Cubic Splines

The cubic spline interpolation problem takes a set of points (called knots or nodes) such that
\[ a = x_0 < x_1 < \ldots < x_n = b \]
and finds
\[
s(x) = \begin{cases} 
  s_0(x) & x \in [x_0, x_1] \\
  s_1(x) & x \in [x_1, x_2] \\
  \vdots & \vdots \\
  s_{n-1}(x) & x \in [x_k, x_{k+1}] 
\end{cases}
\]
where \( s_k(x) \) is a cubic polynomial and
\[
s(x_k) = f(x_k) = f_k, \quad k = 0, \ldots, n
\]
and \( s, s', \) and \( s'' \) are continuous.

We take \( s_k(x) \) to have the form
\[
s_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3.
\]
Thus we must solve for \( 4n \) coefficients.

The first two derivatives of \( s_k \) are
\[
s_k'(x) = b_k + 2c_k(x - x_k) + 3d_k(x - x_k)^2 \]
\[
s_k''(x) = 2c_k + 6d_k(x - x_k).\]

We have that
\[
s_k''(x_k) = 2c_k.
\]

By continuity of \( s'' \)
\[
s_k''(x_{k+1}) = 2c_k + 6d_kh_k = s_{k+1}''(x_{k+1}) = 2c_{k+1}.
\]
where \( h_k = x_{k+1} - x_k \) is the spacing between points.
Solving for $d_k$ yields

$$d_k = (c_{k+1} - c_k)/(3h_k). \quad (1)$$

We have that

$$s_k(x_k) = a_k = f_k$$

and

$$s_k(x_{k+1}) = f_{k+1} = f_k + b_kh_k + c_kh_k^2 + (c_{k+1} - c_k)h_k^3/3$$

so

$$b_k = \delta_k - (2c_k + c_{k+1})h_k/3, \quad \delta_k = (f_{k+1} - f_k)/h_k. \quad (2)$$

Thus we need only solve for the $c_k$. For that, we use first derivative continuity!

We have that

$$s'_k(x_{k+1}) = b_k + 2c_kh_k + 3d_kh_k^2 = s_{k+1}(x_{k+1}) = b_{k+1}.$$  

Using the expressions (2) for $b_k$ and (1) for $d_k$ in terms of $c_k$ and $c_{k+1}$ and the definition of $\delta_k$ in (2) we obtain

$$\delta_k - \frac{h_k}{3}(c_{k+1} + 2c_k) + 2c_kh_k + (c_{k+1} - c_k)h_k = \delta_{k+1} - \frac{h_{k+1}}{3}(c_{k+2} + c_{k+1}).$$

After some algebra we get

$$h_kc_k + 2(h_k + h_{k+1})c_{k+1} + h_{k+1}c_{k+2} = 3(\delta_{k+1} - \delta_k), \quad k = 0, \ldots, n-2 \quad (3)$$

This is $n-1$ linear equations in the $n+1$ variables $c_0, c_1, \ldots, c_n$ necessary to construct the spline.

We need two more conditions, called endpoint conditions.

The simplest are for natural or variational splines. Here we set

$$s''(x_0) = 2c_0 = s''(x_n) = 2c_n = 0.$$  

This leads to a tridiagonal system of equations. The following is the version for 7 knots. It is $5 \times 5$.

$$Tc = b \quad (4)$$
where

\[
T = \begin{pmatrix}
2(h_0 + h_1) & h_1 & 0 & 0 & 0 \\
qquad h_1 & 2(h_1 + h_2) & h_2 & 0 & 0 \\
qquad 0 & h_2 & 2(h_2 + h_3) & h_3 & 0 \\
qquad 0 & 0 & h_3 & 2(h_3 + h_4) & h_4 \\
qquad 0 & 0 & 0 & h_4 & 2(h_4 + h_5)
\end{pmatrix}
\]

\[
c = \begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5
\end{pmatrix}, \quad b = 3 \begin{pmatrix}
\delta_1 - \delta_0 \\
\delta_2 - \delta_1 \\
\delta_3 - \delta_2 \\
\delta_4 - \delta_3 \\
\delta_5 - \delta_4
\end{pmatrix}
\]

This assumes that \(c_0 = c_6 = 0\). This is a particularly nice linear system. It is diagonally dominant and tridiagonal (zero except for the main diagonal, the superdiagonal, and the subdiagonal). It does require pivoting in Gaussian elimination and can be solved in \(O(n)\) operations where \(n\) is the number of knots in the spline.

