Lecture # 33
The Simplex Algorithm—Some loose ends

We are using the simplex algorithm to solve the linear programming problem

\[
\min \; c^T x
\]  
\[
\text{subject to}
\]

\[
Ax = b
\]
\[
x \geq 0
\]

Here \( x \) is an \( n \)-vector and \( A \) is an \( m \times n \) matrix with \( m \leq n \).

To start simplex, we need a basic feasible point, that is a point \( x \) that satisfies (2)–(3). Such a point can be found using a Phase I problem. We introduce variables \( z \) and define the LP

\[
\min \; e^T z
\]

where \( e = (1,1,\ldots,1)^T \), subject to

\[
Ax + Ez = b
\]
\[
x, z \geq 0
\]

Here

\[
E = \text{diag}(e_{11}, \ldots, e_{nn})
\]

with

\[
e_{jj} = \text{sign}(b_j).
\]

A feasible point for this problem is

\[
x = 0, \quad z_j = |b_j|.
\]

This problem has the optimal objective value

\[
e^T z = 0
\]

if and only if \( z = 0 \) which is true if and only the problem has a feasible point.

Clearly, if \( z = 0 \) then (2)–(3) is satisfied.

On the other hand, if \( x \) is a feasible point, then \( (0, x)^T \) is a solution to the above LP.
However, if we terminate the simplex algorithm at a point where
\[ e^T z > 0 \]
then clearly the LP (1)–(2)–(3) has no feasible points.

The following is an example of an application of linear programming in business logistics.

**Example 1 (A Transportation Problem)** We could also call this the Wal-Mart problem.

A retail company has two factories \( F_1 \) and \( F_2 \) and 20 retail outlets \( R_1, \ldots, R_{20} \).

Factory \( F_i \) produces \( a_i \) “goods.” Retail outlet \( R_j \) demands \( b_j \) “goods.” The cost to ship from \( F_i \) to \( R_j \) is \( c_{ij} \). We sent \( x_{ij} \) goods from \( F_i \) to \( R_j \). The optimal \( x = (x_{11}, \ldots, x_{1,20}, x_{21}, \ldots, x_{2,20})^T \) solves the LP

\[
\min \sum_{i=1}^{2} \sum_{j=1}^{20} c_{ij} x_{ij}
\]

subject to
\[
A_1 x \leq a, \quad A_2 x \geq b \\
x \geq 0
\]

Here
\[
A_1 = \begin{pmatrix} 20 & 20 \\ e^T & 0 & 0 & e^T \end{pmatrix}, \quad e = (1,1,\ldots,1)^T
\]

and
\[
A_2 = \begin{pmatrix} I_{20} & I_{20} \end{pmatrix}.
\]

We note that there are 22 constraints. Introducing the slack variable \( z \), we have
\[
\begin{pmatrix} -A_1 \\ A_2 \end{pmatrix} x + z = \begin{pmatrix} -a \\ b \end{pmatrix}.
\]

Thus the LP becomes
\[
\min c^T x + 0^T z
\]

subject to
\[
\begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}
\quad x, z \geq 0
\]
One other problem is how to choose \( s_q \) to make the most progress in minimizing the objective function. This is called the \textit{pricing} strategy. Here are some possibilities.

1. Choose the most negative \( s_q \). (Dantzig’s original strategy)

2. For each component \( t = B^{-1}a_q \), choose entering variable that maximizes \( s_q x_q^+ \).

3. Choose direction

\[
\eta_q = \begin{pmatrix} -B^{-1}a_q \\ e_q \end{pmatrix}.
\]

Choose \( q \) such that

\[
\frac{c^T \eta_q}{||\eta_q||_2}
\]

is a minimum.

\textbf{Degeneracy} The problem where

\[
(x_B)_i = 0
\]

for some \( i \). In this case, the algorithm cannot move. There are techniques for picking different sets of indices, but the algorithm can cycle. This problem used to be rare, but with large LP’s it has become more common!

One trick is to perturb \( b \), solve a nearby problem. Let

\[
b(\epsilon) = b + E \begin{pmatrix} \epsilon \\ \vdots \\ e^m \end{pmatrix}
\]

where \( E \) is nonsingular. Then

\[
x_B(\epsilon) = x_B + B^{-1}E \begin{pmatrix} \epsilon \\ \vdots \\ e^m \end{pmatrix}
\]

\[
= x_B + \sum_{k=1}^{m} (B^{-1}E)_{k} \epsilon^k
\]
We can choose \( E \) so that

\[
\sum_{k=1}^{m} (B^{-1}E)_k \epsilon^k > 0
\]

for all sufficiently small \( \epsilon \). We then perform our simplex step on the perturbed problem.

