{101 \kappa^2} - \frac{100}{10201 \kappa^4} + O\left( \frac{1}{\kappa^6} \right)
\]
\( H_\kappa = \begin{bmatrix}
\frac{1}{101} & 0 & -\frac{1}{101} \\
0 & \frac{1}{100} & 0 \\
-\frac{1}{101} & 0 & 1 + \kappa^2
\end{bmatrix} \), and \( w = [1 \ 0 \ 0]^* \) as a test vector

Davis–Kahan’s \( \tan 2\theta \)
\[
\sin \angle(w, v_1(H_\kappa)) \leq \sin \left( \frac{1}{2} \arctan \left( \frac{2}{100 - 101} \right) \right) = O(1)
\]

Mathias–Veselić (and similarly from Li’s \( \sin \Theta \) theorems)
\[
\sin \angle(w, v_1(H_\kappa)) \leq \frac{\sqrt{\frac{1}{100} \frac{1}{101}}}{\frac{\sqrt{2}}{101 + 101\kappa^2}} \frac{\sqrt{1 - \sqrt{\frac{1}{101 + 101\kappa^2}}}}{\sqrt{1 - \sqrt{\frac{1}{101 + 101\kappa^2}}}} = O\left( \frac{1}{\kappa} \right)
\]

But:
\[
\sin \angle(w, v_1(H_\kappa)) = \frac{\frac{1}{101\kappa^2}}{100\frac{10201\kappa^4}{10201\kappa^4}} + O\left( \frac{1}{\kappa^6} \right)
\]
Introduction

H_\kappa = \begin{bmatrix}
\frac{1}{101} & 0 & -\frac{1}{101} \\
0 & \frac{1}{100} & 0 \\
-\frac{1}{101} & 0 & 1 + \kappa^2
\end{bmatrix}, and \ w = [1 \ 0 \ 0]^* as a test vector

Davis–Kahan’s \tan 2\theta
\sin \angle(w, v_1(H_\kappa)) \leq \sin \left(\frac{1}{2} \arctan \left(\frac{2}{100 - \frac{1}{101}}\right)\right) = O(1)

Mathias–Veselić (and similarly from Li’s \sin \Theta theorems)
\sin \angle(w, v_1(H_\kappa)) \leq \frac{\sqrt{1 - \frac{1}{100}}}{\sqrt{101 + 101\kappa^2}} \frac{\sqrt{2}}{\sqrt{101 + 101\kappa^2}} = O\left(\frac{1}{\kappa}\right)

But: \sin \angle(w, v_1(H_\kappa)) = \frac{1}{101\kappa^2} - \frac{100}{10201\kappa^4} + O\left(\frac{1}{\kappa^6}\right)
Relative eigenvector estimates:

are a function of a residual which detects inaccuracies (favourable)

but, the estimates are unsharp (?)

What, in fact, are we estimating and why do we care?

adapt the metric which is used (relative–energy norm)

prove that the obtained inequality is sharp

We obtain efficient adaptive estimates which are robust in multi-scale situations.
Relative eigenvector estimates:

are a function of a residual which detects inaccuracies (favourable)

but, the estimates are unsharp (?)

What, in fact, are we estimating and why do we care?

adapt the metric which is used (relative–energy norm)

prove that the obtained inequality is sharp

We obtain efficient adaptive estimates which are robust in multi-scale situations.
Relative eigenvector estimates:

- are a function of a residual which detects inaccuracies (favourable)
- but, the estimates are unsharp (?)

What, in fact, are we estimating and why do we care?
- adapt the metric which is used (relative–energy norm)
- prove that the obtained inequality is sharp

We obtain efficient adaptive estimates which are robust in multi-scale situations.
Want $Hv = \lambda v$, $H = LL^* = H^{1/2}H^{1/2}$.

$X, \mathcal{X} = \text{span}(X)$; $\mathcal{X}$ invariant $\iff HX = XM$, that is, $L^*\mathcal{X} = L^{-1}\mathcal{X}, Y \equiv H^{1/2}\mathcal{X} = Z \equiv H^{-1/2}\mathcal{X}$.

