1. Consider the equality constrained least squares problem
\[ \min \frac{1}{2} \|Ex - f\|_2^2 \]
subject to
\[ Ax = b \]
where \( E \in \mathbb{R}^{p \times n} \), \( A \in \mathbb{R}^{m \times n} \), \( f \in \mathbb{R}^p \), and \( b \in \mathbb{R}^m \). We assume that \( p \geq n \geq m \), \( \text{Rank}(A) = m \) and \( \text{Rank}(E) = n \) (these conditions can be weakened). Show that the Kurush–Kuhn–Tucker conditions yield the linear system
\[
\begin{pmatrix}
m & p & n \\
p & 0 & A \\
n & I_p & E
\end{pmatrix}
\begin{pmatrix}
\lambda \\
r \\
x
\end{pmatrix}
= 
\begin{pmatrix}
b \\
f \\
0
\end{pmatrix}
\]
where \( r = f - Ex \).

2. Consider the equality constrained quadratic programming problem
\[ \min \sum_{j=1}^n jx_j^2 \]
subject to
\[ \sum_{j=1}^n x_j = \kappa. \]
(a) Give a formula for the vector \( x = (x_1, x_2, \ldots, x_n)^T \) that solves this problem.
(b) Does this solution minimize the function with respect to the constraints
\[ x_j \geq 0, \quad \sum_{j=1}^n x_j \geq \kappa \]
Be careful how you explain your answer to this.
3. Consider the problem

$$\max (x_1 + 1)^2 + (x_2 + 1)^2$$

subject to

$$x_1^2 + x_2^2 \leq 2, \quad x_2 \leq 1.$$ 

Find its Kurush–Kuhn–Tucker conditions and determine its solution. [Hint: Graphing the constraint region and the contours of the objective function will help.]

4. Consider the quadratic equality constrained problem

$$\min \frac{1}{2} x^T A x - x^T b, \quad A \text{ symmetric}$$

subject to

$$E x = f$$

where $A$ is an $n \times n$ matrix and $E$ is a $p \times n$ matrix. We may assume that $p \leq n$ and that rank($E$) = $p$ (thus satisfying LICQ).

(a) Show that the Kurush–Kuhn–Tucker conditions given by the linear system

$$\begin{pmatrix} A & -E^T \\ E & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}$$

(Usually, we replace $\lambda$ with $-\lambda$ and $-E^T$ with $E^T$ to make the KKT matrix symmetric. You can show that this matrix is almost always indefinite.)

(b) Let $E^T$ have the Q–R decomposition

$$E^T = \begin{pmatrix} p & n-p \\ Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix}$$

where $R$ is a nonsingular $p \times p$ matrix. Give the second order necessary and sufficient conditions for $x^*$ to be a minimum.

(c) The minimum two-norm solution to the constraint equation is given from the following procedure. Solve

$$R^T g_1 = f.$$
\[ x_{\text{min}} = Q_1 g_1. \]

Show that all solutions of the constraint equation are given by

\[ x = x_{\text{min}} + Q_2 g_2, \quad g_2 \in \mathbb{R}^{n-p} \]

Use this equation to reduce (1)–(2) to an unconstrained problem involving

\[ B = Q_2^T A Q_2. \]

Feel free to ask me for hints on this one or any of the others.