**Modifying LPs into standard form.** Here I will do problem 1 on p. 700. Part (a) is

\[
\begin{align*}
\text{max} & \quad 3x_1 + x_2 - 5x_3 + 2 \\
\text{s.t.} & \quad x_1 \geq x_2 \\
& \quad x_2 \leq 0 \\
& \quad -x_1 + 4x_3 \geq 0 \\
& \quad x_1 + x_2 + x_3 = 0
\end{align*}
\]

To start, do a change of variables, let \( z_1 = x_1, z_2 = x_2 + 2, z_3 = x_3 \). Then the LP becomes

\[
\begin{align*}
\text{min} & \quad -3z_1 - z_2 - 5z_3 \\
\text{s.t.} & \quad z_1 \geq z_2 - 2 \\
& \quad z_2 \leq 2 \\
& \quad -z_1 + 4z_3 \geq 0 \\
& \quad z_1 + z_2 + z_3 = 2
\end{align*}
\]

The inequality constraints can be corrected by adding three slack variables \( z_4, z_5 \) and \( z_6 \) so that the constraints become

\[
\begin{align*}
& \quad z_1 - z_2 - z_4 = 0 \\
& \quad z_2 + z_5 = 2 \\
& \quad -z_1 + 4z_3 - z_6 = 0
\end{align*}
\]

where

\( z_4, z_5, z_6 \geq 0 \).

There are no positivity constraints on \( z_1, z_2 \) and \( z_3 \). So we let

\[
z_i = z_i^+ - z_i^-
\]

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where
\[ z_i^+ = \max(z_i, 0), \quad z_i^- = \max(-z_i, 0). \]

This remakes our LP into
\[
\min 3z_1^+ + 3z_1^- + z_2^+ - z_2^- - 5z_3^+ + 5z_3^-
\]
\[
z_1^+ - z_1^- - z_2^+ + z_2^- - z_4 = 0
\]
\[
z_2^+ - z_2^- + z_5 = 2
\]
\[
-z_1^+ + z_1^- + 4z_3^+ - 4z_3^- - z_6 = 0
\]
\[
z_1^+ - z_1^- + z_2^- - z_2^+ + z_3 - z_3^- = 2
\]

If we now let
\[ \mathbf{y} = (z_1^+, z_1^-, z_2^+, z_2^-, z_3^+, z_3^-, z_4, z_5, z_6)^T \]

then we have a problem in \( \mathbf{y} \) where
\[
\mathbf{c} = (-3, 3, -1, 1, 5, -5, 0, 0, 0)^T
\]
\[
\mathbf{b} = (0, 2, 0, 2)^T
\]
\[
\mathbf{A} = \begin{pmatrix}
1 & -1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 4 & -4 & 0 & 0 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\min \mathbf{c}^T \mathbf{y}
\]
\[
\mathbf{A} \mathbf{y} = \mathbf{b}
\]
\[
\mathbf{y} \geq 0
\]

Thus this creates a standard form.
In the second example we have
\[
\min |x_1 + x_2 + x_3|
\]
subject to
\[
x_1 - x_2 = 5
\]
\[
x_2 - x_3 = 7
\]
\[
x_1 \leq 0
\]
\[
x_3 \geq 2
\]
Some variable transformations are necessary to put this in standard form. Let

\[ z_1 = -x_1 \]
\[ z_2 = x_1 + x_2 + x_3 \]
\[ z_3 = x_3 - 2 \]

Then we have

\[
\begin{align*}
\min & \quad |z_2| \\
-2z_1 - z_2 + z_3 & = 7 \\
z_1 + z_2 - 2z_3 & = 11 \\
z_1, z_3 & \geq 0
\end{align*}
\]

Except for the absolute value and the lack of positivity constraint on \( z_2 \), this is in standard form. Here again, we split the variable \( z_2 \) into

\[ z_2 = z_2^+ - z_2^- \]

where

\[ z_2^+ = \max(z_2, 0), \quad z_2^- = \max(-z_2, 0). \]

Then

\[ |z_2| = z_2^+ + z_2^- \]

and we have

\[
\begin{align*}
\min & \quad c^T y \\
\text{subject to} & \quad Ay = b \\
& \quad y \geq 0
\end{align*}
\]

where

\[ c = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \]
\[ A = \begin{pmatrix} -2 & -1 & 1 & 1 \\ 1 & 1 & -1 & -2 \end{pmatrix} \]
\[ b = \begin{pmatrix} 7 \\ 11 \end{pmatrix} \]
\[ y = \begin{pmatrix} z_1 \\ z_2^+ \\ z_2^- \end{pmatrix} \]

Part (c) is

\[ \min |x_1| - |x_2| \]

subject to

\[ x_1 + x_2 = 5 \]
\[ 2x_1 + 3x_2 - x_3 \leq 0 \]
\[ x_3 \geq 4 \]

First, we make all of the variables positive. Let

\[ z_1 = x_1^+, z_2 = x_1^-, z_3 = x_2^+, z_4 = x_2^-, z_5 = x_3 - 4 \]

and add the slack variable \( z_6 \geq 0 \).

Then

\[ |x_1| = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad |x_2| = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \]

This then become the LP

\[ \min c^T z \]

subject to

\[ Az = b \]
\[ z \geq 0 \]

where

\[ c = (1, 1, -1, -1, 0, 0)^T \]
\[ A = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 2 & -2 & 3 & -3 & -1 & 1 \end{pmatrix} \]

\[ b = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \]

\[ z = (z_1, z_2, z_3, z_4, z_5, z_6)^T \]