$\mathcal{X}$ not invariant, $R \equiv HX - XM \neq 0$

$sin \angle(Y, Z) = ||P_Y - P_Z||$ error measure

\[
\begin{bmatrix} X & X_\perp \end{bmatrix}^* H \begin{bmatrix} X & X_\perp \end{bmatrix} = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}
\]

$\Gamma = M^{-1/2}VW^{-1/2}, \|\Gamma\| = \sin \psi, |\lambda - \mu|/\mu \leq \sin \psi$

$|\lambda - \mu|/\mu \leq \sin^2 \psi/\text{relgap}$, $\text{relgap}(\lambda, \mu) = |\lambda - \mu|/(\lambda + \mu)$

Drmač, Drmač and Hari 1997
Strang & Fix—Energy norm identity:

\[ H = H^* \text{ then } \quad \|H^{1/2}(u - v)\|^2 = \lambda\|u - v\|^2 + \mu - \lambda \]

(true also when \( H \) is unbounded!)

where \( \mu = u^* Hu \), \( \|u\| = 1 \) and \( Hv = \lambda v \), \( \|v\| = 1 \)

We need

Mathias–Veselić inequality

Schur–Complement for both lower and upper estimates.
Strang & Fix—Energy norm identity:

\[ H = H^* \text{ then (true also when } H \text{ is unbounded!)} \]

\[ \| H^{1/2} (u - v) \|^2 = \lambda \| u - v \|^2 + \mu - \lambda \]

where \( \mu = u^* Hu, \| u \| = 1 \) and \( H v = \lambda v, \| v \| = 1 \)

this implies

\[ \frac{\mu - \lambda}{\lambda} \leq \frac{\| H^{1/2} (u - v) \|^2}{\| H^{1/2} v \|^2} = \| u - v \|^2 + \frac{\mu - \lambda}{\lambda} \]

We need

Mathias–Veselić inequality

Schur–Complement for both lower and upper estimates.
Strang & Fix—Energy norm identity:

\[ H = H^* \text{ then } \quad \| H^{1/2}(u - v) \|^2 = \lambda \| u - v \|^2 + \mu - \lambda \]

(true also when \( H \) is unbounded!)

where \( \mu = u^*Hu, \| u \| = 1 \) and \( Hv = \lambda v, \| v \| = 1 \)

this implies

\[ \frac{\mu - \lambda}{\lambda} \leq \frac{\| H^{1/2}(u - v) \|^2}{\| H^{1/2}v \|^2} = \| u - v \|^2 + \frac{\mu - \lambda}{\lambda} \]

We need

Mathias–Veselić inequality

Schur–Complement for both lower and upper estimates.
Strang & Fix—Energy norm identity:

\[ H = H^* \text{ then} \]

\[ \| H^{1/2}(u-v) \|^2 = \lambda \| u-v \|^2 + \mu - \lambda \]

(true also when \( H \) is unbounded!)

where \( \mu = u^*Hu \), \( \| u \| = 1 \) and \( Hv = \lambda v \), \( \| v \| = 1 \)

this implies

\[ \frac{\mu - \lambda}{\lambda} \leq \frac{\| H^{1/2}(u-v) \|^2}{\| H^{1/2}v \|^2} = \| u-v \|^2 + \frac{\mu - \lambda}{\lambda} \]

We need

Mathias–Veselič inequality

Schur–Complement for both lower and upper estimates.
Cluster robust eigenvalue estimates

Let $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$ and $\mu_1 \leq \cdots \leq \mu_m$ are $\sigma(M)$ and $Hv_i = \lambda_i v_i$

Relative residual $\Gamma := M^{-1/2}VW^{-1/2} + \text{gap } M \leq \alpha < \beta \leq W$ then $\lambda_j \not\in \sigma(W), j = 1, \ldots, m$ and

$$0 \leq \frac{\mu_j - \lambda_j}{\mu_j} \leq \min \left\{ \|\Gamma\|, \frac{\|\Gamma\|^2}{\|(I-\lambda_j W^{-1})^{-1}\|} \right\} \leq \min \left\{ \|\Gamma\|, \frac{(\lambda_{m+1} + \mu_j)\|\Gamma\|^2}{\lambda_{m+1} - \mu_j} \right\}.$$
Cluster robust eigenvalue estimates

Let $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$ and $\mu_1 \leq \cdots \leq \mu_m$ are $\sigma(M)$ and $Hv_i = \lambda_i v_i$

Relative residual $\Gamma := M^{-1/2} VW^{-1/2} + \text{gap } M \leq \alpha < \beta \leq W$ then $\lambda_j \not\in \sigma(W)$, $j = 1, \ldots, m$ and

$$0 \leq \frac{\mu_j - \lambda_j}{\mu_j} \leq \min \left\{ \|\Gamma\|, \frac{\|\Gamma\|^2}{\| (I - \lambda_j W^{-1})^{-1} \|} \right\} \leq \min \left\{ \|\Gamma\|, \frac{(\lambda_{m+1} + \mu_j)\|\Gamma\|^2}{\lambda_{m+1} - \mu_j} \right\}.$$

