Finite Difference Interpolation

The Newton divided difference formula is that for the points \(x_0, x_1, \ldots, x_n\), if

\[
p_n(x_i) = f(x_i)
\]

\[
p_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \ldots + (x - x_0)(x - x_{n-1})f[x_0, x_1, \ldots, x_n].
\]

This is a simple elegant formula, but we can simplify it even further when the points are evenly spaced.

For that let,

\[
x_k = x_0 + kh, k = 0, 1, 2, \ldots.
\]

The book allows negative indicies, but for now we will not get into that.

**Forward Differences**

\[
\Delta f(x_k) = f(x_{k+1}) - f(x_k)
\]

\[
\Delta^2 f(x_k) = \Delta f(x_{k+1}) - \Delta f(x_k) = f(x_{k+2}) - 2f(x_{k+1}) + f(x_k)
\]

\[
\Delta^n f(x_k) = \Delta^{n-1} f(x_{k+1}) - \Delta^{n-1} f(x_k)
\]

For example

\[
\Delta^3 f(x_0) = f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0).
\]

An induction argument yields the following expression.

\[
\Delta^n f(x_k) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x_{k+n-j}).
\]

As before we set up a table. For instance

<table>
<thead>
<tr>
<th>x</th>
<th>f</th>
<th>\Delta f</th>
<th>\Delta^2 f</th>
<th>\Delta^3 f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.0</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>5</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
How to construct the polynomial. Essentially,

\[ f[x_k, x_{k+1}, \ldots, x_{k+n}] = \frac{\Delta^n f(x_k)}{h^n n!} \]

We invent the shorthand

\[ f_k = f(x_k), \quad \Delta^n f(x_k) = \Delta^n f_k. \]

We then let \( x = x_0 + sh \), thus

\[
\begin{align*}
p_n(x) &= p_n(x + sh) \overset{\text{def}}{=} \tilde{p}_n(s) \\
&= f_0 + sh \frac{\Delta f_0}{h} + \frac{s(s-1)h^2 \Delta^2 f_0}{2! h^2} + \ldots \\
&\quad + \frac{s(s-1) \cdots (s-n+1)h^n \Delta^n f_0}{n! h^n} \\
&= f_0 + s \Delta f_0 + \frac{s(s-1) \Delta^2 f_0}{2!} + \ldots \\
&\quad + \frac{s(s-1) \cdots (s-n+1) \Delta^n f_0}{n!}
\end{align*}
\]

This is a much easier expression to work with.

For instance, for the example above, the resulting polynomial is

\[ \tilde{p}_2(s) = p_2(x_0 + 0.5 \times s) = f_0 + s \Delta f_0 + \frac{s(s-1)}{2!} \Delta^2 f_0. \]

**Error in Interpolation**

The error in interpolation is best expressed using the Newton formula. Let \( p_n(x) \) be the polynomial such that \( p_n \) interpolates \( f \) at \( x_0, x_1, \ldots, x_n \).

To find \( f(t) - p_n(t) \) for a point \( t \neq x_0, x_1, \ldots, x_n \), let \( p_{n+1}(x) \) be the polynomial that interpolates \( f \) at \( x_0, x_1, \ldots, x_n, t \), that is, add \( t \) as an extra point. Then

\[ p_{n+1}(x) = p_n(x) + (x - x_0) \cdots (x - x_n) f[x_0, x_1, \ldots, x_n, t]. \]

Since

\[ f(t) = p_{n+1}(t), \]

we have that

\[ f(t) - p_n(t) = (t - x_0) \cdots (t - x_n) f[x_0, x_1, \ldots, x_n, t]. \]
If we use the fact that for some $\xi \in (a, b)$ where 
\[ a = \min_i \{ \min_i x_i, t \}, \quad b = \max_i \{ \max_i x_i, t \} \]
then 
\[ f[x_0, x_1, \ldots, x_n, t] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \]
and we obtain the formula
\[ f(t) - p_n(t) = \frac{(t - x_0) \cdots (t - x_n)f^{(n+1)}(\xi)}{n!}. \]

**Third and Last Form – Lagrange Form**

Problem, find the polynomial $p_n(x)$ such that 
\[ p_n(x_i) = y_i, \quad i = 0, \ldots, n. \]

Start with the interpolation problem 
\[ \ell_i(x) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases} \]

First $\ell_i(x)$ is a polynomial with $n$ roots 
\[ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n. \]

So for some constant $c$, 
\[ \ell_i(x) = c(x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n) = c \prod_{j \neq i} (x - x_j) \]

Now we need only determine $c$. We have that 
\[ \ell_i(x_i) = c \prod_{j \neq i} (x_i - x_j) = 1 \]
thus 
\[ c = 1/ \prod_{j \neq i} (x_i - x_j). \]

Therefore 
\[ \ell_i(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} = \prod_{j \neq i} \left( \frac{x - x_j}{x_i - x_j} \right). \]
Then the interpolation polynomial may be written as

\[ p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x). \]

Since

\[ p_n(x_j) = y_j \ell_j(x_j) = y_j. \]

Incidentally, this is another proof of existence and uniqueness.

A quick example.

**Example 1** The same problem again.

\[ f(x) = \cos x, \quad x_0 = 0, x_1 = \pi/4, x_2 = \pi/2. \]

We have that

\[
\begin{align*}
\ell_0(x) &= \frac{(x - \pi/4)(x - \pi/2)}{(0 - \pi/4)(0 - \pi/2)} = \frac{8}{\pi^2} (x - \pi/4)(x - \pi/2) \\
\ell_1(x) &= \frac{x(x - \pi/2)}{\pi/4(-\pi/4)} = \frac{16}{\pi^2} x(\pi/2 - x) \\
\ell_2(x) &= \frac{8}{\pi^2} x(x - \pi/4).
\end{align*}
\]

Taking \( y_0 = 1, y_1 = \sqrt{1/2}, y_2 = 0 \), we have

\[ p_2(x) = \frac{8}{\pi^2} (x - \pi/4)(x - \pi/2) + \sqrt{1/2} \frac{16}{\pi^2} x(\pi/2 - x). \]

This is an explicit formula. If we add points, there is no easy way to do it.

On the other hand, if I want the interpolation function for \( \sin x \) on the same points, I can just change the \( y_i \) values to \( y_0 = 0, \ y_1 = \sqrt{1/2} \) and \( y_2 = 1 \).

In that case,

\[ p_2(x) = \sqrt{1/2} \frac{16}{\pi^2} x(\pi/2 - x) + \frac{8}{\pi^2} x(x - \pi/4). \]

Thus it is easier to produce another interpolation function on the same points than with the divided difference formulas.