Other endpoint conditions lead to slightly different systems.

For instance, first derivative conditions lead to complete splines. These specify that

\[
s'(x_0) = f'(x_0), \quad s'(x_n) = f'(x_n).
\]

If we have first derivative information at the endpoints, these lead to better approximations than do the natural splines.

Since

\[
s'(x_0) = b_0 = \delta_0 - (2c_0 + c_1)h_0/3 = f'(x_0),
\]

\[
s'(x_n) = b_{n-1} + 2c_{n-1}h_{n-1} + 3d_{n-1}h_{n-1}^2 = f'(x_n)
\]

Use of (2) and (1) and some algebra yields the two extra conditions

\[
(2c_0 + c_1)h_0/3 = \delta_0 - f'(x_0),
\]

\[
(2c_n + c_{n-1})h_0/3 = f'(x_n) - \delta_{n-1}
\]
Thus for the 7 node example we obtain (4) with

\[
T = \begin{pmatrix}
2h_0 & h_0 & 0 & 0 & 0 & 0 & 0 \\
h_0 & 2(h_0 + h_1) & h_1 & 0 & 0 & 0 & 0 \\
0 & h_1 & 2(h_1 + h_2) & h_2 & 0 & 0 & 0 \\
0 & 0 & h_2 & 2(h_2 + h_3) & h_3 & 0 & 0 \\
0 & 0 & 0 & h_3 & 2(h_3 + h_4) & h_4 & 0 \\
0 & 0 & 0 & 0 & h_4 & 2(h_4 + h_5) & h_5 \\
0 & 0 & 0 & 0 & 0 & h_5 & 2h_5 \\
\end{pmatrix}
\]

\[
c = \begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6 \\
\end{pmatrix}, \quad b = 3 \begin{pmatrix}
d_0 - f'(x_0) \\
d_1 - d_0 \\
d_2 - d_1 \\
d_3 - d_2 \\
d_4 - d_3 \\
d_5 - d_4 \\
f'(x_n) - d_5 \\
\end{pmatrix}.
\]

**Example 1**

\[f(x) = \cos x, \quad x \in [0, \pi/2]\]

Let \(x_k = \pi * k/12\) giving us 7 knots as above. For the natural splines we get

Then

\[
T = \pi/12 \begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 \\
0 & 1 & 4 & 1 & 0 \\
0 & 0 & 1 & 4 & 1 \\
0 & 0 & 0 & 1 & 4 \\
\end{pmatrix}
\]

Some computations yield

\[
b = \begin{pmatrix}
-0.7543 \\
-0.6763 \\
-0.5522 \\
-0.3905 \\
-0.2021 \\
\end{pmatrix}
\]
and thus we get

\[ c = \begin{pmatrix} -0.6205 \\ -0.3994 \\ -0.3652 \\ -0.2489 \\ -0.1308 \end{pmatrix} \]

For the complete splines we get

\[ T = \pi/12 \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \]

and

\[ b = \begin{pmatrix} -0.3905 \\ -0.7543 \\ -0.6763 \\ -0.5522 \\ -0.3905 \\ -0.2021 \\ -0.0342 \end{pmatrix} \]

The solution is

\[ c = \begin{pmatrix} -0.5029 \\ -0.4857 \\ -0.4355 \\ -0.3556 \\ -0.2514 \\ -0.1301 \\ -0.0002 \end{pmatrix} \]

Both the natural and complete splines have the following property.

**Theorem 1** Let \( f \) be any twice continuously differentiable function such that \( f \) satisfies the interpolation condition and

\[ f''(x_0) = f''(x_n) = 0. \]  

(5)
Then if $s$ is the cubic spline satisfying the same conditions then

$$\int_a^b [f''(x)]^2 \, dx \geq \int_a^b [s''(x)]^2 \, dx.$$  

The same conclusion holds if the condition

$$f'(x_0) = y'_0, \quad f'(x_n) = y'_n$$

is substituted for (5).

This theorem account for the “smooth” interpolation property of splines.

Other interesting endpoint conditions include second derivative conditions.

$$s''(x_0) = f''(x_0), \quad s''(x_n) = f''(x_n).$$

Again these assume some knowledge of the second derivative. That knowledge is often not available.

A useful spline if no endpoint information is available is the not-a-knot spline assumption. This assumes third derivative continuity near the boundary. This assumption is just

$$s'''_0(x_1) = s'''_1(x_1), \quad s'''_{n-2}(x_{n-1}) = s'''_{n-1}(x_{n-1}).$$