If $TK + \mu_i := u_i^* Hu_i, \|u_i\| = 1$ then

$$\frac{\|H^{1/2}(u_i - v_i)\|^2}{\|H^{1/2}u_i\|^2} \leq \|u_i - v_i\|^2 + \frac{(\lambda_{m+1} + \mu_i)\|\Gamma\|^2}{\lambda_{m+1} - \mu_i}$$
Cluster robust eigenvalue estimates

Let $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$ and $\mu_1 \leq \cdots \leq \mu_m$ are $\sigma(M)$ and $Hv_i = \lambda_i v_i$

Relative residual $\Gamma := M^{-1/2} VW^{-1/2} + \text{gap } M \leq \alpha < \beta \leq W$ then $\lambda_j \not\in \sigma(W)$, $j = 1, \ldots, m$ and

$$0 \leq \frac{\mu_j - \lambda_j}{\mu_j} \leq \min \left\{ \| \Gamma \|, \frac{\| \Gamma \|^2}{\|(I - \lambda_j W^{-1})^{-1}\|} \right\} \leq \min \left\{ \| \Gamma \|, \frac{(\lambda_{m+1} + \mu_j) \| \Gamma \|^2}{\lambda_{m+1} - \mu_j} \right\}.$$

If $TK + \mu_i := u_i^* Hu_i, \| u_i \| = 1$ then

$$\frac{\| H^{1/2}(u_i - v_i) \|^2}{\| H^{1/2} u_i \|^2} \leq \max_{\nu \in \sigma(H) \backslash \{\lambda_i\}} \frac{\nu \mu_i}{(\nu - \mu_i)^2} \frac{\| \Gamma \|^2}{1 - \| \Gamma \|} + \frac{(\lambda_{m+1} + \mu_i) \| \Gamma \|^2}{\lambda_{m+1} - \mu_i}.$$
Cluster robust eigenvalue estimates II

Chose $\lambda_j$, $1 \leq j \leq m$ and make a shift $H - \lambda_j I$.

Set

$$H_{cong} = \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix} \begin{bmatrix} \Gamma^* (I - \lambda_j W^{-1})^{-1} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix}$$

and use Schur-complement congruence to conclude

$$\lambda_j(H_{cong}) = \lambda_j(H).$$

Use $\|\Gamma^* (I - \lambda_j W^{-1})^{-1} \Gamma\| < 1$ and the standard argument to complete the proof.

Similar techniques yield estimates for other error functions too.
Cluster robust eigenvalue estimates II

Chose $\lambda_j$, $1 \leq j \leq m$ and make a shift $H - \lambda_j I$.

Set

$$H_{cong} = \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix} \left\{ I - \left[ \begin{array}{cc} \Gamma^* (I - \lambda_j W^{-1})^{-1} \Gamma & 0 \\ 0 & 0 \end{array} \right] \right\} \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix}$$

and use Schur-complement congruence to conclude

$$\lambda_j(H_{cong}) = \lambda_j(H).$$

Use $\|\Gamma^* (I - \lambda_j W^{-1})^{-1} \Gamma\| < 1$ and the standard argument to complete the proof.

Similar techniques yield estimates for other error functions too.
Cluster robust eigenvalue estimates II

Chose $\lambda_j$, $1 \leq j \leq m$ and make a shift $H - \lambda_j I$.

Set

$$H_{cong} = \begin{bmatrix} M^{1/2} & 0 \\ 0 & W^{1/2} \end{bmatrix} \left\{ I - \begin{bmatrix} \Gamma^* (I - \lambda_j W^{-1})^{-1} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} M^{1/2} \\ 0 \\ W^{1/2} \end{bmatrix}$$

and use Schur-complement congruence to conclude $\lambda_j(H_{cong}) = \lambda_j(H)$.

Use $\|\Gamma^* (I - \lambda_j W^{-1})^{-1} \Gamma\| < 1$ and the standard argument to complete the proof.

Similar techniques yield estimates for other error functions too.
Assume $\lambda_1 = \lambda_m$ and go back to $I - \lambda M^{-1} = \Gamma^* (I - \lambda W^{-1})^{-1} \Gamma$
then $\lambda \leq W$ implies

$$I - \lambda M^{-1} \geq \Gamma^* \Gamma$$

(*)

This is significant for FEM refinement since

estimates on SVD of $\Gamma$ are computable at the price of one DDOT in PINVIT (Neymeyr 2002)

Temple–Kato + (*) infers

$$F(h_{\mathcal{T}_d}) \geq \frac{|\lambda - \mu|}{\mu} \geq f(h_{\mathcal{T}_d})$$

where $h_{\mathcal{T}_d}$ describes the local mesh size

$\Gamma$ is a basis for an efficient estimator
Assume $\lambda_1 = \lambda_m$ and go back to $I - \lambda M^{-1} = \Gamma^* (I - \lambda W^{-1})^{-1} \Gamma$
then $\lambda \leq W$ implies

$$I - \lambda M^{-1} \geq \Gamma^* \Gamma$$

(*)

This is significant for FEM refinement since estimates on SVD of $\Gamma$ are computable at the price of one DDOT in PINVIT (Neymeyr 2002)
Temple–Kato + (*) infers

$$F(h_{\mathcal{I}_d}) \geq \frac{|\lambda - \mu|}{\mu} \geq f(h_{\mathcal{I}_d})$$

where $h_{\mathcal{I}_d}$ describes the local mesh size
$\Gamma$ is a basis for an efficient estimator
Assume $\lambda_1 = \lambda_m$ and go back to $I - \lambda M^{-1} = \Gamma^*(I - \lambda W^{-1})^{-1}\Gamma$ then $\lambda \leq W$ implies

$$I - \lambda M^{-1} \geq \Gamma^*\Gamma$$

This is significant for FEM refinement since estimates on SVD of $\Gamma$ are computable at the price of one `ddot` in PINVIT (Neymeyr 2002) Temple–Kato + (*) infers

$$F(h_{I_d}) \geq \frac{|\lambda - \mu|}{\mu} \geq f(h_{I_d})$$

where $h_{I_d}$ describes the local mesh size $\Gamma$ is a basis for an efficient estimator
Assume $\lambda_1 = \lambda_m$ and go back to $I - \lambda M^{-1} = \Gamma^* (I - \lambda W^{-1})^{-1} \Gamma$
then $\lambda \leq W$ implies

$$I - \lambda M^{-1} \geq \Gamma^* \Gamma \quad (*)$$

This is significant for FEM refinement since estimates on SVD of $\Gamma$ are computable at the price of one \texttt{DDOT} in PINVIT (Neymeyr 2002)

Temple–Kato + (*) infers

$$F(h_{T_d}) \geq \frac{|\lambda - \mu|}{\mu} \geq f(h_{T_d})$$

where $h_{T_d}$ describes the local mesh size

$\Gamma$ is a basis for an efficient estimator
\[ H_\kappa = \begin{bmatrix} \frac{1}{101} & 0 & -\frac{1}{101} \\ 0 & \frac{1}{100} & 0 \\ -\frac{1}{101} & 0 & 1 + \kappa^2 \end{bmatrix} \]

**Eigenvalues:** lower estimate
\[
\frac{1}{101 + 101\kappa^2} \leq \frac{\mu - \lambda_1(H_\kappa)}{\mu} = \frac{1}{101} \frac{1}{101\kappa^2} + O\left(\frac{1}{\kappa^4}\right)
\]

and upper estimate
\[
\frac{\mu - \lambda_1(H_\kappa)}{\mu} \leq \frac{1}{100} \frac{1}{101 + 101\kappa^2} \leq 2 \frac{1}{\kappa^2}
\]

Similarly for energy norm of eigenvectors since
\[
\frac{\mu - \lambda}{\lambda} \leq \frac{\|H^{1/2}(u - v)\|^2}{\|H^{1/2}v\|^2} = \|u - v\|^2 + \frac{\mu - \lambda}{\lambda}
\]
\[ H_\kappa = \begin{bmatrix} \frac{1}{101} & 0 & -\frac{1}{101} \\ 0 & \frac{1}{100} & -1 \\ -\frac{1}{101} & 0 & 1 + \kappa^2 \end{bmatrix} \]

**Eigenvalues:** lower estimate

\[
\frac{1}{101 + 101\kappa^2} \leq \frac{\mu - \lambda_1(H_\kappa)}{\mu} = \frac{1}{101} \frac{1}{101\kappa^2} + O\left(\frac{1}{\kappa^4}\right)
\]

and upper estimate

\[
\frac{\mu - \lambda_1(H_\kappa)}{\mu} \leq \frac{1}{100} \frac{1}{101 + 101\kappa^2} \leq 2 \frac{1}{\kappa^2}
\]

Similarly for energy norm of eigenvectors since

\[
\frac{\mu - \lambda}{\lambda} \leq \frac{||H^{1/2}(u - v)||^2}{||H^{1/2}v||^2} = ||u - v||^2 + \frac{\mu - \lambda}{\lambda}
\]
$$H_\kappa = \begin{bmatrix} \frac{1}{101} & 0 & -\frac{1}{101} \\ 0 & \frac{1}{100} & 0 \\ -\frac{1}{101} & 0 & 1 + \kappa^2 \end{bmatrix}$$

**Eigenvalues:** lower estimate

$$\frac{1}{101+101\kappa^2} \leq \frac{\mu - \lambda_1(H_\kappa)}{\mu} = \frac{1}{101} \frac{1}{101\kappa^2} + O\left(\frac{1}{\kappa^4}\right)$$

and upper estimate

$$\frac{\mu - \lambda_1(H_\kappa)}{\mu} \leq \frac{1}{100} + \frac{1}{101} \frac{1}{101 + 101\kappa^2} \leq 2 \frac{1}{\kappa^2}$$

Similarly for energy norm of eigenvectors since

$$\frac{\mu - \lambda}{\lambda} \leq \frac{\|H^{1/2}(u - v)\|^2}{\|H^{1/2}v\|^2} = \|u - v\|^2 + \frac{\mu - \lambda}{\lambda}$$
$H_\kappa = \begin{bmatrix}
\frac{1}{101} & 0 & -\frac{1}{101} \\
0 & \frac{1}{100} & 0 \\
-\frac{1}{101} & 0 & 1 + \kappa^2
\end{bmatrix}$

Eigenvalues: lower estimate
$$\frac{1}{101 + 101\kappa^2} \leq \frac{\mu - \lambda_1(H_\kappa)}{\mu} = \frac{1}{101} \frac{1}{101\kappa^2} + O\left(\frac{1}{\kappa^4}\right)$$

and upper estimate
$$\frac{\mu - \lambda_1(H_\kappa)}{\mu} \leq \frac{1}{100} + \frac{1}{101} \frac{1}{100 + 101\kappa^2} \leq 2 \frac{1}{\kappa^2}$$

Similarly for energy norm of eigenvectors since
$$\frac{\mu - \lambda}{\lambda} \leq \frac{\|H^{1/2}(u-v)\|^2}{\|H^{1/2}v\|^2} = \|u - v\|^2 + \frac{\mu - \lambda}{\lambda}$$
A history for \[ \sum_{i=1}^{m} \lambda_i = \min_{X^*X = I_m} \text{Trace}(X^*HX) \]
Let $P \oplus P_\perp = I$ and $0 < M \leq \alpha < \beta \leq W$.

To diagonalise $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$

search for the weak solution of

$$WX - XM - XVX + V^* = 0$$

and compute

$$\tan 2\Theta(P, E(\alpha)) \leq \frac{2}{\frac{\beta - \alpha}{\sqrt{\beta \alpha}}} \|M^{-1/2} VW^{-1/2}\|$$
Let $P \oplus P_\perp = I$ and $0 < M \leq \alpha < \beta \leq W$.

To diagonalise $H = \begin{bmatrix} M & V \\ V^* & W \end{bmatrix}$

search for the weak solution of

$$WX - XM - XVX + V^* = 0$$

and compute

$$\tan 2\Theta(P, E(\alpha)) \leq \frac{2}{\sqrt{\beta - \alpha}} \|M^{-1/2} VW^{-1/2}\|$$
Since $\sin \frac{1}{2} \arctan 2x \leq x \leq \frac{x}{\sqrt{1-x}}$, estimate

$$\sin \Theta(P, E(\alpha)) \leq \sin \frac{1}{2} \arctan \frac{2\sqrt{\beta \alpha}}{\beta - \alpha} \|M^{-1/2} VW^{-1/2}\|$$

is sharper than any of the $\sin \Theta$ theorems!

Works also when $M = 0$ and no assumption on $\|VW^{-1/2}\|$ e.g.

$$S = \begin{pmatrix} 0 & \text{div} \\ -\text{grad} & -\Delta \end{pmatrix}.$$ 

the Stokes operator
Since $\sin \frac{1}{2} \arctan 2x \leq x \leq \frac{x}{\sqrt{1-x}}$, estimate

$$\sin \Theta(P, E(\alpha)) \leq \sin \frac{1}{2} \arctan \frac{2\sqrt{\beta \alpha}}{\beta - \alpha} \|M^{-1/2} VW^{-1/2}\|$$

is sharper than any of the $\sin \Theta$ theorems!

Works also when $M = 0$ and no assumption on $\|VW^{-1/2}\|$ e.g.

$$S = \begin{pmatrix} 0 & \text{div} \\ -\text{grad} & -\Delta \end{pmatrix}.$$
Since $\sin \frac{1}{2} \arctan 2x \leq x \leq \frac{x}{\sqrt{1-x}}$, estimate

$$\sin \Theta(P, E(\alpha)) \leq \sin \frac{1}{2} \arctan \frac{2\sqrt{\beta\alpha}}{\beta - \alpha} \|M^{-1/2}VW^{-1/2}\|$$

is sharper than any of the $\sin \Theta$ theorems!

Works also when $M = 0$ and no assumption on $\|VW^{-1/2}\|$ e.g.

$$S = \begin{pmatrix} 0 & \text{div} \\ -\text{grad} & -\Delta \end{pmatrix}.$$  

the Stokes operator
Error estimates are tuned for multi-scale problems.

detects well the separation of various scales
(“exploding” vs. “converging” eigenvalues)
equally sharp energy norm estimates for
eigenvectors/invariant subspaces

Also: avoidance of locking when modelling thin elastic
structures e.g. Arch model

Applicable for adaptive FEM methods. The paradigm here is:
discrete $H^{-1}$ norm Temple–Kato and Wielandt–Hoffman
efficient estimators for
relative error $\frac{|\lambda - \mu|}{\mu}$
$H^1$-norm error in the eigenvectors
Error estimates are tuned for **multi-scale** problems.

- detects well the separation of various scales ("exploding" vs. "converging" eigenvalues)
- equally sharp energy norm estimates for eigenvectors/invariant subspaces
- Also: **avoidance of locking** when modelling thin elastic structures e.g. Arch model

Applicable for **adaptive** FEM methods. The paradigm here is:
- discrete $H^{-1}$ norm Temple–Kato and Wielandt–Hoffman efficient estimators for relative error $\frac{|\lambda - \mu|}{\mu}$
- $H^1$-norm error in the eigenvectors
Error estimates are tuned for multi-scale problems.

detects well the separation of various scales ("exploding" vs. "converging" eigenvalues)
equally sharp energy norm estimates for eigenvectors/invariant subspaces

Also: avoidance of locking when modelling thin elastic structures e.g. Arch model

Applicable for adaptive FEM methods. The paradigm here is:

discrete $H^{-1}$ norm Temple–Kato and Wielandt–Hoffman

efficient estimators for relative error $\frac{|\lambda - \mu|}{\mu}$

$H^1$-norm error in the eigenvectors
Error estimates are tuned for multi-scale problems.

detects well the separation of various scales
(“exploding” vs. “converging” eigenvalues)
equally sharp energy norm estimates for
eigenvectors/invariant subspaces
Also: avoidance of locking when modelling thin elastic structures e.g. Arch model

Applicable for adaptive FEM methods. The paradigm here is:
discrete $H^{-1}$ norm Temple–Kato and Wielandt–Hoffman
efficient estimators for
relative error $\frac{|\lambda - \mu|}{\mu}$
$H^1$-norm error in the eigenvectors
Error estimates are tuned for multi-scale problems.

detects well the separation of various scales ("exploding" vs. "converging" eigenvalues)
equally sharp energy norm estimates for eigenvectors/invariant subspaces
Also: avoidance of locking when modelling thin elastic structures e.g. Arch model

Applicable for adaptive FEM methods. The paradigm here is:
discrete $H^{-1}$ norm Temple–Kato and Wielandt–Hoffman efficient estimators for relative error $\frac{|\lambda - \mu|}{\mu}$
$H^1$-norm error in the eigenvectors
Error estimates are tuned for multi-scale problems.

detects well the separation of various scales ("exploding" vs. "converging" eigenvalues)
equally sharp energy norm estimates for eigenvectors/invariant subspaces
Also: avoidance of locking when modelling thin elastic structures e.g. Arch model

Applicable for adaptive FEM methods. The paradigm here is:
discrete $H^{-1}$ norm Temple–Kato and Wielandt–Hoffman efficient estimators for
relative error $\frac{|\lambda - \mu|}{\mu}$
$H^1$-norm error in the eigenvectors